Real Analysis

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This book is the lecture notes of an undergraduate course on Real Analysis in Koç University during 1995-2006 given by Prof. Ali Ülger.

Contents

PRELIMINARIES

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2. **Sequences and Subsequences**

0.1 Sets and Mappings

Let X be any set, by 2^X we denote the set of all subsets of X. $A \subseteq X \Leftrightarrow A \in 2^X$.

If $A \subseteq X$ by A^C we shall denote the complement of A in X.

$A^C = \{x \in X, x \notin A\} = X - A$

Now let I be an index set. Suppose that for each $\alpha \in I$, we have a set A_α.

Then the collection of $A_\alpha, \alpha \in I$ is said to be a family of sets.

For such a family, if $I \neq \varnothing$, for $\alpha \in I$, $\cup A_\alpha$ and $\cap A_\alpha$ are defined by:

$$\cup A_\alpha = \{x : \exists \alpha \in I \ni x \in A_\alpha\}$$

$$\cap A_\alpha = \{x : \forall \alpha \in I \ni x \in A_\alpha\}$$

Now suppose that $A_\alpha \subseteq X$. Then,

$$\begin{aligned} (\cup A_\alpha)^C &= \cap A_\alpha^C \\ (\cap A_\alpha)^C &= \cup A_\alpha^C \end{aligned} \qquad \text{(De Morgan's Law)}$$

Now let Y be another set and $f : X \to Y$ be a mapping.

For any $A \subseteq X$ we define the **direct image** of A under f by

$$f\left(A\right) = \{y \in Y : y = f\ \ (x)\ \text{ for some } x \in A\}$$

For any $B \subseteq X$,we define the **inverse image** of B under f by

$$f^{-1}\left(B\right) = \{x \in X : f(x) \in B\}$$

Example 0.1.1 *: If $X = Y = \mathbb{R},\ f(x) = \sin x$,*

$B\ = \{0\}\,, f^{-1}\left(B\right) = \{x \in \mathbb{R}{:}f(x) = 0\} = \pi\mathbb{Z}$

Example 0.1.2 *if $X = Y = \mathbb{R}, f(x) = x^2, f^{-1}([0,\infty[) = \{x \in \mathbb{R} : f(x) \in [0,\infty[\}$*

Properties: Let $\left(A_\alpha\right)_{\alpha\in I}$ be a family of subsets of X and $f : X \to Y$ be any mapping, then:

1. $f\left(\cup_{\alpha\in I} A_\alpha\right) = \cup_{\alpha\in I} f\left(A_\alpha\right)$
2. $f\left(\cap_{\alpha\in I} A_\alpha\right) \subseteq \cap_{\alpha\in I} f\left(A_\alpha\right)$

Example 0.1.3 *(for property 2): Let $X = \mathbb{R}^2, Y = \mathbb{R}$ and if $f : \mathbb{R}^2 \to \mathbb{R},\ f((x, y)) = x$ (the first projection)*
Let $A_1 = \{(x, x) : x \in \mathbb{R}\},\ A_2 = \{(x, 2x) : x \in \mathbb{R}\}$
Now, $f(A_1) = \mathbb{R}$ on the other hand, $f(A_2) = \mathbb{R}$. But $A_1 \cap A_2 = \{(0, 0)\}$
$f\left(A_1 \cap A_2\right) = 0 \neq f(A_1) \cap f(A_2) = \mathbb{R}$

Proposition 0.1.4 *Let $f : X \to Y$ be a mapping and $\left\{B_\alpha\right\}_{\alpha\in I}$be a family of subsets of Y. Then:*

1. $f^{-1}\left(\cup_{\alpha\in I} B_\alpha\right) = \cup_{\alpha\in I} f^{-1}(B_\alpha)$
2. $f^{-1}\left(\cap_{\alpha\in I}\right) B_\alpha = \cap_{\alpha\in I} f^{-1}(B_\alpha)$

Proposition 0.1.5 *Let $f : X \to Y$ be a mapping. $A \subseteq X$, $B \subseteq Y$ two sets. Then:*

1. *$f^{-1}(f(A)) \supseteq A$ (Equality holds if f is 1-to-1)*
2. *$f\left(f^{-1}\left(B\right)\right) \subseteq B$ (Equality holds if f is onto)*

Example 0.1.6 *Let $f : R \to R$, $f(x) = \sin x$ Take $A = \{\pi\} \Longrightarrow f(A) = 0$*
$f^{-1}(\{0\}) = \pi\mathbb{Z} \supset A$

Example 0.1.7 *Let* $f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^2$
$B=[0,+\infty[\implies f^{-1}(B)=\mathbb{R}$
Then $f\left(f^{-1}(B)\right)=f(\mathbb{R})=B$

Example 0.1.8 *Let* $f:R\to R$, $f(x)=\sin x$ *Let* $B=\{0,2\}$ $f^{-1}(B)=\pi\mathbb{Z}$
$f\ \left(f^{-1}(B)\right)=f(\pi\mathbb{Z})=\{0\}\subset\{0,2\}$

Remark: Let $f:X\to Y$ be a mapping.

If $A\subseteq X\implies f(A^c)\neq f(A)^c$ (= if f is bijective)

But for $B\subseteq Y$ $f^{-1}(B^c)=f^{-1}(B)^C$

0.2 Sequences and Subsequences

The set $\mathbb{N}=\{1,2,....\}$ is the set of the positive integers.

Definition 0.2.1 *Let* X *be any set,* $X\neq\emptyset$. *Any mapping* $\Gamma:\mathbb{N}\to X$ *is said to be a* ***sequence in*** X.

Let for each $n\in N$, $x_n=\Gamma(n)$. *Then instead of* $\Gamma(n)$ *we usually write* $(x_n)_{n\in N}$ *and say that* $(x_n)_{n\in N}$ *is a sequence in* X.

The set $\Gamma\left(N\right)=\{x_n:n\in\mathbb{N}\}\subseteq X$ *is the range of* Γ.

Remark: Do not confuse Γ which is a mapping with its range. It is a set **!**

Example 0.2.2 *Let* $\mathbb{X}=\mathbb{R}$, $\Gamma:\mathbb{N}\to\mathbb{R}$

$\Gamma(n)=n^2$ $\Gamma(n)=n$ $\Gamma(n)=\ln(n+1)$ *Sequences in* $\mathbb{R}$.

0.2.1 Infinite subsets of $\mathbb{N}$

Let $\mathcal{F}=\left\{F\in 2^{\mathbb{N}}:F \text{ is an infinite set}\right\}$. What is $card\mathcal{F}$?

Let p & q be two prime numbers, $(p\neq q)$, then $\forall n,m\in N\setminus\{0\}\,,p^n\neq q^m$ $(*)$

Let $p_{0,}\,p_1,\,p_2,\ldots,p_k,\ldots$ be distinct prime numbers.

Let for each $k=0,1,2,\ldots$, $F_k=\left\{p_k^{n+1}:n\in\mathbb{N}\right\}$. If $p_k=2\implies F_k=\{2{,}2^2{,}2^3,\ldots\}$

Hence $(*)$ shows that for $i\neq j$, $F_i\cap F_j=\varnothing$. Moreover, each F_i is an infinite set.

Let also $\mathcal{F}_0 = \{A \in 2^n : A \text{ is finite}\}$

$\mathcal{F}_0 \cap \mathcal{F} = \varnothing$

$\mathcal{F}_0 \cup \mathcal{F} = 2^N$

Proposition 0.2.3 *$\mathcal{F}_0$ is countable.*

Proof 0.2.4 *For every $n \in \mathbb{N}$, $\mathbb{N}^n = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \times \cdots \times \mathbb{N}$ is a countable set.*

So, $Y = \cup_{n\in\mathbb{N}} \mathbb{N}^n$ is also countable.

Now we define a mapping $f : \mathcal{F}_0 \to Y$ as follows:

Let $A \in \mathcal{F}_0$. So A is of the form: $A = \{n_1, n_2, n_3, \ldots, n_k, \ldots\}$

$f(A) = (n_1, n_2, \ldots, n_k) \in \mathbb{N}^k$. Clearly f is $1 - to - 1$. As Y is countable , so is $\mathcal{F}_0$.

Conclusion: *The set $\mathcal{F}$ is uncountable.*
Thus in $\mathbb{N}$ there are uncountably many infinite subsets.

Definition 0.2.5 *A mapping $\Gamma : \mathbb{N} \to \mathbb{N}$ is said to be* ***strictly increasing*** *if whenever $n < m$ we have $\Gamma(n) < \Gamma(m)$*

Example 0.2.6 *Let*
$\Gamma : \mathbb{N} \to \mathbb{N}, \Gamma(n) = 2n$
$\Gamma : \mathbb{N} \to \mathbb{N}, \Gamma(n) = 3n + 1$
$\Gamma : \mathbb{N} \to \mathbb{N}, \Gamma(n) = n^2 + n + 1$
are strictly increasing mappings.

Question: How many strictly increasing mappings $\Gamma : \mathbb{N} \to \mathbb{N}$ do we have?

*If $\Gamma : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then the set $\mathcal{F} = \Gamma(\mathbb{N})$ is an infinite set.

**Now let $\mathcal{F} \subseteq N$ be an infinite set. So $\mathcal{F}$ is of the form $\mathcal{F} = \{n_0, n_1, ...\}$ with $n_0 < n_1 < n_2 < \ldots$

To $\mathcal{F}$, we associate the mapping $\Gamma : \mathbb{N} \to \mathbb{N}$,$\Gamma(k) = n_k$ so that $\Gamma(\mathbb{N}) = \mathcal{F}$.

The above two points "*" and "**" show that there are uncountably many strictly increasing mappings.

0.2.2 Subsequences of a Given Sequence

Definition 0.2.7 *Let $\Gamma : \mathbb{N} \to X$ be any sequence and $\Psi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing mapping. $\Gamma \circ \Psi : \mathbb{N} \to X$ is also a sequence. The sequence $\Gamma \circ \Psi$ is said to be a* ***subsequence of*** *Γ.*

Hence any sequence Γ has uncountably many subsequences.

Practical notation for subsequences: Let $\Gamma = (x_n)_{n \in N}$ be a sequence. ($\Gamma : \mathbb{N} \to X$, $x_n = \Gamma(n)$)

Let $\Psi : \mathbb{N} \to \mathbb{N}$ be a strictly increasing mapping.

Let $n_k = \Psi(k)$ so that $n_0 < n_1 < n_2 < \ldots < n_k < \ldots$ then, $\Gamma \circ \Psi(k) = x_{n_k}$, and $\boxed{k \to \infty \Longrightarrow n_k \to \infty}$

So, $(x_{n_k})_{k \in \mathbb{N}}$ is a subsequence of $(x_n)_{n \in \mathbb{N}}$

If we put $y_k = x_{n_k}$ is a sequence of its own , i.e. $(y_k)_{k \in \mathbb{N}}$ is a sequence.

So, if $(x_n)_{n \in \mathbb{N}}$ is a sequence in a set and $n_0 < n_1 < n_2 < \ldots < n_k$ are given integers.

Taking $y_k = x_{n_k}$ we obtain a new sequence $(y_k)_{k \in \mathbf{N}}$. This later sequence is said to be a **subsequence of** $(x_n)_{n \in N}$.

Observe that $\{x_{n_0}, x_{n_1}, x_{n_2}, ...\} \subseteq \{x_0, x_1, x_2 ....\}$

Example 0.2.8 *If $X = \mathbb{R}$, $x_n = \frac{1}{n^2+1}$ and $n_0 < n_1 < n_2 < \ldots < n_k < \ldots$ is any sequence of integers.*

$y_k = \frac{1}{(n_k)^2+1}$ *, is a subsequence of x_n.*

So, for instance, if $n_k = 3k + 5 \Rightarrow n_0 = 5$, $n_1 = 8$, $n_2 = 11, \ldots$

then, $x_{n_k} = \dfrac{1}{(3k+5)^2+1}$ *and it takes such values for given n_k's.*

$x_0 = 1 \quad x_{n_0} = \dfrac{1}{26}$

$x_1 = \frac{1}{2} \quad x_{n_1} = \dfrac{1}{65}$

$x_2 = \frac{1}{5} \quad x_{n_2} = \dfrac{1}{122}$

Example 0.2.9 • *Consider the sequence $(x_n)_{n\in\mathbb{N}}$ that goes as follows:*

$$0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, \ldots$$

Give at least three subsequences of that sequence.

1. $x_0 \longrightarrow x_{n_0}$
2. $x_5 \longrightarrow x_{n_1}$
3. $x_{10} \longrightarrow x_{n_2}$

- *For the sequence $x_n = (-1)^n$ Find at least three subsequences.*

  $(x_{2n})_{n\in\mathbb{N}}$ *is a subsequence*

  $(x_{2n+1})_{n\in\mathbb{N}}$ *is a subsequence*

  $(x_{3n+2})_{n\in\mathbb{N}}$ *is a subsequence*

Remark:

- Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. If $x_0 = x_1 = x_2 = \ldots = x_k = \ldots$, then we say that, $(x_n)_{n\in\mathbb{N}}$ is a **constant sequence**.
- If there exists $N \in \mathbb{N} \ni \forall n \geq N, x_N = x_n = x_{n+1} = \ldots$ then we say, $(x_n)_{n\in\mathbb{N}}$is **almost constant**.

0.2.3 Exercises I

The letters X, Y, Z will denote sets and the letters f, g, h will denote the mappings.

1. Let $(A_\alpha)_{\alpha\in I}$ be a family in 2^X and $A \in 2^X$.

   Show that $(\cup_{\alpha\in I} A_\alpha)\backslash A = \cup_{\alpha\in I}(A_\alpha\backslash A)$ and $A\backslash(\cup_{\alpha\in I} A_\alpha) = \cap_{\alpha\in I}(A\backslash A_\alpha)$.

2. Let $(A_n)_{n\in\mathbb{N}}$ be a sequence of sets. Let $B_0 = A_0,\ B_1 = A_1\backslash A_0, ..., B_n = A_n\backslash \cup_{k<n} A_k, ...$

   Show that the sets B_n are pairwise disjoint, $\cup_{k\leq n} B_k = \cup_{k\leq n} A_k$ and $\cup_{n\in\mathbb{N}} B_n = \cup_{n\in\mathbb{N}} A_n$.

   Deduce from this another proof of the fact that the countable union of countably many sets is at most countable.

3. Prove that $f : X \to Y$, is one-to-one iff it is left invertible, i.e. there exists a mapping $g : Y \to X$ such that $g \circ f = I_X$.

   Show that such g is onto.

4. Prove that $f : X \to Y$, is onto iff it is right invertible, i.e. there exists a mapping $g : Y \to X$ such that $f \circ g = I_Y$.

   Show that such a g is one-to-one.

5. Let $f : X \to Y$ be a mapping. Let $F : 2^X \to 2^Y$ be the mapping defined by

   $F(A) = f(A)$.

   Show that F is one-to-one (onto) iff f is one-to-one (onto).

6. Let $f : X \to Y$ and $g : Y \to Z$ be two mappings. Show that

   (a) If $g \circ f$ is one-to-one, then f is one-to-one.

   (b) If $g \circ f$ is onto, then g is onto.

   (c) If $g \circ f$ is onto and g is one-to-one, then f is onto.

   (d) If $g \circ f$ is one-to-one and f is onto, then g is one-to-one.

7. Let f and g be as in 6. If f and g are both bijective, then show that $g \circ f$ is bijective and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

8. If $f : X \to Y$, and $g : X \to Z$, are such that the implication

   $(g(x) = g(y) \Rightarrow f(x) = f(y))$ holds, then show that there exists a mapping

   $h : Z \to Y$, such that $h \circ g = f$.

9. If $f : Z \to X$ is a mapping and $g : Y \to X$ is a one-to-one mapping, then show that, there exists a mapping $h : Z \to Y$ such that $f = g \circ h$ iff $f(Z) \subseteq g(Y)$.

10. For $A \subseteq X$, let $\chi_A : X \to \{0,1\}$ be the mapping defined by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$

    Show that the following holds.

    (a) $\chi_A = 0$ iff $A = \emptyset$.

    (b) $\chi_A = 1$ iff $A = X$.

    (c) $\chi_A = \chi_B$ iff $A = B$.

    (d) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \times \chi_B$ and $\chi_{A\cap B} = \chi_A \times \chi_B$.

    (e) $\chi_{A^c} = 1 - \chi_A$.

    (f) $\chi_{A\Delta B} = |\chi_A - \chi_B|$, where $A\Delta B = (A\backslash B) \cup (B\backslash A)$.

    (g) $\chi_{A\Delta B} = \chi_A + \chi_B \pmod 2$

11. Let $F(X; \{0,1\})$ be the set of the mappings $f : X \to \{0,1\}$.

    Show that there exists a bijection between the sets $F(X; \{0,1\})$ and 2^X.

12. Let S be the set of all the sequences in the set $\{0,1\}$. Show that the set S is uncountable.

13. Let $F = \{A \in 2^{\mathbb{N}} : \text{both } A, \text{ and } A^c, \text{ are infinite}\}$. Show that the set F is uncountable.

14. Suppose that the sets X and Y are infinite and $f : X \to Y$ is an onto mapping such that, for each $y \in Y$, the set $f^{-1}(\{y\})$ is countable.

    Show that then $Card(X) = Card(Y)$.

15. Let $F = \{A \in 2^{\mathbb{N}} : A \neq \emptyset \text{ and finite}\}$. Let $\varphi : F \to \mathbb{N}$, $\varphi(A) = \displaystyle\sum_{n\in A} n$.

    Show that φ is onto and that, for each $n \in \mathbb{N}\,(n \geq 1)$, the set $\varphi^{-1}(n)$ is finite.

    From this deduce another proof of the fact that F is countable.

16. Suppose that X is infinite and F is the set of the finite subsets of X. Show that $Card(X) = Card(F)$.

17. Show that A is infinite iff it has a proper subset B such that $Card(A) = Card(B)$.

18. Let $p_1, p_2, \ldots$ be prime numbers. Let $F_k = \{(p_k)^{n+1} : n \in \mathbb{N}\}$.

    Show that the sets $F_1, F_2, \ldots$ are infinite and pairwise disjoint.

    Deduce that any sequence $(x_n)_{n\in\mathbb{N}}$ in a set X has infinitely many subsequences with pairwise disjoint index sets.

19. Let $A_0, A_1, \ldots$ be nonempty subsets of X. Put $A^* = \cap_{n\in\mathbb{N}}\cup_{k\geq n} A_k$ and $A_* = \cup_{n\in\mathbb{N}}\cap_{k\geq n} A_k$.

(a) Show that $A_* \subseteq A^*$ and that $A_* = A^*$ if the sequence of the sets $(A_n)_{n\in\mathbb{N}}$ is monotone.

(b) Let $x \in X$ be a given point. Show that

i. $x \in A^*$ iff $x \in A_n$ for infinitely many $n \in \mathbb{N}$.

ii. $x \in A_*$ iff $x \in A_n$ for all but finitely many $n \in \mathbb{N}$.

(c) Explain the difference between the sentences in 19(b)i and 19(b)ii.

20. Let $(X, \leq)$ be an ordered set such that for any two elements x, y in X, $\sup\{x, y\}$ and $\inf\{x, y\}$ exist.

    Let $f : X \to X$ be a mapping. Show that f is increasing iff $f(\inf\{x, y\}) \leq \inf f(\{x, y\})$, for every x, y in X.

21. Let $(x_n, y_n)_{(n,n)\in\mathbb{N}\times\mathbb{N}}$ be a sequence in $X \times X$ and A and B be two infinite subsets of $\mathbb{N}$.

    Is the sequence $(x_k, y_p)_{(k,p)\in A\times B}$ a subsequence of $(x_n, y_n)_{(n,n)\in\mathbb{N}\times\mathbb{N}}$?

0.3 Some Notes:

Definition 0.3.1 *Let $X\,(\neq \varnothing)$ be a set and $\leq$ be a binary relation in X.*
$\leq$ *is said to be an* ***order relation****, if it is reflexive, antisymmetric, transitive.*

The set X equipped with an order relation is said to be an ordered set.

Example 0.3.2 1. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ *are ordered under the usual "less than", $\leq$.*

2. *Let E be any set and $X = 2^E$*

   For $A, B \in X$ let $A \preceq B$ iff $A \subseteq B$ then, $\preceq$ is a n order relation on X, known as ***inclusion relation****.*

3. *Let $X = F\,(\mathbb{N},\mathbb{N})$ be the set of all mappings: $\Gamma : \mathbb{N} \rightarrow \mathbb{N}$.*

   We define a binary relation $\preceq$ on X as follows:

   $\Gamma \leq \Psi$ *iff* $\Gamma(n) \leq \Psi(n)\ \ \forall\ n \in \mathbb{N}$. *Then $\leq$ is an order relation on $\mathbb{N}$.*

Definition 0.3.3 *Now, let $(X, \preceq)$ be an ordered set and $A \subseteq X$, $(A \neq \varnothing)$. We say that,*

1. *A is* ***bounded from above****, if there is an $m \in X \ni \forall a \in A, a \preceq m$.*

   Such an m is said to be an ***upper bound*** *for A. Of course any $m' \in X$, $m \preceq m'$ is also an upper bound.*

   For instance, $\mathbb{Q}$ and $\mathbb{N}$ are not bounded from above.

   Now let $A = \{x \in \mathbb{Q}, x^2 \preceq 2\}$ Then, A is bounded from above.

2. *A is* ***bounded from below*** *if $\exists n \in X \ni \forall a \in A, n \preceq a$*

   In this case, n is said to be a ***lower bound*** *for A. Of course any $n' \preceq n$ is also a lower bound for A.*

   For instance, $\mathbb{N}$ is bounded from below.

   But $\mathbb{Q}$ and $\mathbb{Z}$ are not bounded from below.

3. *A is* ***bounded****, if $A \subseteq X$ is both bounded from above and below.*

   Hence, A is bounded if $\exists n, m \in X$, $\forall x \in A$, $n \preceq x \preceq m$

4. *A has a* ***greatest element*** *if there is an element $\alpha \in A \ni \forall x \in X$, $x \preceq \alpha$*

   A has a a ***smallest element*** *if there is an element $\beta \in A \ni \forall x \in A$, $\beta \preceq x$*

Example 0.3.4 *If $X = \mathbb{Q}$, $A = \mathbb{N}$ then A has a smallest element namely $\beta = 0$ but it has no greatest element.*
If $X = \mathbb{N}$, $A \subseteq \mathbb{N}$ $A \neq \varnothing$ then A has a smallest element

Definition 0.3.5 *Let $A \subseteq X$ be a set. We say that, A has a* ***least upper bound***

$$\text{if } \exists \alpha \in X \ni \begin{cases} i) & \forall x \in A, x \leq \alpha \\ ii) & \forall \beta \in X satisfying \text{ “} \forall x \in A, x \leq \beta \text{”}, \ \alpha \leq \beta. \end{cases}$$

In this case ,we write, $\alpha = \sup A$ or $\alpha = lubA$

$$\text{Thus, } \alpha = \sup A \Leftrightarrow \begin{cases} (1) & \forall x \in A, x \leq \alpha \\ (2) & \forall \beta \in X, \text{ if for all } x \in A, x \leq \beta \text{ then } \alpha \leq \beta. \end{cases}$$

If $X = \mathbb{Q}$, $A = \{x \in Q : x^2 \leq 2\}$. Does A have a least upper bound?

Definition 0.3.6 *Let $A \subseteq X$ we say that A has a greatest lower bound if*

$$\exists \beta \in X \ni: \begin{cases} (i) & \forall x \in A, x \geq \beta, \\ (ii) & \forall \gamma \in X \text{ satisfying “} \forall x \in A, x \geq \gamma \text{”} \ \gamma \leq \beta. \end{cases}$$

If A has a greatest element α, then $\alpha = \sup A$, conversely, if $\alpha = \sup A$ and $\alpha \in A \Rightarrow \alpha$ is the greatest element of A.

Similarly, if A has a smallest element β then $\beta = \inf A$.

0.3.1 Exercises II

1. Find at least 3 different subsequences of the sequence

$$\begin{array}{cccccccccccccl} x_0 & x_1 & x_2 & x_3 & x_4 & \multicolumn{9}{l}{\dots\dots\dots\dots\dots\dots\dots\dots} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & & & & & & & \\ 0, & 1, & 2, & 3, & 0, & 1, & 2, & 3, & 0, & 1, & 2, & 3, & 0, \dots \end{array}$$

2. Find at least 2 different subsequences of the sequence

$$\begin{array}{cccccccl} x_0 & x_1 & x_2 & x_3 & \multicolumn{4}{l}{\dots\dots\dots\dots} \\ \downarrow & \downarrow & \downarrow & \downarrow & & & & \\ 1, & \frac{1}{2}, & 3, & \frac{1}{4}, & 5, & \frac{1}{6}, & 7, & \frac{1}{8}, \dots \end{array}$$

3. Let x, y, z be 3 real numbers. Put $x^+ = \max\{x, 0\}$ and $x^- = \min\{-x, 0\}$.
   Prove the following:
   (a) $x = x^+ - x^-$
   (b) $|x| = x^+ + x^-$
   (c) $x + y = \max\{x, y\} + \min\{x, y\}$
   (d) $\sup\{x, y\} + z = \sup\{x + z, y + z\}$
   (e) $\min\{x, y\} + z = \min\{x + z, y + z\}$
   (f) $x \leq y$ iff $x^+ \leq y^+$ and $x^- \leq y^-$
   (g) $\sup\{x, y\} = -\inf\{-x, -y\}$
   (h) $\max\{x, y\} = \max\{x - y, 0\} + y = (x - y)^+ + y = \dfrac{|x - y| + x + y}{2}$.
   (i) $\min\{x, y\} = \min\{x - y, 0\} + y = -(x - y)^- + y = \dfrac{x - y - |x - y|}{2}$.

4. Let X be an infinite set. Let F be the set of all the finite subsets of X.
   Show that $CardF = CardX$.

5. Show that $\mathbb{N}$ contains infinitely many infinite sets $A_0, A_1, ..., A_n, ...$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$.

6. Let $a = (a_1, ..., a_n) \in \mathbb{R}^n$ be a fixed element, $1 \leq p < \infty$ and $||x||_p = [|a_1|^p + ... + |a_n|^p]^{\frac{1}{p}}$.
   Show that $\lim_{p\to\infty} ||a||_p = \max\{|a_1|, |a_2|, ..., |a_n|\}$.

7. Let A and B be 2 nonempty subsets of $\mathbb{R}$.
   Let $A + B = \{a + b : a \in A,\ b \in B\}$, $A \times B = \{a \times b : a \in A, b \in B\}$. Show that
   (a) if A and B are bounded from above (or below), then so are the sets $A + B$, $A \times B$, $A \cup B$, $A \cap B$.

(b) if A is bounded from above, then so is every nonempty subsets of A.

8. Let A and B be 2 nonempty subsets of $\mathbb{R}$. Assume that both of them are bounded. Show that

   (a) if $A \subseteq B$, then $\sup A \leq \sup B$ and $\inf A \geq \inf B$.

   (b) $\sup(A + B) = \sup A + \sup B$.

   (c) $\sup\{|a| \times |b| : a \in A, b \in B\} \leq \sup\{|a| : a \in A\} \times \sup\{|b| : b \in B\}$.

   Give an example showing that in 8c in general we do not have equality.

Chapter 1

The Real Number System

1. **Axiomatic definition and basic Properties of $\mathbb{R}$**
2. **Convergence in $\mathbb{R}$ and monotone sequences**
3. **Bolzano-Weierstrass Theorem**
4. **Cauchy sequences**
5. $\limsup$, $\liminf$
6. **Elementary topology of $\mathbb{R}$**

" God created the real numbers, we learn its properties."

1.1 Axiomatic Definition and Basic Properties of $\mathbb{R}$

There exists a set **R** called the set of real numbers, satisfying the following axioms:

Axiom 1.1.1 ***(Algebraic Structure)*** $(\mathbb{R}, +, .)$ *is a field and it contains* $\mathbb{Q}$ *as a subfield.*

We denote the natural element of $\mathbb{R}$ for $+$ by 0.
The inverse for $x \neq 0$ for multiplication by $\frac{1}{x}$, for addition by $-x$.

Axiom 1.1.2 ***(Order Structure)*** *There exists an order relation on* $(\mathbb{R}, \leq)$ *extending that of* $\mathbb{Q}$*, which is total (i.e.,* $\forall x, y \in \mathbb{R}$*,* $x \leq y$ *or* $y \leq x$*) and which is consistent with the algebraic structure. This means that,*

1. $x \leq y \Longrightarrow (\forall z \in \mathbb{R})\, x + z \leq y + z$
2. $x \leq y$ *and* $z \geq 0 \Longrightarrow xz \leq yz$

Axiom 1.1.3 ***(Supremum)*** *Any nonempty set $A \subseteq \mathbb{R}$, which is bounded from above, has a supremum $\alpha \in \mathbb{R}$ i.e. there is a number $\alpha \in \mathbb{R}$ such that:*

1. $\forall\, x \in A, x \leq \alpha$
2. $\forall \varepsilon > 0,\ \exists\, x_\varepsilon \in A,\ x_\varepsilon > \alpha - \varepsilon$

For any $A \subseteq X$,

$$\alpha = \sup A \iff \begin{cases} 1)\ \forall\, x \in A, x \leq \alpha \\ 2)\ \forall\, \beta \in X, \text{ if } \forall x \in A, x \leq \beta, \text{ then } \alpha \leq \beta \end{cases}$$

This α is said to be the supremum of $\mathbb{A}$ and denoted by $\alpha = \sup A$. Thus,

$$\boxed{\alpha = \sup A \iff \begin{cases} 1)\ \forall\, x \in A,\ x \leq \alpha \\ 2)\ \forall\, \varepsilon > 0,\ \exists\, x_\varepsilon \in A, \ni\ x_\varepsilon > \alpha - \varepsilon \end{cases}}$$

Example 1.1.4 *Let $A = \{x \in \mathbb{Q} : x^2 < 2\}$. Then, $A \subseteq \mathbb{R}$. A is bounded from above, hence by the supremum axiom, A has a supremum in $\mathbb{R}$. Let $\alpha = \sup A$.*

Let us see that $x = \sqrt{2}$.

1. $\forall x \in A,\ x < \sqrt{2}$
2. *Let $\varepsilon > 0$ be any number. So, for $x_\varepsilon \in A$, $x_\varepsilon > \sqrt{2} - \varepsilon$.*

$$2 > x_\varepsilon^2 > \underbrace{\left(\sqrt{2} - \varepsilon\right)^2 = 2 - 2\sqrt{2}\varepsilon + \varepsilon^2}$$
$$\underbrace{\varepsilon\left(2\sqrt{2} - \varepsilon\right) > 0}$$
$$2\sqrt{2} - \varepsilon > 0$$
$$\varepsilon < 2\sqrt{2}$$

You can always find $x_\varepsilon \in A \ni x_\varepsilon > \sqrt{2} - \varepsilon$. So, $\sup A = \sqrt{2}$.

Example 1.1.5 *Let $A = \left\{\dfrac{n}{n+1} : n = 1, 2, 3, ..\right\}$. Then A is bounded from above, so $\alpha = \sup A$ exists. Let us see that $\alpha = 1$. Indeed,*

1. $\forall\, n \geq 0,\ \dfrac{n}{n+1} \leq 1$
2. *Let $\varepsilon \succ 0$ be any number, then the inequality $\dfrac{n}{n+1} > 1 - \varepsilon$ has a solution n_ε. Then,*

$$x_\varepsilon = \frac{n_\varepsilon}{n_\varepsilon + 1} > 1 - \varepsilon.$$

Proposition 1.1.6 *A nonempty subset $B \subseteq \mathbb{R}$, which is bounded below has an infimum $\beta \in \mathbb{R}$.*

Proof 1.1.7 *We are going to show that, there is a number $\beta \in \mathbb{R}$ such that:*

1. $\forall\, x \in B,\ \beta \leq x$
2. $\forall\, \varepsilon > 0,\ \exists\ x_\varepsilon \in B : x_\varepsilon < \beta + \varepsilon$

Let $A = \{-x : x \in B\}$. Then A is bounded from above, so by supremum axiom 1.1.3 $\exists\, \alpha \in R \ni \alpha = \sup A$.

$$\Longrightarrow \begin{array}{l} \mathbf{1})\forall x \in B, -x \leq \alpha \\ \mathbf{2})\forall \varepsilon > 0, \exists\ x_{\varepsilon} \in B : -x_{\varepsilon} \geq \alpha - \varepsilon \end{array}$$

This is equivalent to,

$$\Longrightarrow \begin{array}{l} \mathbf{1})\forall\, x \in B, x \geq -\alpha \\ \mathbf{2})\forall\, \varepsilon \succ 0, \exists\ x_{\varepsilon} \in B : -x_{\varepsilon} \leq -\alpha + \varepsilon \end{array}$$

Hence $-\alpha$ is the infimum of B.

At the same time we have proved that,
$\sup(-B) = -\inf(B)$ (for any set B bounded from below)
$\inf(-A) = -\sup(A)$ (for any set A bounded from above)

Proposition 1.1.8 *Given any $x \in \mathbb{R}$, $x \geq 0$, there is a unique $n \in \mathbb{N} \ni n - 1 < x \leq n$.*

Proof 1.1.9 *Let $A = \{n \in \mathbb{N} : n \geq x\}$. Then, $A \neq \varnothing$.*

$A \subseteq \mathbb{N} \Longrightarrow A$ has a smallest element, call it n.

Thus, $n \in A$, but $n - 1 \notin A$. So, $n \geq x$, but $n - 1 < x$, i.e. $n - 1 < x \leq n$

Proposition 1.1.10 ***(Archimedean Property of $\mathbb{R}$):*** *Given any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $N.\varepsilon > 1$.*

Proof 1.1.11 *Observe that, $N.\varepsilon > 1$ is equivalent to $\dfrac{1}{N} < \varepsilon$.*

Let in the Proposition 1.1.8, $x = \dfrac{1}{\varepsilon}$. Then, there is $n \in \mathbb{N}$ such that $N - 1 \leq \frac{1}{\varepsilon} < N$.

Hence, $\dfrac{1}{N} < \varepsilon$.

Proposition 1.1.12 ***(Density of $\mathbb{Q}$ in $\mathbb{R}$):*** *Given any $x \in \mathbb{R}$ and any $\varepsilon > 0$ there is an $r \in \mathbb{Q}$ such that $|x - r| < \varepsilon$.*

Proof 1.1.13 *If $x \in \mathbb{Q}$, then take $r = x$. Suppose $x > 0$.*

By the Proposition 1.1.10, there is an $N \in \mathbb{N} \ni \dfrac{1}{N} < \varepsilon$. Consider the number N_x.
By the Proposition 1.1.8, applies to N_x, there is an integer $n \in \mathbb{N}, \ni\ n < N_x \leq n + 1$.
Let $r = \dfrac{n}{N}$, then $r \in \mathbb{Q}$ and $\dfrac{n}{N} \leq x - r \leq \dfrac{n}{N} + \dfrac{1}{N}$. Thus, $0 \leq x - r \leq \dfrac{1}{N} < \varepsilon$.
Hence, $|x - r| < \varepsilon$. If $x < 0$ then $-x > 0$.
So, by what proceeds, there is $r \in \mathbb{Q}\ \ni\ |-x - r| = |x - (-r)| < \varepsilon$

Proposition 1.1.14 *Given any two real numbers, $a, b \in \mathbb{R}$ with $a < b$, there is at least one $r \in \mathbb{Q} \ni\ a < r < b$*

Proof 1.1.15 *Let $x = \dfrac{a+b}{2}$. Let $\varepsilon > 0$ be $\ni$, $a < x - \varepsilon < x < x + \varepsilon < b$*
$\left(e.g.\ \text{let } 0 < \varepsilon < \dfrac{b-a}{2} \right)$.

Then by the Proposition 1.1.12, there is $r \in \mathbb{Q} \ni |x - r| < \varepsilon \Longrightarrow -\varepsilon < x - r < \varepsilon$.

So, $x - \varepsilon < r < x + \varepsilon$. Hence $a < r < b$. Now,

if we take $b = r$, there is $r_1 \in \mathbb{Q} \ni\ a < r_1 < r$.

if we take $b = r_1$ there is $r_2 \in \mathbb{Q} \ni\ a < r_2 < r_1$.

if we take $b = r_2$ there is $r_3 \in \mathbb{Q} \ni\ a < r_3 < r_2$.

So that in between a and b there are infinitely many rational numbers.

1.2 Intervals

For any $a, b \in \mathbb{R}$, $a \leq b$, we define $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

There are finite, closed or open intervals.

1. $[a, \infty[= \{x \in \mathbb{R} : x \geq a\}$ is closed infinite interval.
2. $]a, \infty[= \{x \in \mathbb{R} : x > a\}$ is open infinite interval.
3. $]a, b[= \{x \in \mathbb{R} : a < x < b\}$ is open finite interval.
4. $[a, b[$, $]a, b]$ are half open half closed intervals.

Definition 1.2.1 *Let $A \subseteq \mathbb{R}$ be a nonempty set. Then, A is an interval $\Leftrightarrow \forall\, a, b \in A$, if $a < b$ and for $r \in \mathbb{R}$, we have $a < r < b$, then $r \in A \Leftrightarrow \forall\, a, b \in A$, $a < b$, $[a, b] \subseteq A$.*

Hence, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $]-1, 0[\cup]1, 2[$ are not intervals.

Properties:

1. If $a = b$, $]a, b[= \emptyset$, and $[a, b] = a$. ($]a, b[$ and $[a, b]$ are degenerated intervals.)
2. A subset A from $\mathbb{R}$ is bounded from below $\Leftrightarrow$ A is contained in an interval of the form $[\alpha, \infty[$.
3. A subset A from $\mathbb{R}$ is bounded from above $\Leftrightarrow$ A is contained in $[-\infty, \beta[$.
4. A subset A from $\mathbb{R}$ is bounded $\Leftrightarrow$ A is contained in $[\alpha, \beta[$ $\Leftrightarrow A \subseteq [-M, M]$ for some $M > 0$.

1.2.1 More About Supremum and Infimum:

- If $A =]0,1[$, then $\sup A = 1$, $\inf A = 0$, $1 \notin A$, $0 \notin A$.
- If $A = [0,1]$, then $\sup A = 1$, $\inf A = 0$, $1 \in A$, $0 \in A$.
- If $A =]0,1]$, then $\sup A = 1$, $\inf A = 0$, $1 \in A$, $0 \notin A$.

Definition 1.2.2 *Let $X \neq \emptyset$ be any set and $f : X \to \mathbb{R}$ be a function. Then, $A = f(X)$ is a subset of $\mathbb{R}$.*

If $f(X)$ is bounded from above, we say that f is bounded from above in X.

In this case, $\alpha = \sup f(X)$ exists. Thus, $\sup_{x\in X} f(x) = \sup_{x\in X}\{f(x) : x \in X\}$.

If $f(X)$ is bounded from below, $\beta = \inf f(X)$ exists. $\beta = \inf_{x\in X} f(x)$.

Example 1.2.3 *Let $X =]0,1]$, $f(x) = \dfrac{1}{x}$. Then, $f(x) = \left\{\dfrac{1}{x} : 0 < x \leq 1\right\}$. It is clear that $f(x)$ is not bounded from above. But bounded from below, i.e. $\inf_{x\in X} f(x) = 1$.*

Example 1.2.4 *Let $X =]0,\infty[$, $f(x) = \dfrac{x}{x+1}$. f is bounded in X.*

$$\sup_{x\in X} f(x) = 1,\ \nexists\, x_0 \in X \ni f(x_0) = 1$$
$$\inf_{x\in X} f(x) = 0,\ \nexists\, y_0 \in X \ni f(y_0) = 0$$

Example 1.2.5 *If $X = \mathbb{N}$, then $f : \mathbb{N} \to \mathbb{R}$ is a sequence. $x_n = f(n)$, if f is bounded then $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence, i.e. $|x_n| \leq M, \forall\, n \in \mathbb{N}$ for some $M > 0$, then the range $A = \{x_0, x_1, ..., x_k, ...\} = f(\mathbb{N})$ is a bounded set.*

For instance, $\begin{cases} x_n = e^n \text{ is not a bounded sequence.} \\ x_n = \dfrac{1}{e^n} \text{ is a bounded sequence.} \end{cases}$

Proposition 1.2.6 *Let X be a nonempty set and $f, g : X \to \mathbb{R}$ be two bounded functions. Then,*

1. $\sup_{x\in X}(f(x) + g(x)) \leq \sup_{x\in X} f(x) + \sup_{x\in X} g(x)$
2. $\inf_{x\in X}(f(x) + g(x)) \leq \inf_{x\in X} f(x) + \inf_{x\in X} g(x)$
3. $\sup_{x\in X}|f(x) \times g(x)| \leq \sup_{x\in X}|f(x)| \times \sup_{x\in X}|g(x)|$

Proof 1.2.7 *Since f and g are bounded, $\alpha = \sup f(x)$, $\beta = \sup g(x)$ exists.*

1. *In particular, $\forall\, x \in X$ $f(x) \leq \alpha$, $g(x) \leq \beta$. Hence, adding them we get,*
   $f(x) + g(x) \leq \alpha + \beta, \forall\, x \in X$.
   Hence, $\sup(f(x) + g(x)) \leq \alpha + \beta = \sup f(x) + \sup g(x)$

2. *to prove 2 apply 1 to* $-f$ *and* $-g$ *as we know that* $\inf(-A) = -\sup(A)$

3. $\forall x \in X, \begin{cases} |f(x)| \leq \sup|f(x)| \\ |g(x)| \leq \sup|g(x)| \end{cases} \Longrightarrow$ *Multiplying them, we get:*

$$|f(x)| \times |g(x)| \leq \sup|f(x)| \times \sup|g(x)|$$

Hence, $|f(x) \times g(x)| \leq \sup|f(x)| \times \sup|g(x)|$

Example 1.2.8 *Let* $X = \mathbb{N}$, $f(n) = (-1)^n$, $g(n) = (-1)^{n+1}$.

Then $\sup f(n) = 1$, $\sup g(n) = 1$, $\sup f(n) + \sup g(n) = 2$. $f(n) + g(n) = 0, \forall n \in \mathbb{N}$. *Hence,* $\sup(f(n) + g(n)) = 0 < 2$.

Example 1.2.9 *Let* $X = \left]0, \frac{\pi}{2}\right[$, $f(x) = \sin x$, $g(x) = \cos x$. *So,* $\sup f(x) = 1$, *and* $\sup g(x) = 1$.

$\sin x \times \cos x = \frac{1}{2}\sin 2x$. *Then* $\sup_{x\in X}\left(\frac{1}{2}\sin 2x\right) = \frac{1}{2}$.

$\forall x \in \left]0, \frac{\pi}{2}\right[$, $\sup_{x\in X} f(x) \times \sup_{x\in X} g(x) > \sup_{x\in X}(f(x) \times g(x))$

Proposition 1.2.10 *Let* $f : X \to \mathbb{R}$ *be a function. Suppose that for some* $\alpha > 0$, $f(x) \geq \alpha, \forall x \in X$. *Then,* $\frac{1}{f(x)} \leq \frac{1}{\alpha}$ *and,* $\sup_{x\in X}\left(\frac{1}{f(x)}\right) = \frac{1}{\inf_{x\in X} f(x)}$

Proof 1.2.11 *Let* $\beta = \sup\left(\frac{1}{f(x)}\right)$

$$\begin{cases} \mathbf{1)}\, \forall x \in X, \ \frac{1}{f(x)} < \beta \\ \mathbf{2)}\, \forall \varepsilon > 0, \exists\, x_\varepsilon \in X, \ni \frac{1}{f(x_\varepsilon)} > \beta - \varepsilon \end{cases}$$

Hence, $\beta \times f(x) \geq 1, \forall x \in X$. *This implies that,* $\inf f(x) \geq \frac{1}{\beta}$, *so* $\frac{1}{\inf f(x)} \leq \beta$

*from **(2)**,* $\frac{1}{f(x_\varepsilon)} > \beta - \varepsilon \Longrightarrow f(x_\varepsilon) < \frac{1}{\beta - \varepsilon}$.

This implies that $\inf f(x) \leq \frac{1}{\beta - \varepsilon}$ *and* $\frac{1}{\inf f(x)} \geq \beta - \varepsilon \Longrightarrow \beta - \varepsilon \leq \frac{1}{\inf f(x)} \leq \beta$.

As $\inf f(x)$ *does not depend on* ε, *letting* $\varepsilon \to 0$, *we get that* $\beta = \frac{1}{\inf f(x)}$.

Proposition 1.2.12 *Let* X, Y *be two sets.* $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$ *be two bounded functions. Then,*

1. $\sup_{x\in X, y\in Y}(f(x) + g(y)) = \sup_{x\in X} f(x) + \sup_{y\in Y} g(y)$

2. $\inf_{x\in X, y\in Y}(f(x) + g(y)) = \inf_{x\in X} f(x) + \inf_{y\in Y} g(y)$

Proof 1.2.13

$$\left.\begin{array}{l}\forall x \in X,\ f(x) \leq \sup_{x\in X} f(x)\\ \forall y \in Y,\ g(y) \leq \sup_{y\in Y} g(y)\end{array}\right\} \Longrightarrow f(x) + \ g(y) \leq \sup_{x\in X} f(x) + \sup_{y\in Y} g(y)$$

This implies that,

$$\begin{array}{l}\sup_{x\in X}\left(f(x)+g(y)\right) \leq \ \sup_{x\in X} f(x) + \sup_{y\in Y} g(y)\\ \sup_{x\in X, y\in Y}\left(f(x)+g(y)\right) \leq \sup_{x\in X} f(x) + \sup_{y\in Y} g(y) *\end{array}$$

But $\sup_{x\in X, y\in Y}\left(f(x)+g(y)\right) \geq f(x)+g(y),\ \forall x \in X,\ \forall\, y \in Y$.

Hence, passing to supremum on X *and* Y*, we get,*

$$\sup_{x\in X, y\in Y}\left(f(x)+g(y)\right) \geq \sup_{x\in X} f(x) + \sup_{y\in Y} g(y) **$$

$*$ *and* $**$ *prove 1. To prove 2, replace* f *by* $-f$ *and* g *by* $-g$.

1.2.2 Exercises I

1. Let E be a nonempty subset of $\mathbb{R}$. Complete the sentence: E is an interval iff......................

   Is $\mathbb{Q}$ an interval? Is $\mathbb{R}\backslash\mathbb{Q}$ an interval?

2. Let X, Y be 2 nonempty sets and $\varphi : X \times Y \to \mathbb{R}$ be a bounded function. Show that we have:

   $\sup_{y\in Y}\left[\sup_{x\in X}\varphi(x,y)\right] = \sup_{x\in X}\left[\sup_{y\in Y}\varphi(x,y)\right] = \sup_{(x,y)\in X\times Y}\varphi(x,y)$

3. Let X be a nonempty set and $g, f : X \to \mathbb{R}$ be 2 bounded functions. Show that

   (a) $\sup_{x\in X}(f(x)+g(x)) \leq \sup_{x\in X}f(x) + \sup_{x\in X}g(x)$.

   (b) $\inf_{x\in X}(f(x)+g(x)) \geq \inf_{x\in X}f(x) + \inf_{x\in X}g(x)$.

   (c) $\inf_{x\in X}(f(x)+g(x)) \leq \inf_{x\in X}f(x) + \sup_{x\in X}g(x)$.

   (d) $\sup_{x\in X}|f(x)\times g(x)| \leq \sup_{x\in X}|f(x)| \times \sup_{x\in X}g(x)$.

   (e) $\sup_{x\in X}|f(x)|^n = (\sup_{x\in X}|f(x)|)^n \ \ (\forall n \in \mathbb{N})$.

   (f) $\sup_{x\in X}\sup_{y\in Y}(f(x)+g(y)) = \sup_{x\in X}f(x) + \sup_{y\in Y}g(y)$.

   (g) For $c \in \mathbb{R}$ fixed,
   $\sup_{x\in X}(f(x)+c) = \sup_{x\in X}f(x)+c$, and $\inf_{x\in X}(f(x)+c) = \inf_{x\in X}f(x)+c$.

   (h) $\sup_{x\in X}(-f(x)) = -\inf_{x\in X}f(x)$.

1.3 Convergence in $\mathbb{R}$ and Monotone Sequences

Definition 1.3.1 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$.*

- $(x_n)_{n\in\mathbf{N}}$ *is convergent if there is a number L that satisfies the condition:*

$$\forall\varepsilon > 0, \exists\ N \in \mathbb{N}, \forall n > N \Longrightarrow |x_n - L| < \varepsilon$$

Equivalently, $\forall\varepsilon > 0, x_n \in\ ]L-\varepsilon, L+\varepsilon[$ for all but finitely many $n \in \mathbb{N}$

- *In this case we write, $x_n \to L$, as $n \to \infty$, $L = \lim_{n\to\infty} x_n$*

- *If $(x_n)_{n\in\mathbb{N}}$ does not converge to any $L \in \mathbb{R}$, then we say that $(x_n)_{n\in\mathbb{N}}$ diverges.*

Example 1.3.2 *Let $x_n = (-1)^n$. There is no $L \in \mathbb{R}$ that satisfies the condition of convergence. Indeed, if it was convergent there would be an $L \in \mathbb{R}$ satisfying the convergence condition. Now let $\varepsilon = \frac{1}{2}$. Then for N corresponding to this ε, $\forall n \geq N$, $|x_n - L| < \frac{1}{2}$.*

Now for n odd $\Rightarrow x_n = -1$ and $|-1-L| < \frac{1}{2}$

for n even $\Rightarrow x_n = 1$ and $|1-L| < \frac{1}{2}$

So we have $\frac{-1}{2} < 1 + L < \frac{1}{2}$ and $\frac{-1}{2} < 1 - L < \frac{1}{2}$, and adding these we get $-1 < 2 < 1$ which is nonsense. Hence x_n is divergent (does not mean that it goes to infinity.) as, $|x_n| = 1\ \forall\ n \in \mathbb{N}$.

Example 1.3.3 *Let $x_n = n^2 + 1$. If it is convergent to some $L \in \mathbb{R}$ then,*
$\forall\,\varepsilon > 0$, $\exists\,N \in \mathbb{N}$, $\forall\,n > N$, $|n^2 + 1 - L| < \varepsilon$. i.e. $L - \varepsilon < n^2 + 1 < L + \varepsilon\,\forall$, $n \in \mathbb{N}$.
But this is not possible, since $\mathbb{N}$ is not bounded from above.

Example 1.3.4 $(x_n)_{n\in\mathbb{N}} = \left\{0, 1, \frac{1}{2}, 3, \frac{1}{4}, ...\right\}$. *This sequence does not converge either.*

Example 1.3.5 *Let $x_n = \frac{1}{n}$, then the Archimedian property just means that $\frac{1}{n} \to 0$, as $n \to \infty$.*

$$\left(\forall\,\varepsilon > 0\ \exists\ N \in \mathbb{N}\ \frac{1}{N} < \varepsilon \Longrightarrow \forall n \geq N,\ \frac{1}{n} < \varepsilon \Longrightarrow \frac{1}{n} \to 0\right)$$

Theorem 1.3.6 *Properties of the Convergent Sequences:*

1. ***Uniqueness of the Limit:*** *A sequence $(x_n)_{n\in\mathbb{N}}$ cannot converge to more than one limit.*

2. ***Boundaries of Convergent Sequences:*** *Every convergent sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *is bounded.*
3. ***Passage to Absolute Value:*** *If* $x_n \to L$, *then* $|x_n| \to |L|$.
4. ***Convergence and Inequalities:*** *If* $x_n \geq c, \forall\, n \in \mathbb{N}$ *and* $x_n \to L$, *then* $L \geq c$.
5. *If* $x_n \leq y_n \forall n \in \mathbb{N}$, $x_n \to L$, $y_n \to S$, *then* $L \leq S$.
6. ***Sandwich Theorem:*** $x_n \leq y_n \leq z_n$ *for all* $n \in \mathbb{N}$, *and* $x_n \to L$, $z_n \to L$, *then* $y_n \to L$.
7. *If* $x_n \to L$, *and* $L \neq 0$, *then* $|x_n| \geq \dfrac{|L|}{2}$ *for all but finitely many* $n \in \mathbb{N}$.
8. *If* $x_n \to L$ *and* $y_n \to S$, *then*
   (a) $x_n + y_n \to L + S$
   (b) $x_n \times y_n \to L \times S$
   (c) *If* $S \neq 0$, $\dfrac{x_n}{y_n} \to \dfrac{L}{S}$

Proof 1.3.7 1. *For a contradiction, suppose that* $x_n \to S$ *and* $x_n \to L$, $(L \neq S)$. *Say* $L < S$. *Let* ε *be small enough to have* $]L-\varepsilon, L+\varepsilon] \cap]S-\varepsilon, S+\varepsilon[= \varnothing$. *So,* $0 < \varepsilon < \dfrac{S-L}{3}$. *Since* $x_n \to L$, $x_n \in]L-\varepsilon, L+\varepsilon[$ *for all but finitely many* n. *As* $x_n \to S$, $x_n \in]S-\varepsilon, S+\varepsilon[$ *for all but finitely many* n, *too. This is not possible. Hence the limit is unique.*

2. *Let,* $x_n \to L$, *as* $n \to \infty$. *So, we have:* $\forall \varepsilon > 0, \exists\, N \in \mathbb{N}, \forall\, n \geq N, |x_n - L| < \varepsilon$.

   Hence, since $||x_n| - |L|| \leq |x_n - L| < \varepsilon, \ \forall\, n \geq N \quad |x_n| \leq |L| + \varepsilon$.

   Let $M = \max\{|x_0|, |x_1|, ..., |x_n|, |L| + \varepsilon\}$. *Then* $\forall n \in \mathbb{N}\ |x_n| \leq M$.

   Remark: *Converse of this result is false. Let* $x_n = (-1)^n$. *Then* $|x_n| \leq 1\ \forall\, n \in \mathbb{N}$, *but* $(x_n)_{n\in\mathbb{N}}$ *does not converge.*

3. *As* $x_n \to L$, *we have* $\forall \varepsilon > 0\ \exists\, N \in \mathbb{N}, \forall n \geq N\ |x_n - L| < \varepsilon$. *As* $||x_n| - |L|| \leq |x_n - L|$, *we see that* $\forall n \geq N\ ||x_n| - |L|| < \varepsilon$. *This means that* $|x_n| \to |L|$.

4. *For a contradiction, suppose that* $L < c$. *Let* ε *be small enough to have* $L + \varepsilon < c$. *(e.g. let* $\varepsilon = \dfrac{c-L}{2}$*). Write the definition of convergence for this* ε. *Then, there is* $N \in \mathbb{N} \ni \forall\, n \geq N$, $|x_n - L| < \varepsilon$. *So,* $L - \varepsilon \leq x_n \leq L + \varepsilon$. *As* $x_n \geq c$, *and* $L + \varepsilon < c$. *Contradiction.*

   In particular, if $x_n \geq 0 \forall n \in \mathbb{N}$, *then* $L \geq 0$.

   Remark:

- *If* $x_n > c$ *and* $x_n \to L$*, we can not say* $L > c$*, all we can say is* $L \geq c$*. e.g. Let* $x_n = \dfrac{1}{n}$*, then* $x_n > 0\,\forall n \geq 1$*, but* $\lim_{n\to\infty} x_n = 0$*.*
- *If* $L \geq c$*, we can not say that* $x_n \geq c$ *for all* $n \in \mathbb{N}$*.*

5. *For a contradiction, suppose* $L > S$*. Let* $\varepsilon > 0$ *be small enough to still have* $L - \varepsilon > S + \varepsilon$*. For this* $\varepsilon > 0$*, we write the fact that* $x_n \to L, y_n \to S$*. Then, there is* $N_1 \in \mathbb{N}, \forall n \geq N_1, |x_n - L| < \varepsilon$*. Then, there is* $N_2 \in \mathbb{N}, \forall n \geq N_2,\ |x_n - S| < \varepsilon$*.*

   Let $N = \max\{N_1, N_2\}$*. So* $\forall n \geq N, L - \varepsilon \leq x_n \leq\ L + \varepsilon$*, and* $S - \varepsilon \leq y_n \leq\ S + \varepsilon$*. As* $S < L - \varepsilon < x_n \leq y_n < S + \varepsilon < L - \varepsilon$ *is not possible, this is contradiction.*

6. *Let* $\varepsilon > 0$*, then* $x_n \to L,\ z_n \to L$*.*
   $$\exists N \in \mathbb{N} \ni \forall n \geq N \quad \begin{aligned} L - \varepsilon \leq x_n \leq L + \varepsilon \\ L - \varepsilon \leq z_n \leq L + \varepsilon \end{aligned}$$
   $\forall n \geq N,\ L - \varepsilon < x_n \leq y_n \leq z_n < L + \varepsilon$*.*

   So, $\forall n \geq N,\ L - \varepsilon < y_n < L + \varepsilon \Longrightarrow |y_n - L| < \varepsilon$*. Then,* $y_n \to L$*.*

7. *Let* $\varepsilon = \dfrac{|L|}{2}$*. So* $\varepsilon > 0$*.*

   Corresponding to this ε *there is an* $N \in \mathbb{N}$ *such that* $\forall n \geq N,\ |x_n - L| < \varepsilon$*.*

   As $||x_n| - |L|| \leq |x_n - L| < \varepsilon$*, we have*
   $$\underbrace{|L| - \varepsilon}_{=\frac{|L|}{2}} < |x_n| < |L| + \varepsilon \qquad \Longrightarrow \forall n \geq N,\ |x_n| \geq \frac{|L|}{2}$$

8. (a)
   $$\begin{aligned} &\forall \varepsilon > 0\,\exists N \in \mathbb{N}\,\forall n \geq N,\ |x_n - L| < \frac{\varepsilon}{2} \\ &\forall \varepsilon > 0 \exists N \in \mathbb{N},\ \forall n \geq N\ |y_n - S| < \frac{\varepsilon}{2} \end{aligned}$$
   Then, $\forall n \geq N,\ |x_n + y_n - (S + L)| \leq |x_n - L| + |y_n - S| < \varepsilon$*.*
   Hence, $x_n + y_n \to S + L$

   (b) $x_n \times\ y_n - L \times S = (x_n - L) \times y_n + L \times y_n - L \times S$*.*
   Hence, $|x_n \times y_n - L \times S| \leq |x_n - L| \times |y_n| + |L| \times |y_n - S|$
   As $(y_n)_{n\in\mathbb{N}}$ *converges, it is bounded, say* $|y_n| \leq M\ \forall n \in \mathbb{N}$
   Then, $n \geq N,\ |x_n \times y_n - L \times S| < M\dfrac{\varepsilon}{2} + |L|\,\dfrac{\varepsilon}{2} \leq M + |L|2$*.*
   Hence, $x_n \times y_n \to L \times S$*.*

   (c) $\dfrac{x_n}{y_n} - \dfrac{L}{S} = \dfrac{x_n S - y_n L}{y_n S}$

   Since $S \neq 0$*, by ref2.3.7* $|y_n| \geq \dfrac{|S|}{2}$ *for all but finitely many* $n \in \mathbb{N}$*. Then,*

$$\left|\frac{x_nS - y_nL}{y_nS}\right| \leq 2\left|\frac{x_nS - y_nL}{|S|^2}\right| \to 2\left|\frac{SL - SL}{|S|^2}\right| = 0.$$

Hence, $\left|\frac{x_n}{y_n} - \frac{L}{S}\right| \to 0$, $\frac{x_n}{y_n} \to \frac{L}{S}$.

1.4 Monotone Sequences

Definition 1.4.1 *A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *is said to be*

1. ***increasing*** *if* $x_0 \leq x_1 \leq ... \leq x_n \leq ...$
2. ***decreasing*** *if* $x_0 \geq x_1 \geq ... \geq x_n \geq ...$

e.g. $x_n = \frac{1}{n}$ is decreasing,
$x_n = \frac{n}{n+1}$ is increasing

Remark:

- Any increasing sequence is bounded from below.
- Any decreasing sequence is bounded from above.
  So an increasing sequence is bounded iff it is bounded from above.
- Also, $(x_n)_{n\in\mathbf{N}}$ is increasing iff $(-x_n)$ is decreasing.

Example 1.4.2 *Let* $x_n = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$. *Then, clearly,* x_n *is increasing.*

$3! \geq 2^2$. *Hence,* $\frac{1}{3!} \geq \frac{1}{2^2}$

$4! \geq 2^3$. *Hence* $\frac{1}{4!} \geq \frac{1}{2^3}$

$5! \geq 2^4$. *Hence* $\frac{1}{5!} \geq \frac{1}{2^4}$

$\vdots$

$n! \geq 2^n$. *Hence* $\frac{1}{n!} \geq \frac{1}{2^{n-1}}$.

$$\begin{aligned} \text{Hence, } x_n \leq \frac{5}{2} + & \underbrace{\frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{n-1}}} \\ & = \frac{1}{2^2}\left[1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^{n-3}}\right] \\ & = \frac{1}{2^2}\frac{1 - (\frac{1}{2})^{n-2}}{1 - \frac{1}{2}} \leq \frac{1}{2} \end{aligned}$$

So, $x_n \leq \frac{5}{2} + \frac{1}{2} \Longrightarrow x_n \leq 3, \;\; \forall n \in \mathbb{N} \Longrightarrow$ x_n *is bounded.*

Example 1.4.3 *Let* $x_n = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$ *and* $y_n = x_n + \frac{1}{n!}$.

So, $y_{n-1} - y_n = x_{n-1} - x_n + \frac{1}{(n-1)!} - \frac{1}{n!} = -\frac{2}{n!} + \frac{1}{(n-1)!} = \frac{n-2}{n!} \geq 0, \ \forall n \geq 2.$

Hence, $y_{n-1} \geq y_n$. *So,* $(y_n)_{n\in\mathbb{N}}$ *is decreasing.*

Example 1.4.4 $1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$. *Then,* $x_1 \leq x_2 \leq ... \leq x_3 \leq ...$

As $n^2 \geq n(n-1)$, $\frac{1}{n^2} \leq \frac{1}{n(n-1)} = \frac{1}{n-1} - \frac{1}{n}$

i.e.

$$\frac{1}{2^2} \leq \frac{1}{1} - \frac{1}{2}$$
$$\frac{1}{3^2} \leq \frac{1}{2} - \frac{1}{3}$$
$$\vdots$$
$$\frac{1}{n^2} \leq \frac{1}{n-1} - \frac{1}{n}$$

Then, $x_n \leq 2 - \frac{1}{n} \leq 2, \ \forall n \geq 1, \ x_n \leq 2.$

Theorem 1.4.5 ***(Convergence of monotone sequences):*** *A monotone sequence* $(x_n)_{n\in\mathbb{N}}$ *is convergent iff it is bounded. In this case,*

1. *if* x_n *is increasing, then* $\lim_{n\to\infty} x_n = \sup\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$
2. *if* x_n *is decreasing, then* $\lim_{n\to\infty} x_n = \inf\{x_0, x_1, x_2, \ldots, x_n, \ldots\}$

Proof 1.4.6 *Suppose* $x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots$

We know that every convergent sequence is bounded.

Conversely, suppose that $(x_n)_{n\in\mathbb{N}}$ *is bounded. Then the set* $A = \{x_0, x_1, x_2, ..., x_n, \ldots\}$ *is bounded. So by the supremum axiom,* $\exists\, \alpha \in \mathbb{R}, \ni \alpha = \sup A$

$$\Longrightarrow \begin{cases} 1) & \forall\, n \in \mathbb{N}, \ x_n \leq \alpha \\ 2) & \forall \varepsilon \geq 0, \ \exists\, x_N \in A \ni x_N > \alpha - \varepsilon \end{cases}$$

$\varepsilon > 0$ *being given. For every* $n \geq N$ $\alpha - \varepsilon < x_N \leq x_n \leq \alpha \leq \alpha + \varepsilon$.

So, $\forall n \in \mathbb{N}, \ |x_n - \alpha| < \varepsilon$, *i.e.* $\lim_{n\to\infty} x_n = \alpha$

Example 1.4.7

1. *Let* $x_n = 1 + \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n}$. *We have seen that* x_n *is not bounded.*

   So it diverges by the theorem 1.4.5.

2. *Let* $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$. *We have seen that* $x_n \leq 3\, \forall n \in \mathbb{N}$. *This sequence is increasing, so it converges.*

   Let $e = \lim_{n\to\infty} x_n$. *Since* $x_n \leq 3 \ \forall n \in \mathbb{N}, \ e \leq 3$.

   As $x_2 = 2.5$, *we see that* $2.5 \leq e \leq 3$.

Remark: If the sequence $(x_n)_{n\in\mathbb{N}}$ is strictly increasing, i.e. $x_0 < x_1 < x_2 < \ldots < x_n < \ldots$ and L is the limit, then $x_n \leq L,\ \forall n \in \mathbb{N}$. Hence $x_n < e$ in above example.

Now, let $y_n = x_n + \frac{1}{n!}$. This sequence is decreasing and bounded from below by 0. So, it converges. Hence, $y_n - x_n = \frac{1}{n!} \to 0$ also converges. $\lim_{n\to\infty} y_n = e$.

Theorem 1.4.8 *The number e is not rational.*

Proof 1.4.9 *Let as in the last example, $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$ and $y_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{n!}$. Then, for any $n \in \mathbb{N}\, x_n < e < y_n$.*

For a contradiction, suppose e is rational. So, $e = \frac{p}{q}$, $(p > 0,\ q > 0,\ p, q \in \mathbb{N})$

Let $n \geq q$, $1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} < \frac{p}{q} < 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + \frac{1}{n!}$.

Multiplying by $n!$; $n! + \frac{n!}{1!} + \frac{n!}{2!} + \cdots + 1 < \frac{pn!}{q} < n! + \frac{n!}{1!} + \frac{n!}{2!} + \cdots + 1 + 1$

$n! + \frac{n!}{1!} + \frac{n!}{2!} + \cdots + 1 = N$

$n! + \frac{n!}{1!} + \frac{n!}{2!} + \cdots + 1 + 1 = N + 1$, *and* $\frac{pn!}{q} = M$

$N < M < N + 1$. Hence, N and M are integers. As there is no integer between two consecutive integers, this is not possible.

Hence e is not rational

This theorem says that $\mathbb{Q}$ is not closed under the "limit" operation. Indeed, although every $x_n \in \mathbb{Q}$, $\lim x_n = e \notin \mathbb{Q}$.

1.4.1 Exercises II

1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ and $c \in \mathbb{R}$ a number. If $0 \le x_n \le c$ for infinitely many n and $x_n \to L$ show that then $L \le c$.

2. Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be two convergent sequences. If $x_n \le y_n$ for all but finitely many $n \in N$, show that then $\lim_{n\to\infty} x_n \le \lim_{n\to\infty} y_n$.

3. Let $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ be two convergent sequences with $0 \le x_n \le y_n$ for all but finitely many $n \in \mathbb{N}$. If $y_n \to 0$,then show that $x_n \to 0$.

4. Show that $1 + \frac{1}{2} + \cdots + \frac{1}{2^{k-1}} = 2(1 - \frac{1}{2^k})$.

5. Let $x_n = 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}}$. Show that $\lim_{n\to\infty} x_n = 2$.

6. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence.

   (a) if $|x_{n+1} - x_n| \le \frac{1}{2}|x_n - x_{n-1}|$ then show that $(x_n)_{n\in\mathbb{N}}$ converges.

   (b) if, for some $M > 0$ and $0 < r < 1$, $|x_{n+1} - x_n| \le Mr^n$ ($\forall\, n \in \mathbb{N}$). Show that $(x_n)_{n\in\mathbb{N}}$ converges.

   (c) if, for some $x \in R$ and $0 < r < 1$, $|x_{n+1} - x| \le r|x_n - x|$ for all $n \in \mathbb{N}$. Show that then $x_n \to x$.

7. If $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence and $L = \lim_{n\to\infty} x_n$ with $L > 0$, show that then $x_n > 0$ for all but finitely many $n \in \mathbb{N}$.

   What about if $L = 0$? Can you say that $x_n \ge 0$ for all but finitely many $n \in \mathbb{N}$?

8. Let $(x_n)_{n\in N}$ be a sequence such that for all $n \ge 1$, $|x_{n+1} - x_n| < \frac{1}{n}$.

   Is this sequence convergent? What difference is there between this and the one in the question 6b above?

1.5 Convergence of Subsequences

1.5.1 Cluster points of a Sequence

Let x_n be a sequence and $n_1 < n_2 < n_3 < \ldots < n_k < \ldots$ be integers, and $y_k = x_{n_k}$. Then, y_k is a sequence itself. Suppose that y_k converges as $k \to \infty$ to some $L \in \mathbb{R}$.

Hence, $\forall \varepsilon > 0$, $\exists\, k_0 \in \mathbb{N}$, $\forall\, k \geq k_0$, $y_k \in \,]L-\varepsilon, L+\varepsilon[$.

Equivalently since $y_k = x_{n_k}$, $\forall\, k \geq k_0$, $x_{n_k} \in \,]L-\varepsilon, L+\varepsilon[$.

This shows that $\forall \varepsilon > 0$, $x_n \in \,]L-\varepsilon, L+\varepsilon[$ for infinitely many $n \in \mathbb{N}$.

Definition 1.5.1 *1. Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $\mathbb{R}$ and $L \in \mathbb{R}$. We say that L is a* ***cluster point*** *of x_n iff for each $\varepsilon > 0$, $x_n \in \,]L-\varepsilon, L+\varepsilon[$ for infinitely many $n \in \mathbb{N}$.*

2. In mathematical languages, L is a cluster point of x_n $\Leftrightarrow$ $\forall \varepsilon > 0, \forall\, p \in \mathbb{N}, \exists\, n \geq p$ such that $x_n \in \,]L-\varepsilon, L+\varepsilon[$.

Proposition 1.5.2 *In the definition 1.5.1, 1⇔2*

Proof 1.5.3 • *Suppose 1 holds. Let $\varepsilon > 0$ and $p \in \mathbb{N}$ be arbitrary.*

As by 1, $x_n \in \,]L-\varepsilon, L+\varepsilon[$ for infinitely many n. Among these, 'n's there are at least one $n \geq p$. So, for this n, $x_n \in \,]L-\varepsilon, L+\varepsilon[$.

• *Suppose 2 holds. Let $\varepsilon > 0$ be arbitrary.*

In 2, let $p = 0$. Then, $\exists\, n_0 \geq 0 : x_{n_0} \in \,]L-\varepsilon, L+\varepsilon[$.

$$\begin{array}{llll} & p = n_0 + 1, & \textit{then by 2} & \exists\, n_1 \geq n_0 + 1 : x_{n_1} \in \,]L-\varepsilon, L+\varepsilon[\\ \textit{Let} & p = n_1 + 1, & \textit{then} & \exists\, n_2 \geq n_1 + 1 : x_{n_2} \in \,]L-\varepsilon, L+\varepsilon[\\ & \vdots & & \end{array}$$

In this way we get, $n_1, n_2, ..., n_k$ such that for all $k \in \mathbb{N}$, $x_{n_k} \in \,]L-\varepsilon, L+\varepsilon[$. So, $x_n \in \,]L-\varepsilon, L+\varepsilon[$ for infinitely many n, $p \in \mathbb{N}$. Actually we have proved the following theorem.

Theorem 1.5.4 *Let $(x_n)_{n\in\mathbb{N}}$ be any sequence in $\mathbb{R}$ and $L \in \mathbb{R}$. Then, L is a cluster point of x_n iff x_n has a subsequence $y_k = x_{n_k}$ that converges to L as $k \to \infty$.*

Proof 1.5.5 *(⇒) Suppose L is a cluster point, so we have $\forall \varepsilon > 0$, $x_n \in \,]L-\varepsilon, L+\varepsilon[$ for infinitely many $n \in \mathbb{N}$.*

Let $\varepsilon = \frac{1}{2^0}$. There are infinitely many n such that $x_n \in \,]L-1, L+1[$. Let n_0 be the smallest of these integers. $x_{n_0} \in \,]L-1,\, L+1[$

Let $\varepsilon = \frac{1}{2}$. There are infinitely many $n \in \mathbb{N}$ such that $x_n \in \left]L - \frac{1}{2}, L + \frac{1}{2}\right[$. Let among these n's $n_1 > n_0$ be any integer such that: $x_{n_1} \in \left]L - \frac{1}{2},\, L + \frac{1}{2}\right[$

Next, let $\varepsilon = \frac{1}{2^2}$. *There are infinitely many* n *such that* $x_n \in \left]L - \frac{1}{4}, L + \frac{1}{4}\right[$.
Let among these n*'s* $n_2 > n_1$ *be any integer. So,* $x_{n_2} \in \left]L - \frac{1}{4}, L + \frac{1}{4}\right[$.
Let $\varepsilon = \frac{1}{2^3}$
$\vdots$
In this way we construct a sequence of integers $n_0 < n_1 < n_2 < \ldots < n_k < \ldots$ *such that,* $\forall k \in \mathbb{N}$, $x_{n_k} \in \left]L - \frac{1}{2^k}, L + \frac{1}{2^k}\right[$.
Hence, $|x_{n_k} - L| < \frac{1}{2^k}$. *As* $k \to \infty, x_{n_k} \to L$.
($\Leftarrow$) *Suppose that* x_n *has a subsequence* y_{n_k} *that converges to* L *as* $k \to \infty$.
So $\forall \varepsilon > 0, \exists\, k_0 \in \mathbb{N}, \forall\, k \geq k_0$, $y_k \in \,]L - \varepsilon, L + \varepsilon[$. *Hence, as we have seen above, this means* $\forall \varepsilon > 0$, $x_n \in \,]L - \varepsilon, L + \varepsilon[$ *for infinitely many* $n \in \mathbb{N}$. *So,* L *is a cluster point.*

Example 1.5.6 *Let* $x_n = (-1)^n$. *This sequence is not convergent, but it has convergent subsequences. Indeed,* $L = 1$ *and* $L = -1$ *are cluster points.*
Let $L = 1$. *Then,* $\forall \varepsilon > 0$, $x_{2n} \in \,]1 - \varepsilon, 1 + \varepsilon[$ *for all* $n \in \mathbb{N}$. *So,* $L = 1$ *is a cluster point.*

Example 1.5.7 $(x_n)_{n \in \mathbb{N}} = 0, 1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \ldots$
Here $L = 1$ *is a cluster point of this sequence. Indeed the subsequence*
$(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \cdots, \frac{1}{2n}, \cdots) \to 0$

Example 1.5.8 *Let* $x_n = 1, 2, 3, 4, 1, 2, 3, 4, 1, 2, 3, 4, \ldots$. *Then,* $1, 2, 3, 4$ *are cluster points.*

Example 1.5.9 *Let* $[0, 1] \cap \mathbb{Q} = \{x_0, x_1, \ldots, x_n, \ldots\}$ *in any order. Consider the sequence* $(x_n)_{n \in \mathbb{N}}$. $\forall\, L \in [0, 1]$, $\forall \varepsilon > 0$, *the interval* $]L - \varepsilon, L + \varepsilon[$ *contains infinitely many rational numbers. So,* $x_n \in \,]L - \varepsilon, L + \varepsilon[$ *for infinitely many* n. *Hence, any* $L \in [0, 1]$ *is a cluster point of* $(x_n)_{n \in \mathbb{N}}$.

Example 1.5.10 $x_n = e^n$, $\forall n \in \mathbb{N}$. x_n *has no cluster points.*

Theorem 1.5.11 *Let* x_n *be a sequence in* $\mathbb{R}$. $x_n \to L$ *iff* L *is the only cluster point of* x_n.

Proof 1.5.12 ($\Longrightarrow$) *If* $x_n \to L$, *then every subsequence of* x_n *converges to the same* L. *So,* $x_{2n} \to L$ *and* $x_{2n+1} \to L$
($\Longleftarrow$) *Suppose that* $x_{2n} \to L$ *and* $x_{2n+1} \to L$. *So, we have:*
$\forall \varepsilon > 0$, $\exists\ N_1 \in \mathbb{N} : \forall\, n \geq N \ \ |x_{2n} - L| < \varepsilon$.
$\forall \varepsilon > 0$, $\exists\ N_2 \in \mathbb{N} \forall\, n \geq N \ \ |x_{2n+1} - L| < \varepsilon$
Let, $N' = \max\{N_1, N_2\}$ *and* $N = 2N' + 1$. *Then* $n \geq N \ \ |x_n - L| < \varepsilon \Longrightarrow x_n \to L$

Lemma 1.5.13 ***Bolzano-Weierstrass Theorem:*** *Every sequence $(x_n)_{n\in\mathbb{R}}$ has a monotone subsequence.*

Proof 1.5.14 *Let*

$$F_0 = \{x_0, x_1, ..., x_n, ...\}$$
$$F_1 = \{x_1, x_2, ..., x_n, ...\}$$
$$\vdots$$
$$F_n = \{x_n, x_{n+1}, ...\}$$

Clearly, $F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$
There are two possibilities:

1. *Every F_n has a smallest element.*

2. *There is $p \in \mathbb{N}$, such that F_p does not have a smallest element.*

- *Suppose 1 holds: So, every F_n has a smallest element. Let x_{n_0} be the smallest element of F_0. Then consider the set F_{n_0+1}. Then F_{n_0+1} has a smallest element call it x_{n_1}. Obviously, $n_1 > n_0$ and $x_{n_0} \leq x_{n_1}$, since $F_0 \subseteq F_{n_0+1}$. Next, let consider the set F_{n_1+1}. Then F_{n_1+1} has a smallest element call it x_{n_2}. So $n_2 > n_1$ and $x_{n_1} \leq x_{n_2}$. Next, let consider the set F_{n_2+1}, it has a smallest element call it x_{n_3}, ... so on. Then, $n_0 < n_1 < n_2 < ... < n_k < ...$ and $x_{n_0} \leq x_{n_1} \leq x_{n_2} \leq \ldots \leq x_{n_k} \leq \ldots$. So, $(x_{n_k})_{k\in\mathbb{N}}$ is an increasing subsequence of the initial subsequence.*

- *Suppose 2 holds: So, for some $p \in \mathbb{N}$, F_p has no smallest element. But then, for any $n \geq p$, $F_p = \{x_{p+1,...,}x_{n-1}\} \cup F_n$. F_n can not have a smallest element either. Hence, $\forall n \geq p$, F_n has no smallest element.*

  Let x_{n_0} be any element in F_p. Consider F_{n_0+1}. F_{n_0+1} has no smallest element. So, there is an element call it $x_{n_1} \in F_{n_0+1} \ni x_{n_1} < x_{n_0}$ (Clearly $n_1 > n_0$). Now consider F_{n_1+1}, it has no smallest element. So, there is an element call it $x_{n_2} \in F_{n_1+1} \ni x_{n_2} < x_{n_1}$. Consider $F_{n_2+1}, \ldots$ and so on. In this way, we get $x_{n_0} > x_{n_1} > x_{n_2} > \ldots$ and $n_0 < n_1 < n_2 < \ldots < n_k < \ldots$ So, $(x_{n_k})_{k\in\mathbb{N}}$ is a decreasing subsequence of $(x_n)_{n\in\mathbb{N}}$.

Theorem 1.5.15 ***(Fundamental Theorem of Real Analysis)*** *Every bounded sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ has at least one convergent subsequence. (Or equivalently at least one cluster point.)*

Proof 1.5.16 *By the lemma 1.5.13, x_n has a monotone subsequence. Since every bounded monotone sequence converges, we conclude that x_n has a convergent subsequence.*

Let $x_n = \sin n$. Then x_n has a convergent subsequence.

1.5.2 Cauchy Sequences

Let $x_{n\in\mathbb{N}}$ be a sequence. Suppose we only know that, $|x_{n+1} - x_n| = \dfrac{1}{2n}$. Does such a sequence converge? Let $x_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$. Then, $|x_{n+1} - x_n| = \dfrac{1}{n+1}$. So, $|x_{n+1} - x_n| \to 0$. But $(x_n)_{n\in\mathbb{N}}$ diverges.

Let $x_n = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2}$. Then $|x_{n+1} - x_n| = \dfrac{1}{(n+1)^2} \to 0$. This time as we know $(x_n)_{n\in\mathbb{N}}$ converges. Now let x_n be a sequence that converges to some $L \in \mathbb{R}$. So we have:

$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}\, \forall\, n \geq N \quad |x_n - L| < \dfrac{\varepsilon}{2}$

Hence,$\forall n \geq N$, $\forall m \geq N$,

$$|x_n - x_m| = |x_n - L + L - x_m| \leq \underbrace{|x_n - L|}_{<\frac{\varepsilon}{2}} + \underbrace{|x_m - L|}_{<\frac{\varepsilon}{2}} < \varepsilon$$

i.e., if x_n converges, we have:

$\forall\varepsilon > 0$, $\exists\, n \in \mathbb{N}$, $\forall n \geq N$, $\forall m \geq N$, $|x_n - x_m| < \varepsilon$

This is a necessary condition for convergence.

Definition 1.5.17 *A sequence in $\mathbb{R}$ is said to be a **Cauchy sequence** if it satisfies the Cauchy condition, that is:*

$\forall\varepsilon > 0,\ \exists\, N \in \mathbb{N} \ni \forall\, n \geq N, \forall m \geq N \ \ |x_n - x_m| < \varepsilon$

n and m are independent from each other.

This condition is equivalent to :

$\forall\varepsilon > 0,\ \exists\, N \in \mathbb{N} \ni \forall n \geq N\, \forall\, p \in \mathbb{N}\ |x_{n+p} - x_n| < \varepsilon.$

Again it is equivalent to $\lim_{n\to\infty,\, m\to\infty} |x_n - x_m| = 0$

Example 1.5.18 *Prove or disprove that the following sequences are Cauchy sequences.*

1. $x_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n}$
2. $x_n = 1 + \dfrac{1}{2} + \cdots + \dfrac{1}{2^n}$
3. $|x_{n+1} - x_n| \leq \dfrac{1}{2^n}$

1. $x_{2n} - x_n = \dfrac{1}{n+1} + \cdots + \dfrac{1}{2n} \geq \dfrac{1}{2n} = \dfrac{1}{2}$

   $|x_{2n} - x_n| \geq \dfrac{1}{2}$. *Let* $\varepsilon = \dfrac{1}{4}$. *Contradiction. Then the sequence is not Cauchy.*

2. $|x_{n+p} - x_n| = \dfrac{1}{2^{n+1}} + \cdots + \dfrac{1}{2^{n+p}} = \dfrac{1}{2^{n+1}}\left(1 + \dfrac{1}{2} + \cdots + \dfrac{1}{2^{p-1}}\right) = \dfrac{1}{2^{n+1}}\left(\dfrac{1-\left(\frac{1}{2}\right)^p}{1-\frac{1}{2}}\right) \leq \dfrac{1}{2^n}$.

   $\forall n \geq 0, \forall p \geq 0, |x_{n+p} - x_n| \leq \dfrac{1}{2^n}$.

   $\dfrac{1}{2^n} \to 0$. *So,* $\forall \varepsilon > 0, \exists N \in \mathbb{N} \forall n \geq N, \dfrac{1}{2^n} < \varepsilon$.

   Hence, $\forall n \geq N, \forall p \in \mathbb{N}, |x_{n+p} - x_n| < \varepsilon$, *so* $(x_n)_{n\in\mathbb{N}}$ *is Cauchy.*

3. *Our sequence* $(x_n)_{n\in\mathbb{N}}$ *satisfies the condition* $|x_{n+1} - x_n| \leq \dfrac{1}{2^n}$.

   Hence,
   $$\begin{aligned} |x_{n+p} - x_n| &= |x_{n+p} + x_{n+p-1} - x_{n+p-1} + x_{n+p-2} - x_{n+p-2} - x_n| \\ &\leq |x_{n+p} - x_{n+p-1}| + |x_{n+p-1} - x_{n+p-2}| + \ldots + |x_{n+1} - x_n| \\ &\leq \frac{1}{2^{n+p-1}} + \frac{1}{2^{n+p-2}} + \cdots + \frac{1}{2^n} \end{aligned}$$

   Since $|x_{n+p} - x_{n+p-1}| \leq \dfrac{1}{2^{n+p-1}}, |x_{n+p-1} - x_{n+p-2}| \leq \dfrac{1}{2^{n+p-2}}, |x_{n+1} - x_n| \leq \dfrac{1}{2^n}$.

   As $\dfrac{1}{2^n} \to 0, \forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N\ \dfrac{1}{2^n} < \varepsilon$. *So* $\forall n \geq N, \forall p \in \mathbb{N}, |x_{n+p} - x_n| < \varepsilon$.

   Hence, $(x_n)_{n\in\mathbb{N}}$ *is Cauchy.*

Note: Concerning any sequence $(x_n)_{n\in\mathbb{N}}$, there are two basic questions:

1. Does $(x_n)_{n\in\mathbb{N}}$ converge?
2. If it does, what is $\lim_{n\to\infty} x_n$?

Proposition 1.5.19 *Every Cauchy sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *is bounded.*

Proof 1.5.20 *As* x_n *is Cauchy, we have* $\forall \varepsilon > 0, \exists N \in \mathbb{N} \forall n, m \geq N, |x_n - x_m| < \varepsilon$. *Fix* $m = N$. *Then,* $|x_n| = |x_n - x_N + x_N| \leq \varepsilon + |x_N|\ \forall n \geq N$. *Hence,* $\sup\{|x_n| : n \in \mathbb{N}\} \leq \sup\{|x_0|, \ldots, |x_N| : \varepsilon + |x_N|\}$. *So,* x_n *is bounded.*

Theorem 1.5.21 *(*$\mathbb{R}$ ***is complete****): A sequence* x_n *in* $\mathbb{R}$ *is convergent iff it is Cauchy.*

Proof 1.5.22 *We have already seen that every convergent sequence is Cauchy.*

Conversely, assume x_n *is Cauchy. So it is bounded. Hence, by Bolzano Weierstrass Theorem* x_n *has a convergent subsequence,* $y_k = x_{n_k}$.

Let $\lim_{k\to\infty} y_k = L$. *Let us see not only* x_{n_k}*, but the whole sequence converges to* L.

Indeed $y_k \to L$ *means that:*

$$\forall \varepsilon > 0, \exists k_0 \in \mathbb{N} \ni \forall k \geq k_0 \ |x_{n_k} - L| < \frac{\varepsilon}{2} \quad *$$

As x_n *is Cauchy, we also have:* $\forall\varepsilon > 0, \exists N \in \mathbb{N} \ni \forall n, m \geq N \ |x_n - x_m| < \frac{\varepsilon}{2}$.

Let $k \geq k_0$ *be such that* $n_k \geq N$.

Then $\forall n \geq N$,

$$|x_n - L| = |x_n - x_{n_k} + x_{n_k} - L| \leq \underbrace{|x_n - x_{n_k}|}_{\leq \frac{\varepsilon}{2} \text{ (by Cauchy)}} + \underbrace{|x_{n_k} - L|}_{\leq \frac{\varepsilon}{2} \text{ (by } * \text{)}} \leq \varepsilon$$

Hence $x_n \to L$.

1.5.3 Exercises III

1. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Show that $x_n \to x$ iff there exists a decreasing sequence $(t_k)_{k\in\mathbb{N}}$, $t_k \geq 0$, $t_k \to 0$ such that $|x_n - x| \leq t_n$ for n large.

2. Let $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ be two convergent sequences with $\lim_{n\to\infty} x_n = a = \lim_{n\to\infty} y_n$. Consider the "mixed sequence" $z_n : x_0, y_0, x_1, y_1, x_2, y_2, \ldots$ Show that $z_n \to a$ too.

3. Show that given any x in $\mathbb{R}$ there exists a sequence of **rational** numbers $(r_n)_{n\in\mathbb{N}}$ and a sequence of **irrational** numbers $(s_n)_{n\in\mathbb{N}}$ such that $r_n \to x$ and $s_n \to x$.

4. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. Assume $x_n \in \mathbb{Z}$ for each $n \in \mathbb{N}$. Show that $(x_n)_{n\in\mathbb{N}}$ is convergent iff $(x_n)_{n\in\mathbb{N}}$ is almost constant i.e. $\exists\, N \in \mathbb{N}, \forall\, n \geq N, \forall\, m \geq N,\ x_n = x_m$.

5. Let $(x_n)_{n\in\mathbb{N}}$ be a positive sequence. If $x_n \to 0$ show that then $\frac{x_n}{1+x_n} \to 0$ too.

   Conversely, if $\frac{x_n}{1+x_n} \to 0$ show that then $x_n \to 0$ too.

6. Let $x_n = \ln(n+1)$. Show that $|x_{n+1} - x_n| \to 0$ as $n \to \infty$.

   Is $(x_n)_{n\in\mathbb{N}}$ Cauchy? Is $(x_n)_{n\in\mathbb{N}}$ convergent?

7. For $0 \leq b \leq a$, find $\lim_{k\to\infty}(a^k + b^k)^{\frac{1}{k}}$.

8. If $x_n > c$ for all $n \in \mathbb{N}$ and $x_n \to x$, can you say that $x > c$?

1.6 lim sup, lim inf

Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. Say $a \leq x_n \leq b$ $(\forall\, n \in \mathbb{N})$. Bolzano Weierstrass' Theorem says that x_n has at least one cluster point, say L. Then, $a \leq L \leq b$. We also know that x_n may have uncountably many cluster points. Let F be the set of all the cluster points. $F \neq \varnothing$ and $F \subseteq [a, b]$.

We are going to show that F has a smallest element which we call $\liminf x_n$, and a largest element which we call $\limsup x_n$. Next, we are going to prove the existence of these cluster points.

Let $F_0 = \{x_0, x_1, x_2, \ldots, x_n\}$
$F_1 = \{x_1, x_2, x_3, \ldots, x_n\}$
$F_2 = \{x_2, x_3, x_4, \ldots, x_n\}$
$\vdots$
$F_n = \{x_n, x_{n+1}, \ldots\}$

$F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$ and $F_n \subseteq [a, b]$ $(\forall n \in \mathbb{N})$.

By the supremum axiom, $\sup F_n$ and $\inf F_n$ exist. Let $y_n = \inf F_n$, $z_n = \sup F_n$. Since $F_1 \supseteq F_2 \supseteq \ldots \supseteq F_n \supseteq \ldots$, $y_0 \leq y_1 \leq \ldots \leq y_n \leq \ldots \leq b$ and $z_0 \geq z_1 \geq \ldots \geq z_n \geq \ldots \geq a$.

Hence we have two monotone bounded sequences: y_n, z_n.

Hence $l = \lim_{n\to\infty} y_n$ and $L = \lim_{n\to\infty} z_n$ exists.

Moreover, $l = \sup_{n\in\mathbb{N}} y_n$ and $L = \inf_{n\in\mathbb{N}} z_n$. As $y_n = \inf_{k\geq n} x_k$, and $z_n = \sup_{k\geq n} x_k$, so that $l = \sup_{n\in\mathbb{N}} inf_{k\geq n} x_k$ and $L = \inf_{n\in\mathbb{N}} \sup_{k\geq n} x_k$

Example 1.6.1 *Let* $x_n = (-1)^n$. *Then* $F_n = \{x_n, x_{n+1}, \ldots\} = \{-1, 1\}$, $\forall n \in \mathbb{N}$.
Hence, $y_n = -1, z_n = 1$. *So,* $y_n \to -1$, *and* $z_n \to 1$.
Hence, $\limsup x_n = 1$, $\liminf x_n = -1$.

Example 1.6.2 *Let* $x_n = 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, \ldots$. *Then* $\forall n \geq 0$, $F_n = \{0, 1, 2, 3, 4\}$. *Then* $y_n = 0$, $z_n = 4$. *Hence,* $\limsup x_n = 4$, $\liminf x_n = 0$.

Theorem 1.6.3 *Let* x_n *be a bounded sequence:* $l = \lim_{n\to\infty} \inf_{n\in\mathbb{N}} x_n$, $L = \lim_{n\to\infty} \sup_{n\in\mathbb{N}} x_n$. *Then*

1. *l and L are cluster points of x_n.*

2. *l is the smallest cluster point and L is the largest cluster point of x_n.*

Proof 1.6.4 *First observe that* $\lim_{n\to\infty} \sup_{n\in\mathbb{N}} (-x_n) = -\lim_{n\to\infty} \inf_{n\in\mathbb{N}} x_n$. *Hence, it is enough to prove the theorem for L.*

To show that L is a cluster point, we have to show that given:
$\boxed{1}$ $\forall \varepsilon > 0$, $x_n \in \,]L - \varepsilon, L + \varepsilon[$ *for infinitely many* $n \in \mathbb{N}$.
Let $\varepsilon > 0$ be given. Since $z_n \to L$, by the definition of the convergence,
$\exists\, N \in \mathbb{N} \ni \forall\, n \geq N$ $|z_k - L| < \varepsilon$. *As* $z_k = \inf\{x_k, x_{k+1}, \ldots\}$, *we conclude that:*
$\boxed{2}$ $\forall k \geq N$, $\exists\, n_k \geq k : x_{n_n} \in \,]L - \varepsilon, L + \varepsilon[$

1 $\Leftrightarrow$ *2* *So, L is a cluster point.*

Let us see that L is the largest of the all cluster points of x_n.

If not, for some cluster point S of x_n you would have $S > L$. Let $\varepsilon > 0$ be such that $]S-\varepsilon, S+\varepsilon[\cap]L-\varepsilon, L+\varepsilon[= \varnothing$. As S is a cluster point $x_n \in]S-\varepsilon, S+\varepsilon[$ for infinitely many $n \in \mathbb{N}$. In particular, $z_n = \sup\{x_n, x_{n+1}, \ldots\} \geq L+\varepsilon \; \forall n \in \mathbb{N}$. As $z_n \to L$, this is not possible. So L is the largest cluster point of x_n.

Main interest of $\limsup$, $\liminf$ is that they always exist whereas $\lim_{n\to\infty} x_n$ exists only exceptionally.

Theorem 1.6.5 *Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then, x_n converges iff*
$\lim_{n\to\infty}\sup_{n\in\mathbb{N}} x_n = \lim_{n\to\infty}\inf_{n\in\mathbb{N}} x_n$

Proof 1.6.6 *If $x_n \to L$, then L is the only cluster point of x_n.*

So $\lim_{n\to\infty}\sup_{n\in\mathbb{N}} = \lim_{n\to\infty}\inf_{n\in\mathbb{N}} = L$.

Conversely, if $\limsup = \liminf$, *then this implies that x_n has only one cluster point, namely $S = \limsup = \liminf$.*

To finish the proof it is enough to prove the following result.

Proposition 1.6.7 *If x_n is **bounded** and has **only one** cluster point, (L) then, $x_n \to L$.*

Proof 1.6.8 *If x_n does not converges to L then, $\exists \varepsilon > 0 \ni x_n \notin]L-\varepsilon, L+\varepsilon[$ for infinitely many $n \in \mathbf{N}$. Suppose $x_n \geq L+\varepsilon$ for infinitely many $n \in \mathbf{N}: n_0 < n_1 < n_2 < \ldots < n_k < \ldots$*

So that ,$y_k \geq L+\varepsilon$, where $y_k = x_{n_k}$.

y_k is a subsequence of x_n. As x_n is bounded so is y_k. Hence by the Bolzano Weierstrass' theorem y_k has a convergent subsequence. $y_{k_p} \to S$, and since $y_k \geq L+\varepsilon$, in particular, $S \neq L$ but as S is also a cluster point of x_n we have contradiction. So, $x_n \to L$.

Remark: If x_n is not bounded, the preceding lemma is false.

Example 1.6.9 *Let $x_n = 1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots$ Then, 0 is the only cluster point of x_n. But x_n does not converges to 0. Hence, a sequence x_n diverges iff x_n is unbounded (for example: $x_n = e^n$) or x_n is bounded but has more than one cluster points.(for example: $x_n = (-1)^n$)*

Theorem 1.6.10 *Let x_n and y_n be two bounded sequences.*

1. $\limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n$
2. $\liminf (x_n + y_n) \geq \liminf x_n + \liminf y_n$
3. *If $x_n \geq 0$ and $y_n \geq 0$, then* $\limsup (x_n y_n) \leq \limsup x_n \times \limsup y_n$
4. *If x_n or y_n converges, then the above inequalities become equality.*

Proof 1.6.11 *Let $A_n = \{x_n, x_{n+1}, \ldots\}$, $B_n = \{y_n, y_{n+1}, \ldots\}$, $C_n = \{x_n + y_n, x_{n+1} + y_{n+1}, \ldots\}$. Then,*

1. $\sup C_n \le \sup A_n + \sup B_n$

   Let $X = \{n, n+1, \ldots\}$, $f : X \to \mathbf{R}$, $f(n) = x_n$, $g : X \to \mathbf{R}$, $g(n) = y_n$. *Hence, passing to limits,* $\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n$

2. $infC_n \ge \inf A_n + \inf B_n$. *Similarly,* $\limsup(x_n y_n) \le \limsup x_n \times y_n$

3. *As* $x_n, y_n > 0$, $x_n \times y_n > 0$, *so* $\sup(x_n \times y_n) \le \sup x_n \times \sup y_n$. *Hence, passing to limits,* $\limsup(x_n y_n) \le \limsup x_n \times \limsup y_n$

4. *Suppose* $x_n \to L$.

   - *Let* $S = \limsup y_n$. *Since* S *is a cluster point of* y_n, $\exists$ *a subsequence*
     $y_{n_k} \to S$. *Then, since* $x_n \to L$, $x_{n_k} \to L$ *too.*
     $x_{n_k} + y_{n_k} \to L + S$. *Hence,* $L + S$ *is a cluster point of* $x_n + y_n$
     Since $\limsup(x_n + y_n) \le \limsup x_n + \limsup y_n = L + S$
     Hence $L + S = \limsup(x_n + y_n)$
   - *Similarly, let* $s = \liminf y_n$. *Since* s *is a cluster point of* y_n, $\exists$ *a subsequence*
     $y_{n_k} \to s$. *Then, since* $x_n \to L$, $x_{n_k} \to L$ *too.*
     $x_{n_k} + y_{n_k} \to L + s$. *Hence,* $L + s$ *is a cluster point of* $x_n + y_n$
     Since $\liminf(x_n + y_n) \ge \liminf x_n + \liminf y_n = L + s$
     Hence $L + s = \liminf(x_n + y_n)$
   - *As above let* $S = \limsup y_n$. *Since* S *is a cluster point of* y_n, $\exists$ *a subsequence*
     $y_{n_k} \to S$. *Then, since* $x_n \to L$, $x_{n_k} \to L$ *too.*
     $x_{n_k} \times y_{n_k} \to L \times S$. *Hence,* LS *is a cluster point of* $x_n \times y_n$
     Since $\limsup(x_n \times y_n) \le \limsup x_n \times \limsup y_n = LS$
     Hence $LS = \limsup(x_n \times y_n)$

Example 1.6.12 *Let* $x_n = (-1)$, $y_n = (-1)^{n+1}$. *Then* $x_n + y_n = 0$.
So, $\limsup(x_n + y_n) = 0 < \limsup x_n + \limsup y_n = 2$

1.7 Elementary Topology of $\mathbb{R}$

Definition 1.7.1 *Let* $A \subseteq \mathbf{R}$ *be any set. We say that, "A is closed in* $\mathbb{R}$*" if whenever we take a sequence* x_n*in A, that converges to some* $L \in \mathbf{R}$, $L \in A$ *that is "A is* ***closed*** *under limit operation."*

There are two problems:

1. Which sets are closed?
2. How stable they are? (i.e. $\cup, \cap$ of closed sets are closed.)

Example 1.7.2 *Prove or disprove that the following subsets of* $\mathbf{R}$ *are closed in* $\mathbf{R}$.

1. $A = [a, b]$
2. $A = [a, \infty[$
3. $A = [a, b[$
4. $A = \mathbf{N}$
5. $A = \mathbf{Z}$
6. $A = \mathbf{Q}$
7. $A = \mathbf{R} \setminus \mathbf{Q}$

Solution:

1. $[a, b]$ *Let* x_n *be a sequence in* A *that converges to some* $x \in \mathbb{R}$. *Is* $x \in A$? *As,* $a \leq x_n \leq b \forall n \in \mathbb{N}$ *and as we have seen,* $a \leq x \leq b$ *so,* A *is closed.*

2. $[a, \infty[$ *Let* x_n *be in* A *and* $x_n \to x$, $x \in \mathbb{R}$. *As* $x_n \geq a \forall n \in N$ *then* $x \geq a$. *So* $x \in A$ *so,* A *is closed.*

3. $[a, b[$ *let* $x_n = b - \frac{1}{n}$ *then* $x_n \in A$ *but* $\lim_{n\to\infty} x_n = b$, $b \notin A$. A *is **not** closed.*

4. $A = \mathbb{N}$, x_n *in* $\mathbb{N}$ *a sequence where* $x_n \to x$ *Since* $x_n \to x$, x_n *is Cauchy.* $\forall \varepsilon > 0$, $\forall n \geq N$, $\forall p \in \mathbb{N}\, |x_{n+p} - x_n| < \varepsilon$. *Take* $0 < \varepsilon < 1$, *as* $x_{n+p} - x_n < 1$ *then,* $x_{n+p} = x_n$. *So,* $\forall n \geq N$, $x_n = x_{n+1} = \ldots = x_{n+p} = \ldots$

   Hence, any convergent sequence in $\mathbb{N}$ *is almost constant, so,* $\mathbb{N}$ *is closed in* $\mathbb{R}$

5. $A = \mathbb{Z}$ *same as above,* $\mathbb{Z}$ *is closed in* $\mathbb{R}$.

6. $A = \mathbb{Q}$: *we have seen that* $\frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!} \in \mathbb{R}$

   $x_n \in \mathbb{Q}$ *but* $x_n \to e, \notin \mathbb{Q}$ *then,* Q *is **not** closed in* $\mathbb{R}$.

7. $A = \mathbb{R} \setminus \mathbb{Q}$ *let* $x_n = \frac{\sqrt{2}}{n+1}$, $\forall n \in \mathbb{N}$ *then* $x_n \in \mathbb{R} \setminus \mathbb{Q}$ *but* $\lim_{n\to\infty} x_n = 0 \in \mathbb{Q}$ *then* $\mathbb{R} \setminus \mathbb{Q}$ *is **not** closed in* $\mathbb{R}$.

Proposition 1.7.3 ***(Properties of closed sets):*** *A and B are two closed sets in $\mathbb{R}$. Then,*

1. $A \cup B$ *is also closed.*
2. $A \cap B$ *is also closed.*

Proof 1.7.4 1. *Let* x_n *be a sequence in* $A \cup B$, *that converges to* $x \in \mathbb{R}$. *We need to show that,* $x \in A \cup B$ *too. For* x_n *there are three possibilities:*

(a) $x_n \in A$ *for all but finitely many* n. *Then,* $x \in A$

(b) $x_n \in B$ *for all but finitely many* n. *Then,* $x \in B$

(c) $x_n \in A$ *for infinitely many* n *and* $x_n \in B$ *for infinitely many* n *Then* $x \in A \cap B$

So at any case, $x \in A \cup B$*, hence* $A \cup B$ *is closed.*

2. *Let* x_n *in* $A \cap B$*, that converges to some* $x \in \mathbb{R} \setminus \mathbb{Q}$*. As* $x_n \in A$*,* $x_n \in B$ *for all* $n \in \mathbb{N}$*. Since* A *and* B *are closed,* $x \in A$ *and* $x \in B$ *then* $x \in A \cap B$ *then* $A \cap B$ *is closed.*

Remark:

1. $\cup_{n\geq 1} \left[\dfrac{1}{n}.1 - \dfrac{1}{n}\right] =]0, 1[$

   This shows that, the union of infinitely many closed sets need not be closed.

2. Any finite set $F \subseteq \mathbb{R}$ is closed.

   $F = \{a_1, a_2, \ldots, a_n\}$

   $F = \{a_1\} \cup \{a_2\} \cup \ldots \cup \{a_n\}$. As each $\{a_i\}$ is a closed set, F is closed.

3. The intersection of any family, (finite or not) of closed sets $(F_\alpha)_{\alpha \in I}$ is closed.

   $F = \cap_{\alpha \in I} F_\alpha$

Theorem 1.7.5 *Let* $A \subseteq \mathbb{R}$ *be a bounded set* ($\neq \emptyset$) *and* $\alpha = \sup A$*,* $\beta = \inf A$*. Then, if* A *is closed,* $\alpha \in A$ *and* $\beta \in A$*.*

Proof 1.7.6 $\alpha = \sup A \Leftrightarrow \begin{cases} \forall x \in A, \alpha \geq x \\ \forall \varepsilon > 0, \exists x_\varepsilon \in A : x_\varepsilon > \alpha - \varepsilon \end{cases}$

Now let $\varepsilon = 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}$*. Denote* $x_n \in A$ *that correspond to* $\varepsilon = \frac{1}{n}$ *so that* $x_n > \alpha - \frac{1}{n}$*. In that way we get a sequence,* $(x_n)_{n\geq 1}$ *in* A *such that,* $\alpha - \frac{1}{n} < x_n \leq \alpha$*. Then,* $x_n \to \alpha$ *as* x_n *in* A *and* A *is closed* $\alpha \in A$*.*

Similarly, $\beta \in A$*.*

Remark: Converse is false.

Example 1.7.7 *Let* $A = \{-1\} \cup]0, 1[\cup \{2\}$

$\sup A = 2$, $x_n = \dfrac{1}{n} \in A$*, but* $\lim_{n\to\infty} x_n = 0 \notin A$

$\inf A = -1$

Finding Approximating sequences: Let $A \subseteq \mathbb{R}$ be a set and $x \in \mathbb{R}$.

Problem: When is there a sequence x_n in A converges to x?

Theorem 1.7.8 *Let* $A \subseteq \mathbb{R}$ *be any set and* $L \in \mathbb{R}$ *be any point. Then there exists a sequence* x_n *in* $A :$ $x_n \to L \Leftrightarrow \forall \varepsilon > 0,\ ]L - \varepsilon, L + \varepsilon[\cap A \neq \varnothing$

Proof 1.7.9 ($\Longrightarrow$) *Suppose that there exists a sequence x_n in A that converges to L. So we have* $\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N,\ x_n \in \,]L-\varepsilon, L+\varepsilon[\, \cap A \neq \varnothing$

($\Longleftarrow$) *Suppose that,* $\forall\varepsilon > 0,]L-\varepsilon, L+\varepsilon[\, \cap A \neq \varnothing$. *So for $\varepsilon = 1$ this intersection is not empty. Take any point in it and call it x_1.*

For $\varepsilon = \dfrac{1}{2}$, this intersection is not empty. Take any point in it and call it x_2.

$\vdots$

For $\varepsilon = \dfrac{1}{n}$, this intersection is not empty. Take any point in it and call it x_n. In this way, construct a sequence x_n such that $x_n \in \left]L - \frac{1}{n}, L + \dfrac{1}{n}\right[\cap A, \forall n \geq 1$. *Hence, $x_n \in A$ and $|x_n - L| < \dfrac{1}{n}$ so $x_n \to L$.*

Example 1.7.10 *We have seen that,* $\forall\varepsilon > 0, \forall x \in \mathbb{R},\ ]x-\varepsilon, x+\varepsilon[\, \cap \mathbb{Q} \neq \varnothing$

Hence, $\forall x \in \mathbb{R},\ \exists\, x_n \in \mathbb{Q} : x_n \to x$

Example 1.7.11 *We have seen that,* $\forall\varepsilon > 0, \forall x \in \mathbb{R}.\ ]x-\varepsilon, x+\varepsilon[\, \cap (\mathbb{R}/\mathbb{Q}) \neq \varnothing$.

Hence, $\forall x \in \mathbb{R},\ \exists\, q_n \in (\mathbb{R}/\mathbb{Q}) :\ q_n \to x$

Exercise: Show that given $x \in \mathbb{R}$ there exists a sequence x_n in the set $A = \left\{a + b\sqrt{2} :\ a, b \in \mathbb{Z}\right\}$. That converges to x.

Definition 1.7.12 *A subset A of $\mathbb{R}$ is said to be an* ***open set*** *if $A^C = \mathbb{R}/A$ is closed in $\mathbb{R}$.*

Example 1.7.13 *$\mathbb{Q}$ is not open since $\mathbb{R}/\mathbb{Q}$ is not closed. So $\mathbb{Q}$ and $\mathbb{R}/\mathbb{Q}$ are neither open nor closed.*

Example 1.7.14 *The set $A =]a, b[$ is an open set, since $\mathbb{R}/A =]\infty, a] \cup [b, \infty[$ is closed. ($]\infty, a]$ is closed and $[b, \infty[$ is closed.) Hence A is open.*

Theorem 1.7.15 *Let $A \subseteq \mathbb{R}$ be any set then A is open $\Leftrightarrow$ $\forall x \in A, \exists\varepsilon > 0$ such that $]x-\varepsilon, x+\varepsilon[\, \subseteq A$.*

Proof 1.7.16 ($\Longrightarrow$) *Suppose A is open and $x \in A$. For a contradiction, suppose $\forall\varepsilon > 0$*

$]x-\varepsilon, x+\varepsilon[\, \subsetneq A$ *i.e.* $\forall\varepsilon > 0,]x-\varepsilon, x+\varepsilon[\, \cap A^C \neq \varnothing$.

By the above theorem there exists a sequence x_n in A^C that converges to x as A^C is closed. $x \in A^C$ so $x \notin A$.

Hence, $\exists\, \varepsilon > 0 :\ ]x-\varepsilon, x+\varepsilon[\, \subseteq A$

($\Longleftarrow$) *Suppose that $\forall x \in A,\ \exists\varepsilon > 0 :\ ]x-\varepsilon, x+\varepsilon[\ \subseteq A$. Let us see that A is open, i.e. A^C is closed. Let $(x_n)_{n\in N}$ be a sequence in A^C that converges to some $x \in R$. If $x \notin A^C$ then $x \in A$. So for some $\varepsilon > 0$, $]x-\varepsilon, x+\varepsilon[\, \subseteq A$. As $x_n \to x$, $x_n \in \,]x-\varepsilon, x+\varepsilon[$ for all but finitely many n. So, $x_n \in A$ for all but finitely many n.(contradiction). So, $x \in A^C$ and A^C is closed. So, A is open.*

Consequences:

1. A countable set can not be open.
2. A subset $A \subseteq \mathbb{R}$ is open iff A is a union of open intervals; $A = \cup_{\alpha \in I}]a_\alpha, b_\alpha[$
3. $A = [a, b[$ is neither open nor closed.
4. If A is open and bounded, if $\alpha = \sup A$, and $\beta = \inf A$ then neither $\alpha \in A$ nor $\beta \in A$

Example 1.7.17 $A =]a, b[$

1.8 Closure and Interior of a Set

Definition 1.8.1 *Given any set $A \subseteq \mathbb{R}$,*

1. *The **closure** of A is the smallest closed set that contains A.*
2. *The **interior** of A is the largest open set contained in A.*

Question: Do these sets exist?
Remark: Let $(x_n)_{n \in \mathbb{N}}$ be any bounded sequence. Let $l = \liminf x_n$ and $L = \limsup x_n$. Then, $\forall \varepsilon > 0,\ \exists\, n \in \mathbb{N},\ \forall\, n \geq N,\ \ L - \varepsilon \leq x_n \leq L + \varepsilon$

Let $\bar{A} = \{x \in \mathbb{R} : x = \lim_{n\to\infty} a_n,\ \text{for some sequence } (a_n)_{n\in\mathbb{N}} \text{ in } A\}$.
Thus, $x \in \bar{A} \Leftrightarrow \exists\, a_n \in A,\ \ a_n \to x$.

Theorem 1.8.2 *For any set A*

1. *$\bar{A}$ is closed in $\mathbb{R}$.*
2. *$\bar{A} \supseteq A$.*
3. *$\bar{A} = A$ iff A is closed in $\mathbb{R}$.*
4. *$\bar{A}$ is the smallest closed set that contains A.*

Proof 1.8.3 *1. First remark that if $A = \emptyset$, $\bar{A} = \varnothing$, then A is closed. So, suppose $A \neq \varnothing$. Let us see that the set O, where $O = \mathbb{R}/\bar{A}$ is open. (As we know, O is open $\Leftrightarrow \forall x \in O, \exists\, \varepsilon > 0,\ ]x - \varepsilon, x + \varepsilon[\subseteq O$).*

Let $x \in O$. We want to prove that there is $\varepsilon > 0$ such that $]x - \varepsilon, x + \varepsilon[\subseteq O$. If this was not the case we would have, $\forall \varepsilon > 0\ \ ]x - \varepsilon, x + \varepsilon[\cap \bar{A} \neq \varnothing$. Then by "Approximation Theorem", there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\bar{A}$ that converges to x.

So, $\forall \varepsilon > 0$, $x_n \in\ ]x - \varepsilon, x + \varepsilon[$ for all but finitely many n. Fix one of these n's. Say $n = p$, so that $x_p \in\ ]x - \varepsilon, x + \varepsilon[$. Then choose $\varepsilon' > 0$ small enough such that

$]x_p - \varepsilon', x_p + \varepsilon'[\ \subseteq \ ]x - \varepsilon, x + \varepsilon[$. *Since* $x_p \in \bar{A}$, *there is a sequence* y_k *in* A *that converges to* x_p. *So,* $y_k \in]x_p - \varepsilon', x_p + \varepsilon'[$ *for all but finitely many* $k \in \mathbb{N}$. *This implies that,* $]x - \varepsilon, x + \varepsilon[\cap A \neq \varnothing \ \forall \varepsilon > 0$. *Again by "Approximation Theorem", there exists a sequence* $(a_n)_{n\in\mathbb{N}}$ *in* A *that converges to* x. *So,* $x \in \bar{A}$ *(Contradiction). So for some* $\varepsilon > 0$, $]x - \varepsilon, x + \varepsilon[\subseteq O$. *Hence* O *is open,* $\bar{A}$ *is closed.*

2. *Let* $x \in \bar{A}$. *Let a constant sequence* $a_0 = a_1 = a_2 = \ldots = a_n = \ldots = x$. *Then* $a_n \to x$. *So,* $x \in \bar{A}$, *hence* $\bar{A} \supseteq A$

3. *If* $\bar{A} = A$, *then* A *is closed, since* $\bar{A}$ *is closed. If* A *is closed, then* $\bar{A} = A$.

4. *Let* B *be a closed set that contains* A. *So if* $a_n \in A$, *and* $(a_n)_{n\in\mathbb{N}}$ *converges to some* x, *then* $x \in B$. *Hence* $\bar{A} \subseteq B$.

Definition 1.8.4 *The set* $\bar{A}$ *is said to be the* ***closure*** *of* A.

Example 1.8.5 *Find* $\overline{\mathbb{Q}}$ *and* $\overline{\mathbb{R}/\mathbb{Q}}$.
$\overline{\mathbb{Q}} = \{x \in \mathbb{R} : x = \lim_{n\to\infty} r_n \text{ for some } r_n \in \mathbb{Q}\} = \mathbb{R}$
$\overline{\mathbb{R}/\mathbb{Q}} = \{x \in \mathbb{R} : x = \lim_{n\to\infty} r_n \text{ for some } r_n \in \mathbb{R}/\mathbb{Q}\} = \mathbb{R}$.

$$\forall A \subseteq \mathbb{R}\ \forall x \in \mathbb{R}, \exists a_n \in A : a_n \to x \Leftrightarrow \forall \varepsilon > 0\,]x - \varepsilon, x + \varepsilon[\cap A \neq \varnothing$$

Example 1.8.6 *Let* $A =]a, b[$. *Then,* $\bar{A} = [a, b]$. *(*$b_n = b - \frac{1}{n} \in A$*)*

Example 1.8.7 *Let* $A = \left\{ \frac{1}{n} : n = 1, 2, ... \right\}$. *Then,* $\bar{A} = A \cup \{0\}$.

Example 1.8.8 *Let* $A = \left\{ \frac{1}{n} + \frac{1}{m} : n = 1, 2, \ldots, \ m = 1, 2, \ldots \right\}$.
Then, $\bar{A} = A \cup \left\{ \frac{1}{m} : m = 1, 2, \ldots \right\} \cup \{0\}$

Example 1.8.9 *Let* $A = \left\{ \frac{1}{n} + \frac{1}{m} + \frac{1}{q} : \begin{array}{c} n = 1, 2, \ldots \\ m = 1, 2, \ldots \\ q = 1, 2, \ldots \end{array} \right\}$.
Then $\bar{A} = A \cup \left\{ \frac{1}{m} + \frac{1}{n} : m = 1, 2, \ldots, \ n = 1, 2, \ldots \right\} \cup \{0\}$

Definition 1.8.10 ***(Interior of a Set*** *A) Let* $A \subseteq \mathbb{R}$ *be any set. Define* $\overset{\circ}{A}$ *as follows:*

$\overset{\circ}{A}$ *is the union of all open intervals* $]a, b[$ *contained in* A. $a = b \Longrightarrow]a, b[= \emptyset$. *So, there is always (empty or not) some open interval in* A. *Of course if* $A = \varnothing$, *then* $A^\circ = \varnothing$. *This set* $\overset{\circ}{A}$ *is called the* ***interior*** *of* A.

Theorem 1.8.11 1. $\overset{\circ}{A}$ *is open.*

2. $A \supseteq \overset{\circ}{A}$

3. $A = \overset{\circ}{A} \Leftrightarrow A$ *is open.*

4. $\overset{\circ}{A}$ *is the largest open set contained in* A.

Proof 1.8.12 1. *As the union of any open sets is open, it is open.*

2. $A \supseteq \overset{\circ}{A}$ *is obvious.*

3. *If* $A = \overset{\circ}{A}$, *then* A *is open, since* $\overset{\circ}{A}$ *is open. If* A *is open, then* A *is a union of open intervals* $(= \overset{\circ}{A})$

4. *Let* $B \subseteq A, \ni B$ *is open. Let* $x \in B$. *Then, as* B *is open* $\exists \varepsilon > 0$ *such that* $]x-\varepsilon, x+\varepsilon[\subseteq B$. *Then,* $]x-\varepsilon, x+\varepsilon[\subseteq A$. *So* $x \in \overset{\circ}{A}$. *Hence* $B \subseteq \overset{\circ}{A}$.

Example 1.8.13 *Find* $\overset{\circ}{\mathbb{Q}}$, $\overset{\circ}{\mathbb{N}}$ *and* $\overset{\circ}{\overline{\mathbb{R}/\mathbb{Q}}}$.

- $\overset{\circ}{\mathbb{Q}} = \varnothing$, *since* $\mathbb{Q}$ *does not contain any open interval.*
- $\overset{\circ}{\overline{\mathbb{R}/\mathbb{Q}}} = \varnothing$, *since* $\overline{\mathbb{R}/\mathbb{Q}}$ *does not contain any open interval.*
- $\overset{\circ}{\mathbb{N}} = \varnothing$, *since* $\mathbb{N}$ *does not contain any open interval.*
- $\overset{\circ}{\mathbb{Z}} = \varnothing$, *since* $\mathbb{Z}$ *does not contain any open interval.*
- $\overset{\circ}{\overline{[a,b]}} =]a,b[$

1.8.1 Exercises IV

1. Let $X = \{a_0, a_1, \ldots, a_n, \ldots\}$, where $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence in $\mathbb{R}$ with $L = \lim_{n\to\infty} a_n$.

   F be the set of the cluster points of $(a_n)_{n\in\mathbb{N}}$. Show that $\overline{X} = X \cup F$.

2. Let $X = \{\tan n : n \in \mathbb{N}\}$. Find $\overline{X}$.

3. Let $X = \mathbb{Q}\backslash\mathbb{N}$. Find $\overline{X}$.

4. Let $X = \mathbb{Q}\backslash\mathbb{Z}$. Find $\overline{X}$.

5. Let $X = \left\{\frac{1}{n} + \frac{1}{m} : n = 1, 2, 3, \ldots, \; m = 1, 2, 3, \ldots\right\}$. Find $\overline{X}$.

6. Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two convergent sequences in $\mathbb{R}$ with

   $L = \lim_{n\to\infty} a_n$ and $S = \lim_{n\to\infty} b_n$. Put $A = \{a_n + b_n : n \in \mathbb{N}\}$, $B = \{a_n + b_m : n \in \mathbb{N}, \; m \in \mathbb{N}\}$. Find $\overline{A}$ and $\overline{B}$.

7. For $A = \{\frac{n}{m} : n \in \mathbb{N}, \; m = 1, 2, 3 \ldots\}$, find $\overline{A}$.

8. Let A and B be two nonempty subsets of $\mathbb{R}$. Show that $\overline{A + B} \supseteq \overline{A} + \overline{B}$ and $\overline{A \times B} \supseteq \overline{A} \times \overline{B}$. Here $A + B = \{a + b : a \in A, b \in B\}$ and $A \times B = \{a \times b : a \in A, \; b \in B\}$.

9. Show that for any countable subset $\overset{\circ}{A} = \emptyset$.

10. Let A be a closed subset of $\mathbb{R}$. Show that $\overset{\circ}{A} = \emptyset$ iff $\overline{\mathbb{R}\backslash A} = \mathbb{R}$.

11. Let A be any nonempty proper subset of $\mathbb{R}$ and $B = \bar{A} \backslash \overset{\circ}{A}$. Show that B is closed and $\overset{\circ}{B} = \emptyset$.

12. Let O be an open subset of $\mathbb{R}$ and A be an arbitrary subset of $\mathbb{R}$. Show that

    (a) If $O \cap A = \emptyset$, then $O \cap \bar{A} = \emptyset$ too.

    (b) We have $O \cap \bar{A} \subseteq \overline{O \cap A} \subseteq \bar{O} \cap \bar{A}$.

13. Let A be a subset of $\mathbb{R}$. Show that $\overset{\circ}{\bar{A}} = \emptyset$ iff given any interval $]a, b[$ there exists a subinterval $]c, d[$ of $]a, b[$ such that $]c, d[\cap A = \emptyset$.

14. Let $A = \{x \in \mathbb{R} : x^2 > 2\}$ and $B = \{x \in \mathbb{R} : x^2 \geq 2\}$. Show that $\bar{A} = B$ and $\overset{\circ}{B} = A$.

15. Let A be a nonempty bounded subset of $\mathbb{R}$ and $\delta(A) = \delta(\bar{A})$. Is $\delta(A) = \delta(\overset{\circ}{A})$?

16. Let O_1 and O_2 be two open sets with $\overline{O_1} = \mathbb{R}$ and $\overline{O_2} = \mathbb{R}$. Show that $\overline{O_1 \cap O_2} = \mathbb{R}$.

Chapter 2

Minkowski and Hölder Inequalities

For $1 \leq p \leq \infty$ there is a unique $q \in \,]1, \infty[\, \ni \dfrac{1}{p} + \dfrac{1}{q} = 1$.

So, $\boxed{pq = p + q}$ the number p is said to be the "conjugate of q"

If $p = 2 \Longrightarrow q = 2$.

If $p = \sqrt{2} \Longrightarrow q = \dfrac{\sqrt{2}}{\sqrt{2} - 1}$.

If $p = 1$, then we take $q = \infty$.

If $q = 1$, then we take $p = \infty$.

So, $\forall\, p \in \,]1, \infty[\,,\ \exists\, q \in \,]1, \infty[\, \ni \dfrac{1}{p} + \dfrac{1}{q} = 1$

Now, let $a, b \in \mathbb{R}^+$. Then, $(a - b)^2 \geq 0$. So, $a^2 + b^2 \geq 2ab$. Equivalently,

$$\frac{1}{2}a^2 + \frac{1}{2}b^2 \geq ab \qquad \text{Eq 3.1}$$

As $\dfrac{1}{2} + \dfrac{1}{2} = 1$, we can expect that, $\dfrac{1}{p}a^p + \dfrac{1}{q}b^q \geq ab$, $\begin{pmatrix} 1 < p < \infty \\ \dfrac{1}{p} + \dfrac{1}{q} = 1 \end{pmatrix}$

Lemma 2.0.14 *For $a, b \in \mathbb{R}^+$ $1 < p < \infty$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$. The inequality, $\dfrac{a^p}{p} + \dfrac{b^q}{q} \geq ab$ holds.*

Proof 2.0.15 *If $a = 0$ or $b = 0$, there is nothing to prove. So, Suppose $a > 0$ and $b > 0$, then dividing the inequality (Eq 3.1) by b^q, we obtain*

$$\frac{a^p b^{-q}}{p} + \frac{1}{q} \geq ab^{1-q} \qquad \text{Eq 3.2}$$

Put $x = ab^{1-q}$ (so, $x > 0$). Then $x^p = a^p b^{p-qp}$. Hence, Eq 3.2 becomes

$$\frac{x^p}{p} + \frac{1}{q} \geq x \qquad \text{Eq 3.3}$$

Let $f : [0, \infty[\; \to \mathbb{R},\; f(x) = \dfrac{x^p}{p} + \dfrac{1}{q} - x$. *(Eq 3.2) is equivalent to*

$$f(x) \geq 0, \forall x \in [0, \infty[\qquad \text{Eq 3.4}$$

Now, $f'(x) = x^{p-1} - 1 = 0$. *So,* $x = 1$. *Hence, at* $x = 1$, f *has an extremum.*

As $f''(x) = (p-1)x^{p-2}$ *and* $f''(x) = p - 1 > 0$, *we conclude that* f *has an absolute minimum at* $x = 1$.

As $f(1) = \dfrac{1}{p} + \dfrac{1}{q} - 1 = 0$, *we see that* $f(x) \geq 0$, $\forall x \in [0, \infty[$. *So, (Eq 3.4) holds, so (Eq 3.1) holds. For* $x = (x_1, x_2, x_3, \ldots, x_n) \in \mathbb{R}^n$, *we define the p-norm of* x^n *as:*

$\|x\| p = \sup \{|x_1|, |x_2|, |x_3|, \ldots, |x_n|\}$. *We want to prove that* $\begin{cases} \|x + y\|_p \leq \|x\|_p + \|y\|_p & \textit{and} \\ \|x \times y\|_1 \leq \|x\|_p \times \|y\|_q \end{cases}$

$(1 < p < \infty$ *and* $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $(xy = x_1y_1 + \cdots + x_ny_n))$

For 1, $\|x + y\|_1 \leq \|x\|_1 + \|y\|_1$ *is obvious.*

For $p = \infty$, $\|x + y\|_\infty \leq \|x\|_\infty + \|y\|_\infty$.

Theorem 2.0.16 ***(Hölder inequality)*** *For* $1 < p < \infty$ *and* $x, y \in \mathbb{R}^n$,
$\|x \times y\|_1 \leq \|x\|_p \times \|y\|_q$.

Proof 2.0.17 *If* $\|x\|_p = 0$ *or* $\|y\|_p = 0$, *then there is nothing to prove. So, suppose* $\|x\|_p > 0$ *and* $\|y\|_p > 0$.

Let $a_i = \dfrac{x_i}{\|x\|_p}$, $b_i = \dfrac{y_i}{\|y\|_p}$. *Then by the preceding inequality,* $\dfrac{1}{p}\dfrac{x_i^p}{\|x\|_p^p} + \dfrac{1}{q}\dfrac{y_i^p}{\|y\|_p^p} \geq \dfrac{|x_i|}{\|x\|_p}\dfrac{|y_i|}{\|y\|_p}$.

Adding them from $p = 1$ *to* $p = n$, *we get:*

$$\frac{1}{p}\frac{\sum_{p=1}^{n}|x_i|^p}{\|x\|_p^p} + \frac{1}{q}\frac{\sum_{p=1}^{n}|x_i|^p}{\|y\|^p q} \geq \frac{\sum_{p=1}^{n}|x_i|^p|y_i|^p}{\|x\|_p\,||y||_q}$$

$$1 \geq \frac{\sum_{i=1}^{n}|x_i|\,|y_i|}{\|x\|_p\,\|y\|_q}$$

i.e. $||xy||_1 \leq \|x\|_p\,||y||_q$. *For* $p = q = 2$, *the Hölder inequality becomes:*

$$\sum_{i=1}^{n}|x_i|\,|y_i| \leq \sqrt{|x_1|^p + \ldots + |x_p|^p}\sqrt{|y_1|^p + \ldots + |y_p|^p} \qquad \text{(Cauchy Schwartz Inequality)}$$

Theorem 2.0.18 ***(Minkowski Inequality)*** $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

Proof 2.0.19 *For* $p = 1$,

$\|x + y\|_1 = |x_1 + y_1| + \ldots + |x_n + y_n| \leq |x_1| + |y_1| + \ldots + |x_n| + |y_n| \leq ||x||_1 + \|y\|_1$

For $p = \infty$, $\|x + y\|_\infty = \max\{|x_1 + y_1|, \ldots, |x_n + y_n|\} \leq \max\{|x_1| + |y_1|, \ldots, |x_n| + |y_n|\}$
$\leq \max\{|x_1|, \ldots, |x_n|\} + \max\{|y_1|, \ldots, |y_n|\}$

For $1 \leq p \leq \infty$, $\|x_i + y_i\|_p = |x_i + y_i|^{p-1} \times |x_i + y_i| \leq |x_i + y_i|^{p-1} |x_i| + |x_i + y_i|^{p-1} |y_i|$, *so that* $\sum_{i=1}^{n} |x_i + y_i|^p \leq \sum_{i=1}^{n} |x_i + y_i|^{p-1} |x_i| + \sum_{i=1}^{n} |x_i + y_i|^{p-1} |y_i|$

Now, by Hölder inequality,

$$\sum_{i=1}^{n} |x_i + y_i|^{p-1} |x_i| \leq \left(\sum_{i=1}^{n} |x_i + y_i| \, (^{p-1})^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}.$$

$$\sum_{i=1}^{n} |x_i + y_i|^{p-1} |y_i| \leq \left(\sum_{i=1}^{n} |x_i + y_i| \, (^{p-1})^q\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}} = \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{q}} \left(\sum_{i=1}^{n} |y_i|^p\right)^{\frac{1}{p}}.$$

Hence, $\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{1} \leq \left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{q}} [\|x_p\| + \|y_p\|]$

Hence, $\left(\sum_{i=1}^{n} |x_i + y_i|^p\right)^{\frac{1}{p}} \leq [\|x_p\| + \|y_p\|]$ *i.e.* $\|x_p + y_p\| \leq \|x_p\| + \|y_p\|$.

Example 2.0.20 $\int_a^b f(x)g(x)dx \leq \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}} \times \left(\int_a^b |g(x)|^p \, dx\right)^{\frac{1}{p}}$.

Chapter 3

Metric Spaces (Basic Concepts)

1. **Metric Spaces: Definition and Properties of the Real Numbers**
2. **Topology of Metric Spaces (Open Sets, Closed Sets, etc.)**
3. **Basic topological concepts. (Interior, Closure, boundary of a set etc.)**
4. **Accumulation Points and Isolated Points of a Set**
5. **Density and Separability**
6. **Relativization**
7. **Lindölf Theorem**

3.1 Metrics and Metric Spaces

Let X be any set. ($\neq \emptyset$). A mapping $d : X \times X \to [0, \infty[$ is said to be a "distance" or "metric" if

1. $\forall x, y \in X,\ d(x, y) = 0 \Longleftrightarrow x = y$
2. $\forall x, y \in X,\ d(x, y) = d(y, x)$
3. $\forall x, y \in X,\ d(x, y) \leq d(x, z) + d(z, y)$ "triangle inequality"

   Then the pair (X, d) is said to be a metric space.

Example 3.1.1 *1.* $X = \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ *or* $\mathbb{R}$ *where metric is* $d(x, y) = |x - y|$. *Then* (X, d) *is a metric space. This metric* d *is said to be the "usual metric" of* $\mathbb{R}$.

2. Let $X = \mathbb{R}^n$ *and for* $1 \leq p \leq \infty$, $d_p(x, y) = \|x - y\|_p$. *Then by Minkowski Inequality,* d_p *is a metric on* $\mathbb{R}^n$.

For $p = 2$, *the metric* $d_2(x, y) = \sqrt{|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2}$ *is said to be the "Euclidean metric" on* $\mathbb{R}^n$.

3. *Let X be any set and d be defined by* $d(x,y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$ *.Then d is a metric, known as the "discrete metric". Every set has at least the discrete metric. So every set is a metric space.*

4. *Let X be any set and $f: X \to \mathbb{R}$ be a one-to-one function. Put $d_f(x,y) = |f(x) - f(y)|$. Then d_f is a metric on X. e.g. $X = \mathbb{R}$, $f(x) = \arctan x$.*

   Then $d_f(x,y) = |\arctan(x) - \arctan(y)|$ is a metric on $\mathbb{R}$.

5. *Let E be any set and $X = \mathbb{B}(E) = \{f : E \to \mathbb{R} : f \text{ is bounded on } E\}$.*

   Let $d_\infty(f,g) = \sup_{x\in E} |f(x) - g(x)|$. d_∞ is called the "supremum metric".

Definition 3.1.2 *Let (X,d) be a m.s., $x \in X$ and $\varepsilon > 0$ given.*

1. *The set $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$ is said to be an **open ball** centered at x with radius ε.*

2. *The set $B'_\varepsilon(x) = \{y \in X : d(x,y) \leq \varepsilon\}$ is said to be a **closed ball**.*

3. *The set $S_\varepsilon(x) = \{y \in X : d(x,y) = \varepsilon\}$ is said to be a **sphere**.*

Example 3.1.3 *Let $X = \mathbb{R}^2$, $d = d_2$.*
$B_\varepsilon(0) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < \varepsilon^2\}$
$B'_\varepsilon(0) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq \varepsilon^2\}$
$S_\varepsilon(0) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = \varepsilon^2\}$

Example 3.1.4 *Let $X = R^2$, $d = d_1$.*
$B_1(0) = \{(x,y) \in \mathbb{R}^2 : |x| + |y| < 1\}$
$d = d_\infty : B_1(0) = \{(x,y) \in \mathbb{R}^2 : d_\infty((x,y),(0,0)) = \max\{|x|,|y|\} < 1\}$.

3.2 Open Sets and Closed Sets

Definition 3.2.1 *Let (X,d) be a metric space and $A \subseteq X$ a set. We say that A **is open** $\iff$ for any $x \in A$, $\exists \varepsilon > 0$ such that $B_\varepsilon(x) \subseteq A$.*

This is equivalent to say that a set A is open iff A is a union of a family of open balls. For $X = \mathbb{R}$, $d(x,y) = |x-y|$ $B_\varepsilon(x) =]x-\varepsilon, x+\varepsilon[$.

Example 3.2.2 *On X put the discrete metric d. Then,*
$B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\} = \begin{cases} \emptyset & \text{if } \varepsilon \leq 1 \\ \{x\} & \text{if } \varepsilon > 1 \end{cases}$. *Hence, any set A is open for this metric. $\forall x \in A$, $\exists \varepsilon > 0$ (take $\varepsilon < 1$) $B_\varepsilon(x) = \{x\} \subseteq A$.*

Proposition 3.2.3 *In any metric space (X,d), every open ball $B_\varepsilon(x)$ is an open set.*

Proof 3.2.4 *Let* $y \in B_\varepsilon(x)$. *So* $d(x,y) < \varepsilon$. *Let* $0 < \varepsilon' < \varepsilon - d(x,y)$. *Then* $B_{\varepsilon'}(y) \subseteq B_\varepsilon(x)$.
Indeed, for $z \in B_{\varepsilon'}(y)$,
$d(x,z) \leq d(x,y) + d(y,z) < d(x,y) + \varepsilon' = d(x,y) + \varepsilon - d(x,y) = \varepsilon$.
Hence, $z \in B_\varepsilon(x)$, *i.e.* $B_{\varepsilon'}(y) \subseteq B_\varepsilon(x)$. *So* $B_\varepsilon(x)$ *is an open set.*

Definition 3.2.5 *Let* (X,d) *be a metric space. Let* τ_d *be the collection of all open subsets of* X. *This collection* τ_d *is said to be the* ***"topology of*** X ***defined by*** d***"***.

Proposition 3.2.6 *(**Basic Properties of** τ_d) Let* (X,d) *be any metric space. Then:*

1. $\emptyset \in \tau_d$ *and* $X \in \tau_d$.
2. $O_1, O_2 \in \tau_d \Longrightarrow O_1 \cap O_2 \in \tau_d$.
3. $(O_\alpha)_{\alpha \in I}$ *is a family in* $\Upsilon_d \Longrightarrow \cup_{\alpha \in I} O_\alpha \in \tau_d$.

Proof 3.2.7

1. $\emptyset$ *is open by "intuition".*
2. *Let* O_1, O_2 *be open sets. Let* $x \in O_1 \cap O_2$. *Then* $x \in O_1$ *and* $x \in O_2$.

   Since O_1 *is open* $\exists\ \varepsilon_1 > 0 : B_{\varepsilon_1}(x) \subseteq O_1$.

   Since O_2 *is open* $\exists \varepsilon_2 > 0 : B_{\varepsilon_2}(x) \subseteq O_2$.

   Let $\varepsilon = \min\{\varepsilon_1, \varepsilon_2\}$. *Then* $B_\varepsilon(x) \subseteq O_1 \cap O_2$. *So,* $O_1 \cap O_2$ *is open.*
3. *Let* $(O_\alpha)_{\alpha \in I}$ *be any family of open sets. Let* $O = \cup_{\alpha \in I} O_\alpha$. *Let* $x \in O$, *then* $x \in O_\alpha$ *for some* $\alpha \in I$. *As* O_α *is open there is an* $\varepsilon > 0$ *such that* $B_\varepsilon(x) \subseteq O_\alpha \subseteq O$. *Hence,* O *is open.*

Remark: The intersection of infinitely many open sets need not to be open. Indeed, let $X = \mathbb{R}$, $d(x,y) = |x-y|$. Then, $\cap_{n \geq 1} \left] -\frac{1}{n}, \frac{1}{n} \right[= \{0\}$ is closed.

Definition 3.2.8 *Let* (X,d) *be a metric space. A set* $F \subseteq X$ *is said to be* ***closed*** *if* F^C *is open.*

Theorem 3.2.9 *(**Characterization of closed sets**) Let* (X,d) *be a metric space and* $A \subseteq X$ *a set. Then,* A *is closed* $\Longleftrightarrow \forall x \in X \backslash A,\ \exists\, \varepsilon > 0 : B_\varepsilon(x) \cap A = \emptyset$.

Proof 3.2.10 ($\Longrightarrow$) *Suppose that* A *is closed. Let* $x \in X \backslash A$ *be any point. Then since* $O = X \backslash A$ *is open* $\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq O$, *i.e.* $B_\varepsilon(x) \cap A = \emptyset$.
($\Longleftarrow$) *Suppose that* $\forall x \in X \backslash A,\ \exists\, \varepsilon > 0 : B_\varepsilon(x) \cap A = \emptyset$.
This means that $\forall x \in X \backslash A,\ \exists\, \varepsilon > 0 : B_\varepsilon(x) \subseteq X \backslash A$. *So* $X \backslash A$ *is open. Hence,* A *is closed.*

Example 3.2.11 *In any metric space (X,d), every finite set $F=\{a_1,\dots,a_n\}$ is closed. By Theorem 3.2.9 it is enough to show the following:*

$\forall x\notin F,\ \exists\,\varepsilon>0: B_\varepsilon(x)\cap F=\emptyset.$

Let $x\notin F$. So $d(x,a_i)\neq 0$. Let $\varepsilon=\dfrac{1}{2}\min\{d(x,a_i): i=1,\dots,n\}$. Then $\varepsilon>0$. Now $B_\varepsilon(x)\cap F=\emptyset$. Hence F is closed.

Example 3.2.12 *Let $X=\mathbb{R},\ d(x,y)=|x-y|$.*

Then $F=[a,b]\,,\ F=\mathbb{N},\ F=\mathbb{Z},\ F=[a,\infty[$ are closed.

Proposition 3.2.13 ***(Properties of Closed Sets)*** *Let (X,d) be any m.s., $\mathcal{F}$ be the collection of the closed sets in X. Then;*

1. *$\emptyset\in\mathcal{F}$ and $X\in\mathcal{F}$.*
2. *$F_1,F_2\in\mathcal{F}\Longrightarrow F_1\cup F_2\in\mathcal{F}$.*
3. *The intersection of any family of closed sets $(F_\alpha)_{\alpha\in I}$ is closed.*

Proof 3.2.14

1. *True by definition.*
2. *Let F_1,F_2 be two closed sets. Let $x\in X\backslash(F_1\cup F_2)$.*

   $\Longrightarrow \exists\varepsilon_1>0: B_{\varepsilon_1}(x)\cap F_1=\emptyset$ and $\exists\varepsilon_2>0: B_{\varepsilon_2}(x)\cap F_2=\emptyset$. Let $\varepsilon=\min\{\varepsilon_1,\varepsilon_2\}$. Then $B_\varepsilon(x)\cap F_1\cup F_2=\emptyset$. Hence, $F_1\cup F_2$ is closed.

3.2.1 Exercises I

1. Let d be a metric on a set X. Show that

   (a) $\forall\, x, y, z \in X,\ |d(x,y) - d(y,z)| \leq d(x,z)$.

   (b) For x_n, y_n, x, y in X, $|d(x_n, y_n) - d(x,y)| \leq d(x_n, x) + d(y_n, y)$

2. Let (X, d) be a metric space. For $x, y \in X$, define d' by $d'(x,y) = \min\{1, d(x,y)\}$.

   (a) Show that d' is also a metric on X.

   (b) For $0 < \varepsilon < 1$, $B_\varepsilon(x,d) = \{y \in X : d(x,y) < \varepsilon\} = B_\varepsilon(x,d')$ where $B_\varepsilon(x,d') = \{y \in X : d'(x,y) < \varepsilon\}$.

   (c) Deduce that $\tau_d = \tau_{d'}$.

3. Let (X, d) be a metric space. For x, y in X, define d' by $d'(x,y) = \dfrac{d(x,y)}{1 + d(x,y)}$. Show that

   (a) The function $\varphi : [0, \infty[\, \to [0, \infty[$ defined by $\varphi(x) = \dfrac{x}{1+x}$ is increasing.

   (b) For any $x, y \in [0, \infty[$, $\dfrac{x+y}{1+x+y} \leq \dfrac{x}{1+x} + \dfrac{y}{1+y}$.

   (c) d' is a metric on X.

   (d) Fix $x \in X$. $\forall\, \varepsilon > 0,\ \exists\, \varepsilon' > 0 : B_{\varepsilon'}(x, d') \subseteq B_\varepsilon(x, d)$.

   (e) $\forall\, \varepsilon > 0,\ \exists\, \varepsilon' > 0 : B_{\varepsilon'}(x, d) \subseteq B_\varepsilon(x, d')$.

   (f) Deduce that $\tau_d = \tau_{d'}$.

3.3 Basic Topological Concepts

Definition 3.3.1 *(Closure of a Set) Let (X, d) be a m.s. and let $A \subseteq X$ be any set. Let $\mathcal{A} = \{B \subseteq X : B \supseteq A,\ B \text{ is closed}\}$. $\mathcal{A} \neq \emptyset$, since at least $X \in \mathcal{A}$. Let $\overline{A} = \cap_{B \in \mathcal{A}} B$. Then, $\overline{A}$ is said to be the **closure of** A.*

Proposition 3.3.2 *1. $\overline{A}$ is closed.*

2. $\overline{A}$ is the smallest closed set containing A.

Proof 3.3.3 *1. $\overline{A}$ is closed, since intersection of any family of closed sets is closed.*

2. $\overline{A} \supseteq A$ since each $B \in \mathcal{A}$ contains A. So $\overline{A}$ is the smallest closed set containing A.

Theorem 3.3.4 ***(Characterization of the closure)*** *Let (X, d) be any m.s., A any set and $x \in X$ be any point. Then $x \in \overline{A} \iff \forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset$.*

Proof 3.3.5 *Let $x \in \overline{A}$. If we had some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \emptyset$, then we would have $A \subseteq X \backslash B_\varepsilon(x)$. Let $B = X \backslash B_\varepsilon(x)$. Then B is closed and $A \subseteq B$. So $\overline{A} \subseteq B$. This means that $B_\varepsilon(x) \cap \overline{A} = \emptyset$. This is not possible since $x \in B_\varepsilon(x) \cap \overline{A}$. This contradiction shows that $\forall \varepsilon > 0\ B_\varepsilon(x) \cap A \neq \emptyset$.*

Conversely, suppose $\forall \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset$. Let us see that $x \in \overline{A}$. Let $B \supseteq A$ and B closed. We have to show that $x \in B$. If this was not the case, we would have $x \in B^C$. As B^C is open, for some $\varepsilon > 0$, $B_\varepsilon(x) \subseteq B^C$. So $B \cap B_\varepsilon(x) = \emptyset$. In particular $A \cap B_\varepsilon(x) = \emptyset$, which is not possible. So $x \in B$, so $x \in \overline{A}$.

Example 3.3.6 *Let $X = \mathbb{R},\ d(x, y) = |x - y|$. Then $\overline{\mathbb{Q}} = \mathbb{R}$ and $\overline{\mathbb{R} \backslash \mathbb{Q}} = \mathbb{R}$.*

Example 3.3.7 *Let $X = \mathbb{R}^n$, d =Euclidean metric. Let us see that $\overline{\mathbb{Q}^n} = \mathbb{R}^n$.*

Let $x \in \mathbb{R}^n$ and $\varepsilon > 0$. We need to show that $B_\varepsilon(x) \cap \mathbb{Q}^n \neq \emptyset$, i.e. $\exists\, r = (r_1, \ldots, r_n) \in \mathbb{Q}^n$ such that $(x_1 - r_1)^2 + \cdots + (x_n - r_n)^2 < \varepsilon^2$. Let $r_i \in \mathbb{Q}$ be such that $|x_i - r_i| < \dfrac{\varepsilon}{\sqrt{n}}$. Then $(x_1 - r_1)^2 + \cdots + (x_n - r_n)^2 < \varepsilon^2$.

Proposition 3.3.8 ***(Properties of the Closure Operation)*** *Let (X, d) be a metric space and A, B be two subsets of X. Then,*

1. *A is closed $\iff \overline{A} = A$.*
2. *$\overline{\overline{A}} = \overline{A}$.*
3. *$A \subseteq B \Longrightarrow \overline{A} \subseteq \overline{B}$.*
4. *$\overline{A \cup B} = \overline{A} \cup \overline{B}$.*
5. *$\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.*

Proof 3.3.9 *1. Always true from the definition.*

2. *Follows from 1.*

3. *If $A \subseteq B$, then $A \subseteq \overline{B}$, too. Hence $\overline{A} \subseteq \overline{B}$.*

4. *As $A \subseteq A \cup B$, $B \subseteq A \cup B$. By 3$\Longrightarrow \overline{A} \subseteq \overline{A \cup B}, \overline{B} \subseteq \overline{A \cup B} \Longrightarrow \overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.*

   For the reverse inclusion; as $A \subseteq \overline{A}$, $B \subseteq \overline{B} \Longrightarrow A \cup B \subseteq \overline{A} \cup \overline{B}$. Since the union of two closed sets is closed, $\overline{A} \cup \overline{B}$ is closed. Hence, by 3, $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

5. *As $A \cap B \subseteq A$, $A \cap B \subseteq B$. By 3$\Longrightarrow \overline{A \cap B} \subseteq \overline{A}$, $\overline{A \cap B} \subseteq \overline{B} \Longrightarrow \overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.*

Example 3.3.10 *We do not have $\overline{A \cap B} = \overline{A} \cap \overline{B}$.*
Let $X = \mathbb{R}$, and d be usual metric. ($|x - y|$).
Let $A = \mathbb{Q}$, $B = \mathbb{R} \backslash \mathbb{Q}$. Then $A \cap B = \emptyset$. So $\overline{A \cap B} = \emptyset$. On the other hand $\overline{A} = \mathbb{R}$, $\overline{B} = \mathbb{R}$. So $\overline{A} \cap \overline{B} = \mathbb{R}$.

Definition 3.3.11 ***(Interior of a Set)*** *Let (X, d) be any metric space and $A \subseteq X$ be any set. Let $\mathcal{A} = \{B \subseteq X : B \text{ is open, } B \subseteq A\}$. $\mathcal{A}$ is nonempty since at least $\emptyset \in \mathcal{A}$. Let $\overset{\circ}{A} = \cup_{B \in \mathcal{A}} B$. This is clearly the largest open set contained in A. This set $\overset{\circ}{A}$ is said to be* ***"the interior of** A**"**.*

Theorem 3.3.12 ***(Characterization of the interior)*** *Let (X, d) be a m.s. $A \subseteq X$, $x \in X$. Then $x \in \overset{\circ}{A} \Longleftrightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A$.*

Proof 3.3.13 ($\Longrightarrow$) *Let $x \in \overset{\circ}{A}$. As $\overset{\circ}{A}$ is open, by definition, $\exists \varepsilon > 0 : B_\varepsilon(x) \subseteq \overset{\circ}{A}$. So $B_\varepsilon(x) \subseteq A$.*
($\Longleftarrow$) *Let $\varepsilon > 0$ be such that $B_\varepsilon(x) \subseteq A$. Then this $B_\varepsilon(x) \in \mathcal{A}$. So $B_\varepsilon(x) \subseteq \overset{\circ}{A} \subseteq A$.*

Example 3.3.14 *1. Let $X = \mathbb{R}$, $d(x, y) = |x - y|$.*

   $$\overset{\circ}{\mathbb{Q}} = \emptyset,\ \overset{\circ}{\overline{\mathbb{R} \diagdown \mathbb{Q}}} = \emptyset,\ \overset{\circ}{\mathbb{Z}} = \emptyset,\ \overset{\circ}{\mathbb{N}} = \emptyset,\ \overset{\circ}{\overline{[a, b]}} =]a, b[.$$

2. *Let $X = \mathbb{R}^n$, $d = d_2$. $A = \mathbb{R} \times \{0\} \Longrightarrow \overset{\circ}{A} = \emptyset$.*

3. *Let $X = \mathbb{R}^n$, $d = d_2$. $A = \mathbb{R}^{n-1} \times \{0\} \Longrightarrow \overset{\circ}{A} = \emptyset$.*

Proposition 3.3.15 ***(Properties of the interior operation)***

1. *A is open $\Longleftrightarrow \overset{\circ}{A} = A$.*

2. $\overset{\circ}{\overset{\circ}{A}} = \overset{\circ}{A}$.

3. $A \subseteq B \Longrightarrow \overset{\circ}{A} \subseteq \overset{\circ}{B}$

4. $\overset{\circ}{(A \cup B)} \supseteq \overset{\circ}{A} \cup \overset{\circ}{B}$

5. $\overset{\circ}{(A \cap B)} = \overset{\circ}{A} \cap \overset{\circ}{B}$

Proof 3.3.16 *1. Always true from the definition.*

2. *Follows from 1.*

3. *Let* $A \subseteq B$. *Then* $\overset{\circ}{A} \subseteq B$. *Hence* $\overset{\circ}{A} \subseteq \overset{\circ}{B}$ *since* $\overset{\circ}{B}$ *is the largest open set contained in* B.

4. $A \subseteq A \cup B,\ B \subseteq A \cup B \Longrightarrow \overset{\circ}{A} \subseteq \overset{\circ}{(A \cup B)}$
   $\overset{\circ}{B} \subseteq \overset{\circ}{(A \cup B)} \Longrightarrow \overset{\circ}{A} \cup \overset{\circ}{B} \subseteq \overset{\circ}{(A \cup B)}$.

5. *As* $A \cap B \subseteq A,\ A \cap B \subseteq B, \Longrightarrow \overset{\circ}{(A \cap B)} \subseteq \overset{\circ}{A},\ \overset{\circ}{(A \cap B)} \subseteq \overset{\circ}{B} \Longrightarrow \overset{\circ}{(A \cap B)} \subseteq \overset{\circ}{A} \cap \overset{\circ}{B}$.

   Conversely, $\overset{\circ}{A} \cap \overset{\circ}{B} \subseteq A \cap B$. *As* $\overset{\circ}{A} \cap \overset{\circ}{B}$ *is open and contained in* $A \cap B$, $\overset{\circ}{A} \cap \overset{\circ}{B} \subseteq \overset{\circ}{(A \cap B)}$.

Example 3.3.17 *Let* $X = \mathbb{R}$, $d(x, y) = |x - y|$.
$A = \mathbb{Q},\ B = \mathbb{R} \backslash \mathbb{Q} \Longrightarrow \overset{\circ}{A} = \emptyset,\ \overset{\circ}{B} = \emptyset$. *But* $\overset{\circ}{(A \cup B)} = \mathbb{R}$.

Proposition 3.3.18 ***(Interior-closure connection)*** *Let* (X, d) *be a m.s.* $A \subseteq X$. *Then,*

1. $\left(\overline{A}\right)^C = \overset{\circ}{\left(A^C\right)}$.

2. $\left(\overset{\circ}{A}\right)^C = \overline{\left(A^C\right)}$.

Proof 3.3.19 *1. Let* $x \in X$ *be any point. Then,* $x \in \overset{\circ}{\left(A^C\right)} \Longleftrightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A^C \Longleftrightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \cap A = \emptyset \Longleftrightarrow x \notin \overline{A} \Longleftrightarrow x \in \left(\overline{A}\right)^C$.

2. *Follows directly from 1.*

Definition 3.3.20 *Let* (X, d) *be a m.s.* $A \subseteq X$. *The set* $\partial A = \overline{A} \backslash \overset{\circ}{A}$ *is said to be the* ***boundary*** *of* A.

Example 3.3.21 *Let* $X = \mathbb{R}^2$, $d = d_2$. *Let* $A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. *Then,*
$\overline{A} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *and* A *is open. So* $\overset{\circ}{A} = A \Longrightarrow \partial A = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

Example 3.3.22 *Let* $X = \mathbb{R}$, $d =$*usual metric,* $A = \mathbb{Q}$. *Then* $\overline{A} = \mathbb{R}$. $\overset{\circ}{A} = \emptyset$. *So* $\partial A = \mathbb{R}$.

Example 3.3.23 *Let* $X = \mathbb{R}$, $d =$*usual metric,* $A = [a, b[$. *Then* $\overline{A} = [a, b]$, $\overset{\circ}{A} =]a, b[$. *So* $\partial A = \{a, b\}$.

Proposition 3.3.24 ***(Properties of the Boundary)***

1. *For any set* $A \subseteq X$, ∂A *is always a closed set. Since* $\partial A = \overline{A} \setminus \overset{\circ}{A} = \overline{A} \cap \left(\overset{\circ}{A}\right)^c = \overline{A} \cap \overline{(A^c)}$ *is closed.*

2. $\partial A = \partial \left(A^C\right)$. *Clear from 1.*

3. *A is closed iff* $\partial A \subseteq A$.

4. *A is open iff* $\partial A \cap A = \emptyset$.

3.4 Accumulation and Isolated Points of a Set

Definition 3.4.1 *Let* (X, d) *be a m.s. and* $A \subseteq X$ *be a set. We want to classify the points of* $\overline{A}$. *For* $x \in \overline{A}$ *and* $\varepsilon > 0$ *arbitrary,* $B_\varepsilon(x) \cap A \neq \emptyset$. *So, only one of the following may happen:*

1. $\exists \varepsilon > 0 : B_\varepsilon(x) \cap A = \{x\}$. *In this case we say that x is an* ***isolated point*** *of A.*

2. $\forall \varepsilon > 0 : B_\varepsilon(x) \cap A \setminus \{x\} \neq \emptyset$. *In this case we say that x is an* ***accumulation point*** *of A.*

Remark: An isolated point of A is always in the set A but an accumulation point may or may not be in A.

Example 3.4.2 *Let* $X = \mathbb{R}$, $d(x, y) = |x - y|$

1. $A = \mathbb{N}$. *Then* $\overline{A} = \overline{\mathbb{N}} = \mathbb{N}$. *For* $n \in \mathbb{N}$, *if* $0 < \varepsilon < 1$, $]n - \varepsilon, n + \varepsilon[\cap \mathbb{N} = \{n\}$. *So, every point of* $\mathbb{N}$ *is an isolated point.*

2. $A = \mathbb{Q}$. *Then* $\overline{\mathbb{Q}} = \mathbb{R}$. *Let* $x \in \mathbb{R}$ *and* $\varepsilon > 0$ *be arbitrary.*

   Since $]x - \varepsilon, x + \varepsilon[$ *contains infinitely many rational numbers, certainly* $]x - \varepsilon, x + \varepsilon[\cap \mathbb{Q} \setminus \{x\} \neq \emptyset$.

   Hence, every $x \in \mathbb{R}$ *is an accumulation point of* $\mathbb{Q}$.

3. $A = \left\{\frac{1}{n} : n = 1, 2, 3, \ldots\right\}$. *Then* $\overline{A} = A \cup \{0\}$. $\forall \varepsilon > 0$, $]-\varepsilon, \varepsilon[\cap A$ *is an infinite set.*

   So, 0 *is an accumulation point.*

   For $x = \frac{1}{n}$, *for* $0 < \varepsilon < \min\left\{\frac{1}{n} - \frac{1}{n+1}, \frac{1}{n-1} - \frac{1}{n}\right\}$, $\left]\frac{1}{n} - \varepsilon, \frac{1}{n} + \varepsilon\right[\cap A = \left\{\frac{1}{n}\right\}$.

   So, that any $x = \frac{1}{n}$ *is an isolated point.*

Let, for any $A \subseteq X$, $A' = \left\{x \in \overline{A} : x \text{ is an accumulation point}\right\}$. Then clearly $\overline{A} = A \cup A'$. Thus A is closed $\Longleftrightarrow A' \subseteq A$.

Definition 3.4.3

- *If $A = A'$, then we say that A is **perfect**.*
- *If every $x \in A$ is an isolated point, then we say that A is **discrete**.*

Example 3.4.4 *$A = [a, b]$ is a perfect set.*

Example 3.4.5 *$A = \mathbb{N}$, $A = \mathbb{Z}$ is discrete. Any finite set is discrete.*

Example 3.4.6 *Let $A = \left\{\frac{1}{n} + \frac{1}{m} : n = 1, 2, \ldots, m = 1, 2, \ldots\right\}$. Then*
$\overline{A} = A \cup \left\{\frac{1}{n} : n = 1, 2, \ldots\right\} \cup \{0\}$.
$A' = \left\{\frac{1}{n} : n = 1, 2, \ldots\right\} \cup \{0\}$. $A'' = \{0\}$. $A''' = \emptyset$.

Proposition 3.4.7 *Let $x \in \overline{A}$ be a point. Then, x is an accumulation point $\Longleftrightarrow \forall\varepsilon > 0$, $B_\varepsilon(x) \cap A$ is infinite.*

Proof 3.4.8 *$(\Longleftarrow)$ is trivial.*

$(\Longrightarrow)$ For a contradiction suppose, for some $\varepsilon > 0$, $B_\varepsilon(x) \cap A$ is finite, say $B_\varepsilon(x) \cap A = \{x_0, \ldots, x_n\}$. Let $O = B_\varepsilon(x) \backslash F$, where $F = \{x_0, \ldots, x_n\} - \{x\}$. Then O is open and $x \in O$. So for some $\varepsilon' > 0$, $B_{\varepsilon'}(x) \subseteq O$. Hence, $B_{\varepsilon'}(x) \subseteq B_\varepsilon(x)$ and $B_{\varepsilon'}(x) \cap F = \emptyset$ or $B_{\varepsilon'}(x) \cap F = \{x\}$. Hence, $B_{\varepsilon'}(x) \cap A \backslash \{x\} = \emptyset$. This contradicts the definition of accumulation point. So, $\forall\varepsilon > 0$, $B_\varepsilon(x) \cap A$ is infinite.

3.4.1 Exercises II

All the sets below are the subsets of $\mathbb{R}$ and on $\mathbb{R}$ the metric is its usual metric.

1. Show that, for any set A, A' is a closed set.

2. If $A \neq \emptyset$ and bounded and infinite, show that then $A' \neq \emptyset$.

3. If A is uncountable, then show that $A' \neq \emptyset$.

4. If A is open, then show that A has no isolated point.

5. If A is dense $\mathbb{R}$, then show that A has no isolated point.

6. For $A = \left\{ \frac{1}{n} + \frac{1}{m} + \frac{1}{p} : n = 1, 2, 3, \ldots,\ m = 1, 2, 3, \ldots,\ p = 1, 2, 3, \ldots \right\}$, find $\overline{A}$, A', A'', A''' and A''''.

7. Show that for any set A, $\partial\overline{A} \subseteq \partial A$ and $\partial A^\circ \subseteq \partial A$.

8. Let $a \in \overline{A}$. Show that;

   (a) a is an accumulation point of A iff A has a sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n \neq a$ for all n such that $x_n \to a$.

   (b) a is an isolated point of A iff every sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges to a is eventually constant.

9. If $a \in \partial A$, then show that there exist two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converging to a such that $x_n \in A$ for all $n \in \mathbb{N}$ and $y_n \notin A$ for all $n \in \mathbb{N}$.

10. Show that a subset A of $\mathbb{R}$ can have at most countably many isolated points. Deduce that every discrete subset of $\mathbb{R}$ is countable.

3.5 Density and Separability

Definition 3.5.1 *Let (X, d) be a m.s. If $M \subseteq X$ and $\overline{M} = X$, then we say that M is **dense** in X.*

Example 3.5.2 *$\mathbb{Q}$ is dense in $\mathbb{R}$. $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$. $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.*

Theorem 3.5.3 *Let (X, d) be a m.s and $M \subseteq X$ be a set. Then, The followings are equivalent:*

1. $\overline{M} = X$
2. $\forall O \subseteq X$, *where O is open:* $O \cap M \neq \emptyset$.
3. $\forall x \in X,\ \forall \varepsilon > 0 : B_\varepsilon(x) \cap M \neq \emptyset$.

Definition 3.5.4 *A metric space (X, d) is said to be **separable**, if there is a countable set $M \subseteq X$ such that $\overline{M} = X$.*

Example 3.5.5

1. *Let $X = \mathbb{R}$, $d(x, y) = |x - y|$. Then as $\overline{\mathbb{Q}} = \mathbb{R}$ and $\mathbb{Q}$ is countable, $\mathbb{R}$ is separable.*
2. *Let X be any set. On X put the discrete metric $d(x, y) = \begin{cases} 1 \text{ if } x \neq y \\ 0 \text{ if } x = y \end{cases}$. Then in this metric space (X, d) every set A is at the same time closed and open. So, $\forall A \subseteq X$, $\overline{A} = A$. Hence, the only dense set is X itself.*

   Conclusion: *(X, d) is separable $\Longleftrightarrow$ X is countable.*
3. *$(\mathbb{R}^n, d_2)$ is separable since $\overline{\mathbb{Q}^n} = \mathbb{R}^n$*

Theorem 3.5.6 *Let (X, d) be a m.s. Suppose that we have in this m.s. an uncountable family $(O_\alpha)_{\alpha\in I}$ of nonempty, pairwise disjoint, open sets. (i.e., $\forall \alpha \neq \beta$, $O_\alpha \cap O_\beta = \emptyset$). Then such a metric space cannot be separable.*

Proof 3.5.7 *Let $M \subseteq X$ be any set such that $\overline{M} = X$. By Theorem 3.5.3, $M \cap O_\alpha \neq \emptyset$. Let $x_\alpha \in M \cap O_\alpha$ be any point. As $O_\alpha \cap O_\beta = \emptyset$, for $\alpha \neq \beta$, $x_\alpha \neq x_\beta$. Let $N = \{x_\alpha : \alpha \in I\}$. Then N is uncountable and $N \subseteq M$. This means that any dense subset M of X is uncountable. So (X, d) cannot be separable.*

Example 3.5.8 *Let E be an infinite set. $X = \mathbb{B}(E) = \{\varphi : E \to \mathbb{R} : \varphi \text{ is bounded}\}$. For $d(\varphi, \psi) = \sup|\varphi(x) - \psi(x)|$, X becomes a m.s.*

Let us see that the m.s. (X, d) is not separable. As, E is infinite, 2^E is uncountable. $\forall A \in 2^E$, let $\varphi_A = \chi_A$. Then $\varphi_A \in X$. For $A \neq B$, $d(\varphi_A, \varphi_B) = \sup_{x\in E}|\chi_A(x) - \chi_B(x)| = 1$. Let $B_{\frac{1}{2}}(\varphi_A)$ be the open ball in X with radius $\dfrac{1}{2}$ and center at φ_A. Put $O_A = B_{\frac{1}{2}}(\varphi_A)$. Then $O_A \cap O_B = \emptyset$ for $A \neq B$. Hence, $(O_A)_{A\in 2^E}$ is an uncountable family of nonempty, pairwise disjoint, open sets in X. So (X, d) is not separable.

If $E = \mathbb{N}$, then $\mathbb{B}(E) = \{\varphi : \mathbb{N} \to \mathbb{R} : \varphi \text{ is bounded}\}$ = the space of bounded sequences. The set $\mathbb{B}(\mathbb{N})$ is denoted by l^∞.

3.5.1 Exercises III

All the sets below are subsets of $\mathbb{R}$ and the metric on $\mathbb{R}$ is its usual metric.

1. Find an uncountable closed set K such that $\overset{\circ}{K} = \emptyset$.

2. If $K \neq \emptyset$, $F \neq \emptyset$, both are closed and one of them is bounded, show that then, $K + F$ is closed.

3. Show that the sets $F_1 = \mathbb{Z}$ and $F_2 = \sqrt{2}\mathbb{Z}$ are closed but $F_1 + F_2$ is dense in $\mathbb{R}$.

   Hint: First, show that every subgroup G of $(\mathbb{R}, +)$ is either dense or discrete.

4. Let A be any set $(\neq \emptyset)$ and B any open set $(\neq \emptyset)$. Show that the set $A + B$ is open.

5. Let $A \neq \emptyset$ be any set. Show that $\overline{A} = \cap_{k\geq 1} \left(A + \left]-\frac{1}{k}, \frac{1}{k}\right[\right)$.

6. From 4 and 5 deduce that every closed subset A of $\mathbb{R}$ is the intersection of countably many open sets and that every open set O is the union of countably many closed sets.

7. Let $A = \cup_{n\geq 1} \left[n + \frac{1}{n}, n + 1 - \frac{1}{n}\right]$, $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. Show that both A and $\mathbb{N}^*$ are closed, $A \cap \mathbb{N}^* = \emptyset$ but $d(A, \mathbb{N}^*) = \inf_{a\in A p\in\mathbb{N}^*} |a - p| = 0$.

8. Let $I \neq \emptyset$ be an interval, A any set $(\emptyset \neq A \neq \mathbb{R})$. Show that if $I \cap A \neq \emptyset$ and $I \cap A^C \neq \emptyset$, then $I \cap \partial A \neq \emptyset$.

9. Let $I = [a, b]$ and $O_0, O_1, \ldots, O_n, \ldots$ be open sets such that $I \subseteq \cup_{n\in\mathbb{N}} O_n$. Show that there exists $N \in \mathbb{N}$ such that $I \subseteq \overset{N}{\cup}_{n=0} O_n$.

3.6 Relativization

Let $X = \mathbb{R}^2$, $d = d_2$. Consider $\mathbb{R}$ as a subset of $\mathbb{R}^2$. (We identify $\mathbb{R}$ with x-axis, i.e. $\mathbb{R} \equiv \mathbb{R} \times \{0\}$) Observe that for $(x,0)$, $(y,0)$ in $\mathbb{R} \times \{0\}$. $d_2((x,0),(y,0)) = |x-y|$.

So that, the metric induced by d_2 on $\mathbb{R}$ is just $d(x,y) = |x-y|$. Hence, we have two metric spaces $(\mathbb{R}, d)$, $(\mathbb{R}^2, d_2)$ with $\mathbb{R} \subseteq \mathbb{R}^2$, $d_{2|_{\mathbb{R}}} = d$.

Now, observe also that for $(x,0)$ in $\mathbb{R} \times \{0\}$, $B_\varepsilon((x,0)) \cap \mathbb{R} =]x-\varepsilon, x+\varepsilon[\times \{0\}$. Now, let $A \subseteq \mathbb{R}$ be a set. We can consider A as a subset of $\mathbb{R}$ or as a subset of $\mathbb{R}^2$.

In the abstract case, we have a m.s. (X,d) a subset $M \subseteq X$, so that (M,d) is also a m.s. Let $A \subseteq M$. Let $\overline{A}^M$ be the closure of A in M, and $\overline{A}$ be the closure of A in X.

Question: How are these sets related?

- For $x \in X$ and $\varepsilon > 0$, $B_\varepsilon(x) = \{y \in X : d(x,y) < \varepsilon\}$.
- For $x \in M$ and $\varepsilon > 0$, $\tilde{B}_\varepsilon(x) = \{y \in M : d(x,y) < \varepsilon\}$

Then, it is clear that $\tilde{B}_\varepsilon(x) = B_\varepsilon(x) \cap M$ for $x \in M$.

Proposition 3.6.1 *For $A \subseteq M$, $\overline{A}^M = \overline{A} \cap M$.*

Proof 3.6.2 *Let $x \in M$. Then, $x \in \overline{A}^M \iff \forall \varepsilon > 0,\ \tilde{B}_\varepsilon(x) \cap A \neq \emptyset \iff B_\varepsilon(x) \cap M \cap A = B_\varepsilon(x) \cap A \neq \emptyset \iff x \in \overline{A} \cap M$. Hence, if $A \subseteq M$: A is closed in M means that $\overline{A}^M = A$.*

A is closed in X means that $\overline{A} = A$. For $x \in M$, $\tilde{B}_\varepsilon(x)$ is open in M, but it is not open in X, unless M is open in X.

Proposition 3.6.3 *Let (X,d) be a m.s. $M \subseteq X$ and $A \subseteq M$. Then,*

1. *A is closed in $(M,d) \iff$ there exists a closed set $F \subseteq X$ such that $A = F \cap M$.*
2. *A is open in $(M,d) \iff$ there exists an open set $O \subseteq X$ such that $A = O \cap M$.*

Proof 3.6.4
1. *We have seen that $\overline{A}^M = \overline{A} \cap M$. So if A is closed in M, then $\overline{A}^M = A$, so that $A = \overline{A} \cap M$, so we take $F = \overline{A}$.*

   Conversely, if $A = F \cap M$ for some closed set $F \subseteq X$, then let us see that A is closed in M. So, let $x \in M \backslash A$. Then $x \notin F$. As F is closed in X, there is an $\varepsilon > 0 : B_\varepsilon(x) \cap F = \emptyset$. Then $\tilde{B}_\varepsilon(x) = B_\varepsilon(x) \cap M$ is a neighborhood of x in M and $\tilde{B}_\varepsilon(x) \cap A = \emptyset$. Hence $x \notin \overline{A}^M$ and $\overline{A}^M = A$, so A is closed in M.

2. *Apply 1 to $B = M \backslash A$.*

Example 3.6.5 *Let $X = \mathbb{R}$, $d(x,y) = |x-y|$. Let $M = [0,1[$. Then the set $A = \left[0, \frac{1}{2}\right[$ is open in M, but not in X. Indeed, $\left[0, \frac{1}{2}\right[= \left]-1, \frac{1}{2}\right[\cap M$, where $\left]-1, \frac{1}{2}\right[$ is open. The set $A = \left[\frac{1}{2}, 1\right[$ is closed in M. Indeed $A = \left[\frac{1}{2}, 1\right[= \left[\frac{1}{2}, 2\right] \cap M$, where $\left[\frac{1}{2}, 2\right]$ is closed.*

3.6.1 Exercises IV

1. Let $(X, d_1), (Y, d_2)$ be 2 m.s. and $Z = X \times Y$.

   For $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in Z, put $d(z_1, z_2) = d_1(x_1, x_2) + d_2(y_1, y_2)$ and $d'(z_1, z_2) = \max\{d_1(x_1, x_2), d_2(y_1, y_2)\}$

   (a) Show that d and d' are metrics on Z and $d' \leq d$.

   (b) Let $z_n = (x_n, y_n)$ be a sequence in Z. Show that $z_n \to z = (x, y)$ in (Z, d) iff $x_n \to x$ in (X, d_1) and $y_n \to y$ in (Y, d_2).

   (c) Let $A_1 \subseteq X, A_2 \subseteq Y$. Show that in the m.s. (Z, d) we have $\overline{A_1 \times A_2} = \overline{A_1} \times \overline{A_2}$.

   (d) Let $A_1 \subseteq X, A_2 \subseteq Y$. Show that

      i. $A_1 \times A_2$ is closed in the m.s. (Z, d) iff A_1 is closed in (X, d_1) and A_2 is closed in (Y, d_2).

      ii. $A_1 \times A_2$ is open in (Z, d) iff A_1 is open in (X, d_1) and A_2 is open in (Y, d_2).

   (e) Show that (Z, d) is separable iff both spaces (X, d_1) and (Y, d_2) are separable. Deduce that $(\mathbb{R}^n, d_1)$ is separable.

   (f) Show that the m.s. (Z, d) is complete iff the spaces (X, d_1) and (Y, d_2) are complete.

2. Let $(X, d_1), (Y, d_2)$ and $Z = X \times Y$ be as in the question 1 above. Let $A \subseteq Z$ be any set.

   Show that A is closed in (Z, d) iff A is closed in (Z, d'). Deduce that $\tau_d = \tau_{d'}$.

3.7 Lindölf Theorem

Let (X, d) be a metric space.

Lemma 3.7.1 *If (X, d) is separable, then there exists countably many open sets $(B_n)_{n\in\mathbb{N}}$ such that every open set $O \subseteq X$ is a union of some of B_n's.*

Proof 3.7.2 *Let $M = \{a_n : n \in \mathbb{N}\}$ be a dense subset in X. Consider the collection $\{B_r(a_n) : n \in \mathbb{N},\ r > 0,\ r \in \mathbb{Q}\}$. This is a countable set of open sets. Let us see that these sets satisfy the conclusion of the lemma. Let $O \subseteq X$ be any open set. As $\overline{M} = X$, $O \cap M \neq \emptyset$. Let $F = \{a_n \in M : a_n \in O\}$ i.e., $F = O \cap M$. Since $a_n \in O$ and O is open, there is a rational number $r_n > 0 : B_{r_n}(a_n) \subseteq O$. Let us see that $\cup_{a_n \in F} B_{r_n}(a_n) = 0$.*

Let $x \in O$. Since O is open, for some $\varepsilon > 0$, $B_\varepsilon(x) \subseteq O$. Since $\overline{M} = X$, $B_{\frac{\varepsilon}{3}}(x) \cap M \neq \emptyset$. Hence, for some $a_n \in F$, $a_n \in B_{\frac{\varepsilon}{3}}(x)$. Let $r_n > 0$ rational such that $\dfrac{\varepsilon}{3} < r_n < \dfrac{\varepsilon}{2}$. Then $x \in B_{r_n}(a_n)$. So, $x \in \cup_{a_n \in F} B_{r_n}(a_n)$. Hence $O = \cup_{a_n \in F} B_{r_n}(a_n)$.

Example 3.7.3 *$X = \mathbb{R}$, $d(x, y) = |x - y|$. The collection $\{]a, b[: a, b \in \mathbb{Q}\}$ is countable. Moreover any open set O of $\mathbb{R}$ is a union of some of these intervals.*

Corollary 3.7.4 *If (X, d) is separable, then $\forall M \subseteq X$ the space (M, d) is separable.*

Proof 3.7.5 *Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of open sets as in Lemma 3.7.1.*

Let $F = \{n \in \mathbb{N} : B_n \cap M \neq \emptyset\}$. Let $x_n \in B_n \cap M$ be any point for each $n \in F$. Let us see that $A = \{x_n : n \in F\}$ is dense in M. Let $\tilde{O} = O \cap M$ be any nonempty open set of M. As O is a union of B_n's, then $\tilde{O}$ contains at least one point from A. So A is dense in M.

Theorem 3.7.6 ***(Lindölf)*** *Let (X, d) be a separable m.s. and $(O_\alpha)_{\alpha\in I}$ any family of open sets. Then I has a countable subset $\bar{I}$ such that $\cup_{\alpha\in I} O_\alpha = \cup_{\alpha\in\bar{I}} O_\alpha$.*

Proof 3.7.7 *Let $(B_n)_{n\in\mathbb{N}}$ be a sequence of open sets as in Lemma 3.7.1. So any O_α is a union of some of these B_n's. Now, $\forall n \in N$ let $I_n = \{\alpha \in I : O_\alpha \supseteq B_n\}$. Some of the I_n's are empty but not all. Let $F = \{n \in \mathbb{N} : I_n \neq \emptyset\}$. For each $n \in F$, let $\alpha \in I$ be any point (Hence $O_{\alpha_n} \supseteq B_n$). Let $\bar{I} = \{\alpha_n : n \in F\}$. Let us see that $\cup_{\alpha\in I} O_\alpha = \cup_{\alpha_n\in\bar{I}} O_{\alpha_n}$. The inclusion $\supseteq$ is trivial. To prove the other inclusion, let $x \in \cup_{\alpha\in I} O_\alpha$. Then $x \in O_\alpha$ for some $\alpha \in I$. This O_α is a union of some B_n's. So our x is in one of these B_n's. Then $x \in B_n \subseteq O_\alpha$. Then, as $B_n \subseteq O_{\alpha_n}$, $x \in O_{a_n}$. So, $\cup_{\alpha\in I} O_\alpha = \cup_{\alpha_n\in\bar{I}} O_{\alpha_n}$.*

Chapter 4

Convergence in a Metric Space

1. **Limit and Cluster Points of a Sequence**
2. **Cauchy Sequences and Completeness**
3. lim sup **and** lim inf **Again**

4.1 Limit and Cluster Points of a Sequence

Let (X, d) be a metric space and $(x_n)_{n\in\mathbb{N}}$ be a sequence in X. Then we say that;

x_n converges to $a \in X$ if we have: $\forall \varepsilon > 0,\ \exists N \in N : \forall n \geq N,\ d(x_n, a) < \varepsilon$.

In this case we write $\lim_{n\to\infty} x_n = a$. Then $x_n \to a \iff \forall \varepsilon > 0,\ x_n \in B_\varepsilon(a)$ for all but finitely many n.

Lemma 4.1.1 *Limit is unique. If $x_n \to a$ and $x_n = b$, then $a = b$.*

Proof 4.1.2 *Suppose $a \neq b$. Let $\varepsilon = \frac{d(a,b)}{3}$. Then $B_\varepsilon(a) \cap B_\varepsilon(b) = \emptyset$. As, $x_n \to a$, $x_n \in B_\varepsilon(a)$ for all but finitely many $n \in \mathbb{N}$. So $B_\varepsilon(b)$ can contain at most x_n for finitely many n. So $x_n \nrightarrow b$. Hence $a = b$.*

Example 4.1.3 *Let $X = \mathbb{R}^m$, let $d = d_p,\ 1 < p < \infty$. Then any sequence x_n in X is of the form $x_n = (a_{n,1}, a_{n,2}, \ldots a_{n,m})$. What does "$x_n \to a$ in $\mathbb{R}^m$" mean?*

For $a = (a_1, \ldots, a_m)$, $d_p(x_n, a) = (|a_{n,1} - a_1|^p + \ldots + |a_{n,m} - a_m|^p)^{\frac{1}{p}}$. Then,

$x_n \to a \Leftrightarrow d_p(x_n, a) \to 0 \Leftrightarrow a_{n,i} \to a_i,\ \forall\ i = 1, 2, \ldots, m$. Thus $x_n \to a$ in $\mathbb{R}^m$ iff "x_n converges to a coordinate-wise."

For $p = 1$, $d_1(x_n \to a) = (a_{n,1} - a_n) + \ldots + |a_{n,m} - a_m| \to 0 \Leftrightarrow a_{n,i} \to a_i$ in $\mathbb{R}$.

For $p = \infty$, $d_\infty(x_n, a) = \max\{|a_{n,i} - a_i| : 1 \leq i \leq m\} \to 0 \Leftrightarrow a_{n,i} \to a_i$ in $\mathbb{R}$.

Example 4.1.4 *Let E be any set. $X = \mathbb{B}(E) = \{f : E \to \mathbb{R} : f$ is bounded$\}$. On X we put the metric $d(f, g) = \sup_{x\in E}|f(x) - g(x)|$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in X.*

What does "$f_n \to f$" mean?

$$f_n \to f \Leftrightarrow \sup_{x\in E}|f_n(x) - f(x)| \to 0 \Leftrightarrow \forall\ \varepsilon > 0, \exists\ N \in \mathbb{N}, \forall\ n \geq N,\ \forall x \in E,\ |f_n(x) - f(x)| < \varepsilon$$

Uniform Convergence

Theorem 4.1.5 *Let (X, d) be a m.s. $A \subseteq X$ be a set and $x \in X$ be a point. Then $x \in \bar{A}$ iff there is a sequence a_n in $A : a_n \to x$. So $\bar{A} = \{x \in X : x$ is the limit of some sequence in $A\}$*

Proof 4.1.6 *We have seen that $x \in \bar{A} \Leftrightarrow \forall\ \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset$.*

($\Rightarrow$) Suppose $x \in \bar{A}$. Hence $\forall\ \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \emptyset$.

Let $\varepsilon = 1$. Then take any point in $B_1(x) \cap A$, call it a_1.

Let $\varepsilon = \frac{1}{2}$. Then take any point in $B_{\frac{1}{2}}(x) \cap A$, call it a_2.

Let $\varepsilon = \frac{1}{3}$. Then take any point in $B_{\frac{1}{3}}(x) \cap A$, call it a_3

$\vdots$

Let $\varepsilon = \frac{1}{n}$. Then take any point in $B_{\frac{1}{n}}(x) \cap A$, call it a_n.

In this way we construct a sequence a_n in A such that $a_n \in B_{\frac{1}{n}}(x) \cap A$, i.e. $d(a_n, x) \leq \frac{1}{n}$. So, $d(a_n, x) \to 0$, i.e. $a_n \to x$.

($\Leftarrow$) Conversely suppose that for some sequence a_n in A, we have that a_n converges to x. Then $\forall\, \varepsilon > 0$, $a_n \in B_\varepsilon(x)$ for all but finitely many $n \in \mathbb{N}$. Hence, $B_\varepsilon(x) \cap A \neq \emptyset$. Since $a_n \in A$, $\forall\ n \geq 0$.

Corollary 4.1.7 *Let (X, d) be m.s. $A \subseteq X$. Then,*

A is closed $\Leftrightarrow \forall$ convergent sequence a_n in A, $\lim_{n\to\infty} a_n \in A$.

Proof 4.1.8 *A is closed $\Leftrightarrow \bar{A} = A$. Then apply the Theorem 4.1.5.*

Example 4.1.9 *Let $X = \mathbb{R}^2$ with the Euclidean metric. $d = d_2$. Let F be the graph of the parabola $y = \frac{1}{x}$ for $x > 0$. Is F closed in $\mathbb{R}^2$? $F = \{(x, \frac{1}{x}) : x > 0\}$*

Let $b_n = (a_n, \frac{1}{a_n})$ that converges in $\mathbb{R}^2$ to some $(x, y) \in \mathbb{R}^2$. Then, by the result of the preceding theorem, $a_n \to x$ and $\frac{1}{a_n} \to y$. As $\frac{1}{a_n} \to y$, x cannot be zero. So, $y = \frac{1}{x}$. Hence $(x, y) = (x, \frac{1}{x}) \in F$. So F is closed.

4.2 Cluster Points of a Sequence

Definition 4.2.1 *Let (X, d) be a m.s., $(x_n)_{n\in\mathbb{N}}$ a sequence in X. We say that a point $a \in X$ is a **cluster point** of the sequence iff for any $\varepsilon > 0$, $x_n \in B_\varepsilon(x)$ for infinitely many $n \in \mathbb{N}$.*

*Thus a is a **cluster point of** x_n if $\forall \varepsilon > 0$, $\forall\, k \in \mathbb{N}\, \exists n > k : x_{n_k} \in B_\varepsilon(x)$.*

Proposition 4.2.2 *If $x_n \to x$. Then x is the only cluster point of $(x_n)_{n\in\mathbb{N}}$.*

Remark: The converse of this proposition is false. Indeed, let $x_n = 1, \frac{1}{2}, 3, \frac{1}{4}, 5, \ldots$. Then 0 is the only cluster point but $x_n \nrightarrow 0$.

Proposition 4.2.3 *Let x_n be a sequence in a m.s. (X,d) and $a \in X$. Then a is the only cluster point of x_n iff x_n has a subsequence that converges to a.*

Proof 4.2.4 *Suppose a is a cluster point of x_n. So $\forall\, \varepsilon > 0,\ x_n \in B_\varepsilon(a)$ for infinitely many n.*

Let $\varepsilon = \frac{1}{2^0}$. Let n_0 be any integer such that $x_{n_0} \in B_{\frac{1}{2^0}}(a)$.

Let $\varepsilon = \frac{1}{2}$. Let $n_1 > n_0$ be any integer such that $x_{n_1} \in B_{\frac{1}{2^1}}(a)$.

$\vdots$

Let $\varepsilon = \frac{1}{2^k}$. Let $n_k > n_{k-1}$ be any integer such that $x_{n_k} \in B_{\frac{1}{2^k}}(a)$.

In this way we construct a subsequence x_{n_k} of x_n such that $d(x_{n_k}, a) < \frac{1}{2^k} \to 0$ i.e. $x_{n_k} \to a$ as $k \to \infty$.

Conversely, suppose that x_n has a subsequence x_{n_k} that converges to a. Then $\forall\, \varepsilon > 0,\ x_{n_k} \in B_\varepsilon(a)$ for all but finitely many k or equivalently $\forall\, \varepsilon > 0,\ x_n \in B_\varepsilon(a)$ for infinitely many n. Hence a is a cluster point.

4.3 The set of the cluster points of a sequence x_n

Let x_n be a sequence in a m.s. (X,d). Put

$F_0 = \{x_0, x_1, \ldots\}$

$F_1 = \{x_1, x_2, \ldots\}$

$\vdots$

$F_n = \{x_n, x_{n+1}, \ldots\}$

$\vdots$

and let $F = \cap_{n\in\mathbb{N}}(\overline{F_n})$.

Example 4.3.1 *In $\mathbb{R}$. $x_n = (-1)^n$, then $F_n = \{-1, 1\}$. So $\overline{F_n} = \{-1, 1\}$.*

Hence $F = \cap_{n\in\mathbb{N}}(\overline{F_n}) = \{-1, 1\}$

Let $x_n = 1, \frac{1}{2}, 3, \frac{1}{4}, 5, \ldots$ Then $\overline{F_n} = F_n \cup \{0\}$. Hence $F = \cap_{n\in\mathbb{N}}(\overline{F_n}) = \{0\}$.

Let $x_n = n$. Then $F_n = \{n, n+1, \ldots\}$. So $\overline{F_n} = F_n$, $F = \cap_{n\in\mathbb{N}}(\overline{F_n}) = \emptyset$.

Let x_n be an enumeration of the rational numbers in $[0,1]$, then $F = \cap_{n\in\mathbb{N}}(\overline{F_n}) = [0.1]$

Theorem 4.3.2 *Let x_n be a sequence in any m.s. (X,d) and $F = \cap_{n\in\mathbb{N}}(\overline{F_n})$ as above. Then $a \in X$ is a cluster point of x_n, iff $a \in F$.*

Proof 4.3.3 *Let $a \in F$ be a point. So, $a \in \overline{F_n}\ \forall n \in \mathbb{N}$. Hence $\forall \varepsilon > 0$, $\forall n \in \mathbb{N}$: $B_\varepsilon(a) \cap F_n \neq \emptyset$, i.e. $\forall \varepsilon > 0$, $x_n \in B_\varepsilon(a)$ for infinitely many n.*

Suppose a is a cluster point of x_n. Then $\forall \varepsilon > 0$, $x_n \in B_\varepsilon(a)$ for infinitely many n. Hence $B_\varepsilon(a) \cap F_n \neq \emptyset \forall n \in \mathbb{N}$. So $a \in \overline{F_n}\ \forall n \in \mathbb{N}$. Hence $a \in F$.

Question: When we have $F \neq \emptyset$?

Example 4.3.4 *Let $X = \mathbb{B}(E)$ (E any infinite set), $\mathbb{B}(E)$ =space of bounded functions. $d(x, y) = \sup_{x\in E}|x(x), y(x)|$*

Let $\varphi_0, \varphi_1, \ldots, \varphi_n, \ldots$ be distinct points of E and $\varphi_n = \chi_{\{\varphi_n\}}$, where $\chi_A = \left\{ \begin{array}{l} 1\ x \in A \\ 0\ x \notin A \end{array} \right.$. Consider the sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in X. (Observe that $|\varphi_n(x)| \leq 1$, $\forall\, x \in E$).

Does this sequence have a cluster point in X?

Now, observe that, for $n \neq m$, $d(\varphi_n, \varphi_m) = \sup_{x\in E}|\chi_{\{\varphi_n\}} - \chi_{\{\varphi_m\}}| = 1$. This implies φ_n has no cluster points. Hence, for this sequence $F = \emptyset$. On the other hand, in $(\mathbb{R}, d)$ Bolzano-Weierstrass says that for any bounded sequence x_n, the set $F \neq \emptyset$. Hence, in $(\mathbb{R}, d)$ if x_n is a bounded sequence, then F is a nonempty, closed, bounded set.

Hence $\alpha = \inf F$ and $\beta = \sup F$ exist and $\alpha, \beta \in F$.

Thus $\alpha = \liminf x_n$, $\beta = \limsup x_n$

4.4 Bolzano-Weierstrass in $\mathbb{R}^m$

A sequence x_n in $\mathbb{R}^m$, $x_n = (a_{n,1}, a_{n,2}, \ldots, a_{n,m})$, is bounded iff

$\|x\|_n = \sqrt{(a_{n,1})^1 + \ldots + (a_{n,m})^2} \leq M\ \ \forall\, n \in \mathbb{N}$ (for some $M > 0$).

So iff each $(a_{n,i})_{n\in\mathbb{N}}$ is bounded. ($i = 1, 2, ..., m$).

A subsequence of x_n is of the form: $x_{n,k} = (a_{n_k,1}, a_{n_k,2}, \ldots, a_{n_k,m})$

Theorem 4.4.1 *Every bounded sequence x_n in $\mathbb{R}^m$ has at least one cluster point.*

Proof 4.4.2 *$(a_{n,1})$ is bounded in $\mathbb{R}$ so by Bolzano-Weierstrass in $\mathbb{R}$. $(a_{n,1})$ has a convergent subsequence $(a_{n^1_k,1})$ that converges to some $a_1 \in \mathbb{R}$. Then, $(a_{n^1_k,2})$ is also bounded in $\mathbb{R}$. By Bolzano-Weierstrass in $\mathbb{R}$. $(a_{n^1_k,2})$ has a convergent subsequence $(a_{n^2_k,2})$ that converges to some $a_2 \in \mathbb{R}$. Then $(a_{n^{m-1}_k,m})$ is bounded. By Bolzano-Weierstrass, it has a convergent subsequence $(a_{n^m_k,m})$ that converges to $a_m \in \mathbb{R}$.*

Let $x_{n_k} = (a_{n^m_k,1}, a_{n^m_k,2}, \ldots, a_{n^m_k,m})$. Then (x_{n_k}) is a subsequence of (x_n) and $x_{n_k} \to (a_1, \ldots, a_m)$.

4.5 Cauchy Sequences and Completeness

4.5.1 Complete Metric Spaces

Let (X, d) be a m.s. Let $(x_n)_{n\in\mathbb{N}}$ be sequence in X. If $x_n \to a$, then we have: $\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}, \forall\ n \geq N\ d(x_n, a) < \frac{\varepsilon}{2}$. So,

$$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}\ \forall\ n \geq N\ \forall\ m \geq N\ d(x_n, x_m) < \varepsilon. \qquad \text{Eq 5.1}$$

Hence any convergent sequence satisfies equation Eq 5.1

Definition 4.5.1 *Any sequence x_n satisfying the condition*

$$\forall\ \varepsilon > 0\ \exists\ N \in \mathbb{N}\ \forall\ n \geq N\ \forall\ m \geq N\ d(x_n, x_m) < \varepsilon \qquad \text{Eq 5.2}$$

*is said to be **"Cauchy."** Then every convergent sequence is Cauchy. The converse is false.*

Example 4.5.2 *Let $X = \mathbb{R}$, $d(x, y) = |Arc\tan x - Arc\tan y|$. Now, let $x_n = n^2$. Then $d(x_n, x_m) = |Arc\tan n^2 - Arc\tan m^2|$. So that $\lim_{n\to\infty, m\to\infty} d(x_n, x_m) \to 0$.*

So, $\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}, \forall\ n \geq N,\ \forall\ m \geq N,\ d(x_n, x_m) < \varepsilon$. Hence x_n is Cauchy in this m.s. $(\mathbb{R}, d)$.

But, there is no $a \in \mathbb{R}$ such that $d(x_n, a) = |Arc\tan n^2 - Arc\tan a| \to 0$.

Example 4.5.3 *Let $X = \mathbb{R}^{(\mathbb{N})} = \{\varphi : \mathbb{N} \to \mathbb{R} : \varphi(n) \neq 0$ for all but finitely many $n\}$, $d(x, y) = \sup_{n\in\mathbb{N}} |\varphi(n) - \psi(n)|$.*

Let $\varphi_n = (1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, 0, \ldots, 0, \ldots)$. Then,

$\varphi_{n+p} - \varphi_n = (0, 0, \ldots, 0, \frac{1}{n+1}, \ldots, \frac{1}{n+p}, 0, \ldots, 0, \ldots)$ *so that,*

$d(\varphi_n, \varphi_{n+p}) = \sup_{k\in\mathbb{N}} |\varphi_n(k) - \varphi_{n+p}(k)| = \frac{1}{n+1} \to 0$ *as* $n \to \infty\ \forall\ p \in \mathbb{N}$.

So, φ_n is Cauchy. But there is no $\varphi = (a_1, a_2, \ldots, a_n, 0, \ldots, 0, \ldots) \in X$ for which $d(\varphi_n, \varphi) \to 0$.

Example 4.5.4 $X = \mathbb{Q}$, $(x, y) = |x - y|$, $x_n = 1 + \frac{1}{1!} + \cdots + \frac{1}{n!}$.

Then x_n is Cauchy in $\mathbb{Q}$, but does not converge in $\mathbb{Q}$.

Definition 4.5.5 *A m.s. (X, d) is said to be **complete** if every Cauchy sequence x_n in X converges to some $x \in X$.*

Example 4.5.6 $(\mathbb{R}, d)$ *$d(x, y) = |x - y|$ is complete.*

1. *$\forall\ m \geq 1$, $(\mathbb{R}, d)$ $d = d_2$ =Euclidean metric is a complete m.s. Indeed, $x_n = (x_{n,1}, x_{n,2}, \ldots, x_{n,m})$. x_n is Cauchy iff each component sequence $(x_{n,i})$ is Cauchy in $(\mathbb{R}, d)$. So, converges to $x_i \in \mathbb{R}$, then $x_n \to (x_1, x_2, \ldots, x_m)$*

2. *Let E be any set and $X = \mathbb{B}(E) = \{Y : E \to \mathbb{R}$: any bounded function$\}$. For $\phi, \psi \in X$, let $d(\phi, \psi) = \sup_{x\in E}|\phi(x), \psi(x)|$*

Theorem 4.5.7 *$(\mathbb{B}(E), d)$ is complete.*

Proof 4.5.8 *Let ϕ_n be a Cauchy sequence in $\mathbb{B}(E)$. So we have:*

$$\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}\ \forall\, n \geq N, \forall\, m \geq N,\ d(\phi_n(x) - \phi_m(x)) < \varepsilon \qquad \text{(Eq 5.3)}$$

In particular, for each $x \in E$, the sequence $\phi_n(x)$ is Cauchy $\mathbb{R}$. So, since $\mathbb{R}$ is complete, $(\phi_n(x))_{n\in\mathbb{N}}$ converges to some $\alpha_x \in \mathbb{R}$. Let $\phi : E \to \mathbb{R}$ be defined by $\phi(x) = \alpha_x$.

Let us see that:

1. *ϕ is bounded, i.e. $\phi \in \mathbb{B}(E)$*

2. *$d(\phi_n, \phi) \to 0$.*

In Eq 5.3, let $m = N$ so that $\sup_{x\in E}|\phi_n(x) - \phi_N(x)| \leq \varepsilon$.
Then $\sup_{x\in E}|\phi_n(x)| \leq \varepsilon + \sup_{x\in E}|\phi_N(x)|,\ \forall\, n \geq N$.
i.e., $\forall\ x \in E,\ |\phi_n(x)| \leq \varepsilon + \sup|\phi_N(x)|,\ \forall\ n \geq N$.

Letting $n \to \infty$, we get that: $\forall\, x \in E\ |\phi(x)| \leq \varepsilon + \sup|\phi_N(x)|$. Hence ϕ is bounded. So $\phi \in \mathbb{B}(E)$.

Now, let us see that $d(\phi_n, \phi) \to 0$. In Eq 5.3, let $x \in E$ be any element so that:

*$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N},\ \forall\ n \geq N, \forall\ m \geq N,\ d(\phi_n(x) - \phi_m(x)) < \varepsilon$ (Observe that N does **not** depend on X)*

Now, let $m \to \infty$, we get that: $\forall\varepsilon > 0,\ \exists\ N \in \mathbb{N}, \forall\ n \geq N,\ \forall\ x \in E\ |\phi_n(x) - \phi(x)| < \varepsilon$
$\implies \forall\varepsilon > 0,\ \exists\ N \in \mathbb{N},\ \forall\ n \geq N,\ \forall\ x \in E,\ \sup|\phi_n(x) - \phi(x)| < \varepsilon$.
Hence, $d(\phi_n, \phi) \to 0$. So, $(\mathbb{B}(E), d)$ is complete.

4.5.2 Cluster Points of a Cauchy Sequence

Theorem 4.5.9 *Let (X, d) be a m.s. and $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then either x_n has no cluster points, or it has only one cluster point. In this later case, it converges to this cluster point.*

Proof 4.5.10 *Suppose x_n has a cluster point $a \in X$. Let us see that $x_n \to a$. Let us write what we have:*

1. *$\forall\ \varepsilon > 0,\ \exists\ N \in \mathbb{N}, \forall n \geq N, \forall m \geq N,\ d(x_n, x_m) < \dfrac{\varepsilon}{2}$*

2. *$\forall\ \varepsilon > 0,\ \ x_n \in B_{\frac{\varepsilon}{2}}(a)$ for infinitely many $n \in \mathbb{N}$*

Let $m \geq N$ *be such that* $x_m \in B_{\frac{\varepsilon}{2}}(a)$. *Then;*

$$\forall\ n \geq N \quad d(x_n, a) \leq \underbrace{d(x_n, x_m)}_{\substack{< \frac{\varepsilon}{2} \\ (1)}} + \underbrace{d(x_m, a)}_{\substack{< \frac{\varepsilon}{2} \\ (2)}} < \varepsilon$$

Hence $x_n \to a$.

Chapter 5

Compactness

5.1 Definition and Characterization of Compact Sets

5.1.1 Introduction

$E = [0, 2\pi]$ and $X = \mathbb{B}(E) = \{f : E \to \mathbb{R} : f \text{ is bounded}\}$.

On X we put the metric $d(\Phi, \Psi) = \sup_{x\in E}|\Phi(x) - \Psi(x)|$. Let $\Phi_n(x) = \cos(nx)$. Then $\forall\ n \in N\ \forall\ x \in [o, 2\pi]\ \ |\cos nx| \leq 1$.

Question: Does Φ_n have a subsequence Φ_{n_k} that converges in $(\mathbb{B}(E), d)$?

No.

Now, let $e_1, e_2, \ldots, e_n, \ldots$ be arbitrary points in $[0, 2\pi]$ but $\Psi_n = \chi_{e_n}$. Then $\Phi_n \in \mathbb{B}(E)$ $d(\Psi_n, \Psi_m) = 1$, $\forall\ n \neq m$. Hence **no** sequence of Ψ_n is Cauchy. Hence Ψ_n has no convergent subsequence. Observe that $\forall\ n \in N$, $\forall\ x \in E$ $|\Psi_n(x)| \leq 1$. Hence the analog of the Bolzano-Weierstrass Theorem is **not true** in $(\mathbb{B}(E), d)$.

Question: Characterize the subsets K of $\mathbb{B}(E)$, for which Bolzano-Weierstrass holds.(i.e. every sequence Φ_n in K has a convergent subsequence.)

Definition 5.1.1 *Let (X, d) be a m.s. $K \subseteq X$ and $(O_\alpha)_{\alpha\in I}$ be a family of open sets. If $K \subseteq \cup_{\alpha\in I} O_\alpha$ then we say that $(O_\alpha)_{\alpha\in I}$ is any* ***open covering*** *of K.*

(i.e. $K \subseteq (O_\alpha)_{\alpha\in I} \Rightarrow K \subseteq O_{\alpha_1} \cup O_{\alpha_2} \cup \ldots \cup O_{\alpha_n}$.)

Definition 5.1.2 *A subset K of X is said to be* ***compact*** *if whenever $(O_\alpha)_{\alpha\in I}$ is an open covering of K, then there exists finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in I$ such that $K \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$. (i.e. $K \subseteq \cup_{\alpha\in I} O_\alpha \Longrightarrow K \subseteq O_{\alpha_1} \cup O_{\alpha_2} \cup \ldots \cup O_{\alpha_n}$)*

Example 5.1.3 *Let* $X = \mathbb{R}$, $d(x,y) = |x-y|$. *Let* $K = \mathbb{N}$, $O_n = \left]n - \frac{1}{2}, n + \frac{1}{2}\right[$. *Then* $\mathbb{N} \subseteq \cup_{n\in N} O_n$ *but whatever* $n_1, \ldots, n_k$ $\mathbb{N} \nsubseteq O_{n_1} \cup \ldots \cup O_{n_k}$. *So* $\mathbb{N}$ *is not compact.*

Example 5.1.4 *Let* $K = \left]0,1\right]$ *and* $O_n = \left]\frac{1}{n}, 1 + \frac{1}{n}\right[$. *Then* $K \subseteq \cup_{n\geq 1} O_n$ *but whatever* $n_1, \ldots, n_k$, $K \nsubseteq O_{n_1} \cup \ldots \cup O_{n_k}$. *so* K *is not compact.*

$$K \text{ is compact} \Leftrightarrow \begin{cases} 1) & K \text{ is closed} \\ & \text{and} \\ 2) & \text{Any sequence } (x_n) \text{ in } K \text{ has a convergent subsequence} \end{cases}$$

5.1.2 Properties of Compact Sets:

Proposition 5.1.5 *Let* (X,d) *be a m.s. Any compact set is closed in* X.

Proof 5.1.6 *Let* K *be compact. Take a point* $x \in X\backslash K$. *We want to show that for some* $\varepsilon > 0$, $B_\varepsilon(x) \cap K = \emptyset$. *Since* $x \notin K$, *for every* $y \in K$, $d(x,y) > 0$. *Let* $\varepsilon_y = \frac{d(x,y)}{2}$. *Then* $K \subseteq \cup_{y\in K} B_{\varepsilon_y}(y)$. *As* K *is compact and* $B_{\varepsilon_y}(y)$ *is open, there exist* $y_1, \ldots, y_n$ *such that* $K \subseteq B_{\varepsilon_{y_1}}(y_1) \cup \ldots \cup B_{\varepsilon_{y_n}}(y_n)$. *Let* $0 < \varepsilon < \frac{1}{2}\min\{\varepsilon_{y_1}, \ldots, \varepsilon_{y_n}\}$. *Then* $B_\varepsilon(x) \cap K = \emptyset$, *since* $B_\varepsilon(x) \cap \left[B_{\varepsilon_{y_1}}(y_1) \cup \ldots \cup B_{\varepsilon_{y_n}}(y_n)\right] = \emptyset$. *Hence* K *is closed in* X.

Theorem 5.1.7 1. *The union of two compact sets* K_1, K_2 *of* X *is compact.*

2. *The intersection of a compact set* K *with a closed set* F *is compact.*

Proof 5.1.8 1. *Let* $K = K_1 \cup K_2$. *Let for some open family* $(O_\alpha)_{\alpha\in I}$, $K \subseteq \cup_{\alpha\in I} O_\alpha$. *Then* $K_1 \subseteq \cup_{\alpha\in I} O_\alpha$ *and* $K_2 \subseteq \cup_{\alpha\in I} O_\alpha$. *As* K_1 *and* K_2 *are compact,* $K_1 = O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$ *and* $K_2 = O_{\beta_1} \cup \ldots \cup O_{\beta_m}$ *for some* $\alpha_1, \ldots, \alpha_n$ *and* $\beta_1, \ldots, \beta_m$ *in* I. *Then* $K \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n} \cup O_{\beta_1} \cup \ldots \cup O_{\beta_m}$. *Hence* K *is compact.*

2. *Let* $(O_\alpha)_{\alpha\in I}$ *be an open family such that* $K \cap F \subseteq \cup_{\alpha\in I} O_\alpha$. *Then* $K \subseteq (\cup_{\alpha\in I} O_\alpha) \cup F^C$. *As* K *is compact and* F^C *is open* $K \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n} \cup F^C$ *for some* $\alpha_1, \ldots, \alpha_n \in I$. *Hence* $K \cap F \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$. *So* $K \cap F$ *is compact.*

Example 5.1.9 *Let* (X,d) *any m.s. Then,*

1. *A one-point set* $K = \{a\}$ *is compact.*

2. *If* $K = \{a_1, \ldots, a_n\}$ *then* K *is compact.*

3. *If* (X,d) *is discrete, then any compact set* $K \subseteq X$ *is finite.*

   Indeed, in any discrete metric space every set A *is open and closed. In particular,* $\forall\ x \in X$, *the set* $\{x\}$ *is open. If* K *is compact, from* $K \subseteq \cup_{x\in K}\{x\}$ *we get that* $K \subseteq \{x_1\} \cup \ldots \cup \{x_n\}$ *for some* $x_1, \ldots, x_n \in K$. *Hence* K *is finite.*

4. *For any metric space, if $x_n \in X$ and $x_n \to x$, then the set $K = \{x_n : n \in \mathbb{N}\} \cup \{x\}$ is a compact set.*

   Indeed, suppose that $K \subseteq \cup_{\alpha\in I} O_\alpha, O_\alpha$'s are open in X. Then $x \in O_{\alpha'}$ for some $\alpha' \in I$. Hence $\exists\ N \in N : x_n \in O_{\alpha'}\ \forall\ n \geq N$. Then we say $x_0 \in O_{\alpha_0}, x_1 \in O_{\alpha_1}, \ldots, x_N \in O_{\alpha_N}$. Hence, $K \subseteq O_{\alpha_1} \cup O_{\alpha_2} \cup \ldots \cup O_{\alpha_N} \cup O_{\alpha'}$. So K is compact. X is compact $\Longleftrightarrow$ Whenever $X = \cup_{\alpha\in I} O_\alpha$, we have $X = O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$ for some $\alpha_1, \ldots, \alpha_n \in I$.

5. *Now, let (X, d) be a m.s. and $K \subseteq X$ a set. Then K is compact $\Longleftrightarrow$ (K, d) is a compact m.s.*

   $K \subseteq \cup_{\alpha\in I} O_\alpha \Rightarrow K = \alpha \in I \cup (O_\alpha \cap K) = \cup_{\alpha\in I} \tilde{O}_\alpha$, $\tilde{O}_\alpha = O_\alpha \cap K$ is open in (K, d).

   Hence (K, d) is compact$\Leftrightarrow$whenever $K = \cup_{\alpha\in I} \tilde{O}_\alpha$, $\tilde{O}_\alpha$ is open in (K, d),

   $K = \tilde{O}_{\alpha_1} \cup \ldots \cup \tilde{O}_{\alpha_n}$ for some $\alpha_1, \ldots \alpha_n \in I$. (or $K \subseteq O_{\alpha_1} \cup \ldots \cup O_{\alpha_n}$).

6. *If $K \subseteq M \subseteq X$. Then to say that K is compact in (M, d) is equivalent to saying that K is compact in X.*

   Since in either cases, this is equivalent to say that (K, d) is compact. So compactness is absolute notion.

Example 5.1.10 $\mathbb{R} \subseteq \mathbb{R}^2 \subseteq \ldots \subseteq \mathbb{R}^n \subseteq \ldots$. *$K \subseteq R$ be a set. On $\mathbb{R}^n$ we put d_2. Then,*

$$\begin{aligned} K \text{ is compact in } \mathbb{R} &\Leftrightarrow K \text{ is compact in } \mathbb{R}^2 \\ &\Leftrightarrow K \text{ is compact in } \mathbb{R}^3 \\ &\ \ \vdots \\ &\Leftrightarrow K \text{ is compact in } \mathbb{R}^n \\ &\ \ \vdots \end{aligned}$$

Proposition 5.1.11 *Any compact m.s. (K, d) is separable.*

Proof 5.1.12 *Let $\varepsilon > 0$ be given. Then $K \subseteq \cup_{x\in K} B_\varepsilon(x)$. As K is compact, there exists a **finite set** $F_\varepsilon = \{x_1, \ldots, x_i\}$ such that $K \subseteq B_\varepsilon(x_1) \cup \ldots \cup B_\varepsilon(x_i)$. Let $\varepsilon = 1$, then $\varepsilon = \frac{1}{2}$, then $\varepsilon = \frac{1}{3}, \ldots$ So that for $\varepsilon = \frac{1}{n}$ we have a finite set $F_n \subseteq K$ such that $K \subseteq \cup_{x\in F_n} B_{\frac{1}{n}}(x)$. Let $A = \cup_{n\geq 1} F_n$. Then A is countable and $A \subseteq K$. Let us see that $\bar{A} = K$.*

Indeed if we had $y \in K \backslash \bar{A}$, there would be an $\varepsilon > 0$ such that $B_\varepsilon(y) \cap A = \emptyset$ i.e. $B_\varepsilon(y) \cap F_n = \emptyset\ \forall\ n \geq 1$. Let n be large enough to have $\frac{1}{n} < \frac{\varepsilon}{2}$. As $y \in K \subseteq \cup_{x\in F_n} B_{\frac{1}{n}}(x)$, $d(x, y) < \frac{1}{n}$ for some $x \in F_n$. But then $d(x, y) > \varepsilon$, so $B_\varepsilon(y) \cap F_n \neq \emptyset$, which is not possible. So $\bar{A} = K$ and K is separable.

Proposition 5.1.13 *Let (X, d) be any m.s. $K \subseteq X$ compact, $F_n \subseteq X$ closed such that $K \supseteq F_0 \supseteq F_1 \supseteq \ldots \supseteq F_n \supseteq \ldots$ and each $F_n \neq \emptyset$ then $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$ (Nested Interval Theorem)*

Proof 5.1.14 *For a contradiction, suppose that* $\cap_{n\in\mathbb{N}} F_n = \emptyset$. *Then* $\cup_{n\in\mathbb{N}} F_n^C = X$. *So as* $K \subseteq X$, $K \subseteq \cup_{n\in\mathbb{N}} F_n^C$. *As* K *is compact,* $K \subseteq F_0^C \cup \ldots \cup F_n^C$ *for some* $n \in \mathbb{N}$. *But* $F_0^C \cup \ldots \cup F_n^C = F_n^C$, *since* $F_0 \supseteq \ldots \supseteq F_n$. *so* $K \subseteq F_n^C$. *Hence* $F_n \subseteq K^C$. *as* $F_n \subseteq K$, $F_n \subseteq K \cap K^C = \emptyset$, *so* $F_n = \emptyset$, *which is not possible.*

Hence $\cap_{n\in\mathbb{N}} F_n \neq \emptyset$.

Example 5.1.15 *Let* $X = \mathbb{R}$, $d(x,y) = |x-y|$. $F_n = [n, +\infty[$, *then* $F_0 \supseteq \ldots \supseteq F_n \supseteq \ldots$ *each* F_n *is closed,* $F_n \neq \emptyset$ *but* $\cap_{n\in\mathbb{N}} F_n = \emptyset$.

Our next aim is to prove the following theorem:

Theorem 5.1.16 *Let* (X,d) *be m.s. and* $K \subseteq X$ *a set.*

$$\text{Then } K \text{ is compact} \iff \begin{cases} 1)\, K \text{ is closed} \\ \text{and} \\ 2)\, \text{Every sequence } x_n \text{ in } K \text{ has a convergent subsequence } x_{n_k} \end{cases}$$

Lemma 5.1.17 *Let* (X,d) *be a m.s. and* $K \subseteq X$ *a compact set. Then every sequence* (x_n) *in* K *has a convergent subsequence* (x_{n_k}). *(Equivalently every sequence* x_n *has at least one cluster point* $x \in K$*).*

Proof 5.1.18 *Let* x_n *be a sequence in* K. *Put* $F_n = \{x_n, x_{n+1}, \ldots\}$ *and* $F = \cap_{n\in\mathbb{N}}(\overline{F_n})$. *We have to show that* $F \neq \emptyset$. *As* K *is closed and* $F_n \subseteq K$, $\overline{F_n} \subseteq K$. *So that we have* $K \supseteq \overline{F_0} \supseteq \ldots \supseteq \overline{F_n} \supseteq \ldots$ *and* $\overline{F_n} \neq \emptyset\ \forall n \in \mathbb{N}$. *Hence by the Nested Interval Theorem* $F = \cap_{n\in\mathbb{N}}(\overline{F_n}) \neq \emptyset$. *So* x_n *has a cluster point.*

Corollary 5.1.19 *Any compact m.s.* (X,d) *is* ***complete****.*

Proof 5.1.20 *Let* $(x_n)_{n\in\mathbb{N}}$ *be a Cauchy sequence in* X. *As* (X,d) *is compact,* $(x_n)_{n\in\mathbb{N}}$ *has a cluster point* $x \in X$. *But then, since any Cauchy sequence in any m.s. that has a cluster point converges to this point, we conclude that* $x_n \to x$. *So* (X,d) *is complete.*

Example 5.1.21 *Consider the m.s.* $(\mathbb{Q}, d)$, $d(x,y) = |x-y|$. *Let* $K = \{x \in \mathbb{Q} : 0 \leq x \leq 1\}$. *Then* K *is not compact, since* (K,d) *is* ***not complete****.*

Example 5.1.22 *Let* $x_n = \frac{1}{3}\left[\sum_{k=0}^{n} \frac{1!}{k!}\right]$ *is Cauchy,* $x_n \in K$ *but* $(x_n)_{n\in\mathbb{N}}$ *has no cluster point in* K.

5.2 Second Characterization of Compact Sets

Definition 5.2.1 *A m.s.* (X, d) *(or a subset of it) is said to be* ***sequentially compact*** *if every sequence* $(x_n)_{n\in\mathbb{N}}$ *in* X *has a cluster point* $x \in X$.

Example 5.2.2 *In* $(\mathbb{R}, d)$ *every closed and bounded set* K *is sequentially compact.*

Lemma 5.2.3 *Every sequentially compact m.s.* (X, d) *is separable.*

Proof 5.2.4 *For each* $\varepsilon > 0$, *let* A_ε *be a maximal subset of* X *such that for any* $x \neq y$ *in* A_ε, $d(x, y) \geq \varepsilon$.*(*)*

Claim: A_ε *is finite. If it was not, we could choose infinite distinct points* $x_0, x_1, \dots, x_n, \dots$ $(x_i \neq x_j \,\forall\, i \neq j)$. *Then, since* (X, d) *is sequentially compact,* $(x_n)_{n\in\mathbb{N}}$ *would have a convergent subsequence* $y_k = x_{n_k}$. *Then* $(y_k)_{k\in\mathbb{N}}$ *would be Cauchy. But this is not possible since for* $k \neq k'$ $d(y_k, y_{k'}) \geq \varepsilon$. *So* A_ε *is finite.*

Next, for $\varepsilon = \dfrac{1}{n}$, *denote the corresponding* A_ε *as* A_n*(so* $\forall\, x, y \in A_n$, $x \neq y$, $d(x, y) \geq \dfrac{1}{n}$*). Let* $A = \cup_{n\geq 1} A_n$. *Then* A *is countable, since each* A_n *is finite. Let us see that* $\bar{A} = X$. *If not, then there is an* $a \in X$ *such that* $a \notin \bar{A}$. *Hence for some* $\varepsilon > 0$, $B_\varepsilon(a) \cap A_n = \emptyset$. *Let* n *be such that* $\dfrac{1}{n} < \varepsilon$. *Since* $B_\varepsilon(a) \cap A = \emptyset$, $B_\varepsilon(a) \cap A_n = \emptyset$, *too. Hence* $\forall\, x \in A_n$, $d(a, x) \geq \varepsilon > \dfrac{1}{n}$. *But then* $\bar{A}_n = A_n \cup \{a\}$ *satisfies (*). However this is not possible since* A_n *is a maximal set satisfying (*). Hence* $\bar{A} = X$ *and* X *is separable.*

Theorem 5.2.5 ***(Main Theorem):*** *Let* (X, d) *be a m.s. and* $K \subseteq X$ *a subset of it. Then* K *is compact iff* K *is closed and every sequence* $(x_n)_{n\in\mathbb{N}}$ *in* K *has a cluster point.*

Proof 5.2.6 *We have already proved the implication* $(\Rightarrow)$.

$(\Leftarrow)$ *Suppose that* K *is closed and every sequence in* K *has a cluster point. Then by the Lemma 5.2.3, the m.s.* (K, d) *is separable. Now, to prove that* K *is compact, let* $(O_\alpha)_{\alpha\in I}$ *be a family of open sets in* X *such that* $K \subseteq \cup_{\alpha\in I} O_\alpha$. *Then* $K = \cup_{\alpha\in I}(O_\alpha \cap K)$ *and each* $\tilde{O}_\alpha = O_\alpha \cap K$ *is open in* (K, d). *Now, as* (K, d) *is separable, by "Lindölf Theorem"* I *has a countable subset* J *such that* $K = \cup_{\alpha\in J}(O_\alpha \cap K)$. *Relabelling* O_α*'s we can assume that* $J = \mathbb{N}$, *so that* $K = \cup_{n\in\mathbb{N}}(O_n \cap K)$. *Hence* $K \subseteq \cup_{n\in\mathbb{N}} O_n$. *Then replacing* O_n *by* $\tilde{O}_n = O_1 \cup \ldots \cup O_n$, *we can assume that* $O_0 \subseteq O_1 \subseteq \ldots \subseteq O_n \subseteq \ldots$

Let us see that K *is contained in some* O_n. *If not then, for every* $n \in \mathbb{N}$, $K \nsubseteq O_n$, *so there is* $x_n \in K \backslash O_n$. *In that way we get a sequence* $(x_n)_{n\in\mathbb{N}}$ *in* K. *By hypothesis,* $(x_n)_{n\in\mathbb{N}}$ *has a cluster point* $x \in K$. *As* $K \subseteq \cup_{n\in\mathbb{N}} O_n$, $x \in O_N$ *for some* $N \in \mathbb{N}$. *As* O_N *is open and* $x \in O_N$, $x_n \in O_N$ *for infinitely many* $n \in \mathbb{N}$. *Then* $\exists\, n > N$ *such that* $x_n \in O_N$ *but this is not possible since* $x_n \notin O_n$, *so* $x_n \notin O_p \;\forall\, p \leq n$, *a contradiction.*

Hence, $K \subseteq O_n$ *for some* $n \in \mathbb{N}$, *i.e.* K *is compact.*

Example 5.2.7 *1. **Compact subsets of** $\mathbb{R}$**:** A subset K of $\mathbb{R}$ is compact iff K is closed and bounded.*

Proof: *Let K be compact. We have seen that every compact set in any m.s. is closed, so K is closed. Let us see that K is bounded. If it was not $\forall\, n \in \mathbb{N}\ \exists\ x_n \in K : |x_n| > n$. Such a sequence CANNOT HAVE a convergent subsequence. Hence K is bounded.*

Conversely, suppose that K is closed and bounded. Then by Theorem 5.2.5 and Bolzano Weierstrass Theorem, K is compact.

2. ***Compact subsets of** $(\mathbb{R}^n, d)$ **where** d **is Euclidean metric:*** *A subset K of $\mathbb{R}^n$ is compact iff K is closed and bounded (i.e. $sup_{n\in N}||x||_2 < \infty$). (Same proof as above.)*

3. ***Compact subsets of a discrete m.s.** (X, d)**:** A subset K of a discrete m.s. (X, d) is compact iff K is finite.*

   e.g. if $K \subseteq \mathbb{Z}$ or $K \subseteq \mathbb{N}$, then K is compact iff K is finite.

4. *Let E be an infinite set and $X = \mathbb{B}(E)$ the space of bounded functions with the metric $d(\varphi, \psi) = \sup_{x\in E}|\varphi(x) - \psi(x)|$.*

5. *Let $K = B'_1(0) = \{\varphi \in X : \sup_{x\in E}|\varphi(x) \leq 1|\}$.*

Remark: In any m.s. (X, d) a set $K \subseteq X$ is said to be bounded iff
$\delta(K) = \sup_{x,y\in K} d(x, y) < \infty \Leftrightarrow \exists\, R > 0 : K \subseteq B_R(x)$ for some $x \in X \Leftrightarrow$ For some $x_0 \in K, \sup_{x\in K}\ d(x_0, x) < \infty$. It is clear that $K = B'_1(0)$ is closed and bounded in (X, d). But K is NOT COMPACT!

Let $x_1, \ldots, x_n, \ldots \in E$ ($x_i \neq x_j$, for $i \neq j$). Let $\varphi_n = \chi_{\{x_n\}}$. Then;
$d(\varphi_n, \varphi_m) = \sup_{x\in E}|\varphi_n(x) - \varphi_m(x)| = 1,\ \forall\, n \neq m$ (*).

On the other hand, $\varphi_n \in K,\ \forall\ n \in \mathbb{N}$, since $|\varphi_n(x)| \leq 1,\ \forall\ x \in E$, (*) shows that no subsequence of φ_n is Cauchy, so no convergent subsequence of $(\varphi_n)_{n\in\mathbb{N}}$ is convergent. Hence K is not compact.

5.3 Totally Bounded Sets

Definition 5.3.1 *Let (X, d) be a m.s. A subset $K \subseteq X$ is said to be **totally bounded** if given any $\varepsilon > 0$, there exist finitely many points $x_1, \ldots, x_n \in K$ such that $K \subseteq B_\varepsilon(x_1) \cup \ldots \cup B_\varepsilon(x_n)$.*

Theorem 5.3.2 *Let (X, d) be a **complete** m.s. and $K \subseteq X$ a set. Then;*
K is compact $\Longleftrightarrow$ K is closed and is totally bounded.

To prove this theorem, we need to prove the following lemma:

Lemma 5.3.3 ***(Contour's Nested Interval Theorem)*** *Let (X, d) be a complete m.s. and $F_0 \supseteq F_1 \supseteq \ldots \supseteq F_n \supseteq \ldots$ be nonempty closed sets, such that $\delta(F_n) \to 0$ as $n \to \infty$. Then the intersection $\cap_{n\in N} F_n$ is nonempty and contains just one point. (Here $\delta(F_n) = \sup\{d(x, y) : x, y \in F_n\}$ is the " diameter of F $_n$")*

Proof 5.3.4 *Let for each $n \in \mathbb{N}$, $x_n \in F$ be an arbitrary point. Observe that, as $F_0 \supseteq F_1 \supseteq \ldots \supseteq F_n \supseteq \ldots$, $\{x_n, x_{n+1}, \ldots\} \subseteq F_n$. Let $\varepsilon > 0$ be arbitrary. As $\delta(F_n) \to 0$ there is $N \in \mathbb{N}$ such that $\forall\, n \geq N \quad \delta(F_n) < \varepsilon$. Hence $\forall\, n \geq \mathbb{N} \;\; \forall\, p \in \mathbb{N} \;\; d(x_n, x_{n+p}) \leq \delta(F_n) < \varepsilon$. This shows that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. As (X, d) is complete. $(x_n)_{n\in\mathbb{N}}$ converges to a point $a \in X$. As $\{x_n, x_{n+1}, \ldots, x_{n+p}, \ldots\} \subseteq F_n$ and F_n is closed, $a \in F_n \,\forall\, n \in \mathbb{N}$. So $a \in \cap_{n\in\mathbb{N}} F_n$. As $\delta(F_n) \to 0$, the intersection $\cap_{n\in\mathbb{N}} F_n$ cannot contain any point $x \neq a$.*

Remark: Let $X = \mathbb{R}, d(x, y) = |x - y|$. Let $F_n = \{n, n + 1, \ldots, n + p, \ldots\}$. Then F_n is closed and $F_0 \supseteq F_1 \supseteq \ldots \supseteq F_n \supseteq \ldots$ So, $\cap_{n\in\mathbb{N}} F_n = \emptyset$. (Hence $\delta(F_n) \nrightarrow 0$).

Proof 5.3.5 *proof of theorem 5.3.2*

1. *Suppose that K is compact. Then as we have seen that K is closed. Let $\varepsilon > 0$ be any number. Then, obviously $K \subseteq \cup_{x\in K} B_\varepsilon(x)$. As K is compact and each $B_\varepsilon(x)$ is open, $K \subseteq B_\varepsilon(x_1) \cup \ldots B_\varepsilon(x_k)$ for some $x_1, \ldots, x_k \in K$. So K is totally bounded.*

2. *Conversely, suppose that K is closed and totally bounded. i.e.*

   $\forall\, \varepsilon > 0,\ \exists\, x_1, \ldots, x_k \in K : K \subseteq B_\varepsilon(x_1) \cup \ldots \cup B_\varepsilon(x_k)$. *Let us see that every sequence $(x_n)_{n\in\mathbb{N}}$ in K has a cluster point.*

   In (Eq 5.2) let $\varepsilon = 1$. Then $\exists\, a_1, \ldots, a_p \in K : K \subseteq B_\varepsilon(a_1) \cup \ldots \cup B_\varepsilon(a_p)$. So, for some infinite $F_1 \subseteq \mathbb{N}$, $(x_n)_{n\in F_1}$ is contained in one of these balls, say $B_1(a)$.

   Let $K_1 = B_1'(a_1) \cap K$. Then K_1 is totally bounded.

   So with $\varepsilon = \dfrac{1}{2}$, $K \subseteq B_{\frac{1}{2}}(b_1) \cup \ldots B_{\frac{1}{2}}(b_q)$ for some $b_1, \ldots, b_q \in K_1$.

   As $(x_n)_{n\in F_1} \subseteq K_1 \subseteq B_{\frac{1}{2}}(b_1) \cup \ldots \cup B_{\frac{1}{2}}(b_q)$, there is an infinite set $F_2 \subseteq F_1$ such that $(x_n)_{n\in F_2}$ is contained in one of these balls, say $B_{\frac{1}{2}}(b_1)$.

   Let $K_2 = K_1 \cap B'_{\frac{1}{2}}(b_1)$. Then K_2 is totally bounded.

   With $\varepsilon = \dfrac{1}{3}$, $\exists\, c_1, \ldots, c_k \in K_2$, $K_2 \subseteq B_{\frac{1}{3}}(c_1) \cup \ldots \cup B_{\frac{1}{3}}(c_k)$

   $\vdots$

   And so on. In this way, we construct $K \supseteq K_1 \supseteq \ldots \supseteq K_n \supseteq \ldots$ and $\delta(K_n) \leq \dfrac{2}{n} \to 0$. Hence $\cap_{n\in\mathbb{N}} K_n = \{a\}$ for some $a \in K$. Now, let

   $n_1 \in F_1$ be such that $x_{n_1} \in K_1$
   $n_2 \in F_2,\ n_2 > n_1$ be such that $x_{n_2} \in K_2$
   $\vdots$
   $n_k \in F_k,\ n_k > n_{k-1}$ be such that $x_{n_k} \in K_k$

Then by the proof of Lemma 5.3.3, $y_k = x_{n_k} \to a$. Thus, a is a cluster point of $(x_n)_{n\in N}$. So K is compact.

Example 5.3.6 *Let* $l^1 = \{x = (x_n)_{n\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}} : \sum_{n=0}^{\infty} |x| < \infty\}$. *For* $x \in l^1$,

$||x||_1 = \sum_{n=0}^{\infty} |x_n|$ *and* $d(x, y) = ||x - y||_1$. *Then* (l^1, d) *is a m.s. This m.s. is complete. Let* $K \subseteq l^1$ *be a set. Then,*

$$K \text{ is compact} \Longleftrightarrow \begin{cases} 1^\circ)\ K \text{ is closed} \\ 2^\circ)\ K \text{ is bounded:} \sup_{x\in K} ||x||_1 < \infty \\ 3^\circ)\ \lim_{k\to\infty} \sup x = (x_n)_{n\in N} \in K \displaystyle\sum_{k=n}^{\infty} |x_k| = 0 \end{cases}$$

5.4 Exercises

1. Let (X,d) be a metric space and A be a nonempty subset of X. For $x \in X$ put $d(x,A) = \inf_{a \in A} d(x,a)$. Show that

   (a) $d(x,A) = 0$ iff $x \in \overline{A}$.

   (b) $\forall\, \varepsilon > 0$ the set $F_\varepsilon = \{x \in X : d(x,A) \le \varepsilon\}$ is closed and $A \subseteq F_\varepsilon$.

   (c) $\forall\, \varepsilon > 0$ the set $O_\varepsilon = \{x \in X : d(x,A) \le \varepsilon\}$ is open and $A \subseteq F_\varepsilon$.

2. Let (X,d) be a metric space and A, B be 2 subsets of X $(A \neq \emptyset, B \neq \emptyset)$. Put $d(A,B) = \inf_{(x,y) \in A \times B} d(x,y)$. Show that

   (a) if A is closed, B is compact and $A \cap B = \emptyset$, then $d(A,B) > 0$.

   (b) Assume A, B are compact and disjoint. Let $\varepsilon = \dfrac{d(A,B)}{3}$. Show that the sets $O_\varepsilon = \{x \in X : d(x,A) < \varepsilon\}$ and $\tilde{O}_\varepsilon = \{x \in X : d(x,B) < \varepsilon\}$ are open, $A \subseteq O_\varepsilon, B \subseteq \tilde{O}_\varepsilon$ and $O_\varepsilon \cap \tilde{O}_\varepsilon = \emptyset$.

3. Let (X,d) be a m.s. and $(x_n)_{n \in \mathbb{N}}$ be a convergent sequence in X with $a = \lim_{n \to \infty} x_n$. Show that the set $K = \{x_n : n \in \mathbb{N}\} \cup \{a\}$ is compact.

4. Let (X,d) be a m.s. and $K_0 \supseteq K_1 \supseteq \ldots \supseteq K_n \supseteq \ldots$ be nonempty compact sets. Show that the set $K = \cap_{n \in \mathbb{N}} K_n$ is nonempty.

5. Let $X = \mathbb{R}$, $d(x,y) = |x-y|$ and $F_n = \{n, n+1, n+2, \ldots\}$. Show that

   $F_0 \supseteq F_1 \supseteq \ldots \supseteq F_n \supseteq \ldots, F_n$ is closed and $F_n \neq \emptyset$ for each $n \in \mathbb{N}$ but $\cap_{n \in \mathbb{N}} F_n = \emptyset$.

6. Let l' be the space of all the mappings $\varphi : \mathbb{N} \to \mathbb{R}$ such that $\displaystyle\sum_{n=0}^{\infty} |\varphi(x)| < \infty$. For $\varphi, \psi \in l'$ put $d(\varphi,\psi) = \displaystyle\sum_{n=0}^{\infty} |\varphi(n) - \psi(n)|$. Show that

   (a) d is a metric on l'.

   (b) The set $K = \{\varphi \in l' : \displaystyle\sum_{n=0}^{\infty} |\varphi(n)| \le 1\}$ is closed and bounded.

   (c) Let, for each $n \in \mathbb{N}$, $f_n = \chi_{\{n\}}$. Then, $f_n \in K$ and for $n \neq m, d(f_n, f_m) = 2$.

   (d) $(f_n)_{n \in \mathbb{N}}$ has no subsequence which is Cauchy.

   (e) Show that although K is closed and bounded, it is not compact.

7. On $\mathbb{R}^2$ consider the Euclidean metric d_2. Let $F \subseteq \mathbb{R}^2$ be closed and $O \subseteq \mathbb{R}^2$ be an open set. Let $K \subseteq \mathbb{R}$ be a compact set. Show that

(a) The set $A = \cup_{x\in K}\{y \in \mathbb{R} : (x, y) \in F\}$ is a closed set.

(b) The set $B = \cap_{x\in K}\{y \in \mathbb{R} : (x, y) \in O\}$ is an open set.

8. Let K_1, K_2 be 2 nonempty compact subsets of R, $d(x, y) = |x - y|$. Show that the set $K_1 + K_2$ is also compact.

9. Let K be a compact subset of $\mathbb{R}^n$, $(d = d_2)$ and $\varepsilon > 0$. Show that the set $K_\varepsilon = \{x \in R^n : d(x, K) \leq \varepsilon\}$ is also compact.

10. Let $(X_1, d_1), (X_2, d_2)$ be two m.s., $Y = X_1 \times X_2$ and $d((x_1, x_2), (\tilde{x}_1, \tilde{x}_2)) = \max\{d_1(x_1, \tilde{x}_1), d_2(x_2, \tilde{x}_2)\}$.

11. Let $K_1 \subseteq X_1$ and $K_2 \subseteq X_2$ be compact sets. Show that the set $K = K_1 \times K_2$ is compact in (Y, d)

Chapter 6

Continuity

1. **Definition and First Properties**
2. **Global Conditions**
3. **Uniform Continuity / Lipschitzean Function./ Isometry**
4. **Uniform Extension Theorem**
5. **The Distance Function, Urysohn Lemma**
6. **Equivalence of Metrics**
7. **Completion of a m.s.**

6.1 Definition

In this chapter the "scene" will be as follows: (X, d), (Y, d') will be 2 m.s. $A \subseteq X$ a set, $f : A \to Y$ a function and $a \in A$ a point.

Definition 6.1.1 *The function f is said to be* ***"continuous at a"*** *if we have $\forall\ \varepsilon > 0\ \exists\ \eta > 0,\ \forall\ x \in A,\ d(x, a) < \eta \Longrightarrow d'(f(x), f(a)) < \varepsilon$*

Note: If f is continuous at every $a \in A$, then we say that f is continuous on A.

Example 6.1.2 *1. Let $f : [-1, 1] \to \mathbb{R}$, $f(x) = \begin{cases} -1 & \text{if } -1 \le x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } 0 < x \le 1 \end{cases}$*

Let us see that if f is continuous at $a = 0$.

Let $0 < \varepsilon < 1$. Then $\forall\ \eta > 0,\ \exists\ x_\eta \in A = [-1, 1] : d(x_\eta, a) < \eta$, but $|f(x_\eta) - f(a)| > \varepsilon$. This shows that f is not continuous at $a = 0$.

2. *If a is an isolated point of A, then every function $f : A \to Y$ is continuous at a.*

   Indeed if $a \in A$ is an isolated point, then for some $\eta > 0$, $B_\eta(a) \cap A = \{a\}$. By the definition of continuity at a: $\forall\, \varepsilon > 0\ \exists\, \delta > 0 :\ \forall\, x \in A \cap B_\delta(a),\ d'(f(x), f(a)) < \varepsilon$.

Theorem 6.1.3 ***(Characterization of the Continuity)*** *The function $f : A \to Y$ is continuous at a iff for every sequence $(x_n)_{n\in\mathbb{N}}$ in A converging to a, $f(x_n) \to f(a)$ in Y.*

Proof 6.1.4 *Suppose f is continuous at a. So we have:*
$\forall\, \varepsilon > 0,\ \exists\, \eta > 0,\ \forall\, x \in B_\eta(a) \cap A,\ d'(f(x), f(a)) < \varepsilon$. Now, let $(x_n)_{n\in N}$ be a sequence in A that converges to a. So we have: $\forall\, \varepsilon' > 0\ \exists\, N \in N\ \ \forall\, n \geq N\ d(x_n, a) < \varepsilon'$. Hence if we take $\varepsilon' = \eta$, then $\forall\, n \geq N\ x_n \in B_\eta(a) \cap A$, so $d'(f(x_n), f(a)) < \varepsilon$. Hence $f(x_n) \to f(a)$.

Conversely, suppose that whenever $x_n \to a$, $(x_n \in A)$, $f(x_n) \to f(a)$ (*)
Let us see that f is continuous at a.
If it was not continuous at a, we would have: $\exists\, \varepsilon > 0,\ \forall\, \eta > 0,\ \exists\, x_\eta \in B_\eta(a) \cap A : d'(f(x_\eta), f(a)) \geq \varepsilon$
Let $\eta = 1, \dfrac{1}{2}, \dots, \dfrac{1}{n}$ and denote by x_n : the point x_η that corresponds to $\eta = \dfrac{1}{n}$ so that $x_n \in B_{\frac{1}{n}}(a) \cap A$ and $d'(f(x_n), f(a)) \geq \varepsilon$. Hence $d(x_n, a) < \dfrac{1}{n} \to 0$ (i.e. $x_n \to a$) but $f(x_n) \nrightarrow f(a)$. Hence (*) implies continuity.

Example 6.1.5 *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$*

Is f continuous at $a = 0$?
Let $x_n = \dfrac{1}{n\pi + \frac{\pi}{2}}$. Then $x_n \to 0$, as $x \to \infty$. But
$f(x_n) = \sin(n\pi + \frac{\pi}{2}) = \cos n\pi = (-1)^n \nrightarrow f(0)$. So f is not continuous at $a = 0$. This shows that whenever we choose $\alpha \in \mathbb{R}$ and define f as $f(x) = \begin{cases} \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ \alpha & \text{if } x = 0 \end{cases}$

f is discontinuous at $a = 0$.

Example 6.1.6 *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$*

Let $x_n \in \mathbb{R}$, $x_n \to 0$. Then, $\dfrac{\sin x_n}{x_n} \to 1 \neq f(0)$. So f is not continuous at $a = 0$. But if we define f as $f(x) = \begin{cases} \dfrac{\sin x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ then f becomes continuous at $a = 0$.

Proposition 6.1.7 ***(Operations on Continuous Functions)*** *Let $Y = \mathbb{R}$, and let metric be $d'(x, y) = |x - y|$ and $f, g : A \to \mathbb{R}$ be two functions. Then,*

1. *If f and g are continuous at a, then $f+g$ is continuous at a.*
2. *If f and g are continuous at a, then $f \times g$ is continuous at a.*
3. *If $g(a) \neq 0$ and f and g are continuous at a, then $\dfrac{f}{g}$ is continuous at a.*
4. *If f is continuous at a, then so is $|f|$.*
5. *If f and g are continuous at a, then so are $\max\{f,g\}$ and $\min\{f,g\}$.*

Proof 6.1.8 *Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A that converges to a. Then,*

1. $$\begin{array}{lcll} f(x_n) & + & g(x_n) & \rightarrow f(a)+g(a), \text{ so } f+g \text{ is continuous at } a \\ \downarrow & & \downarrow & \\ f(a) & & g(a) & \end{array}$$

2. $$\begin{array}{lcll} f(x_n) & \times & g(x_n) & \rightarrow f(a)\times g(a), \text{ so } f\times g \text{ is continuous at } a. \\ \downarrow & & \downarrow & \\ f(a) & & g(a) & \end{array}$$

The rest of the proof is left to reader.

Remark: $\max\{f,g\}(x) = \max\{f(x), g(x)\}$

WARNING: If $|f|$ is continuous at a we cannot say that f is continuous at a.

Let $f : [-1,1] \to \mathbb{R}$ $f(x) = \begin{cases} -1 & \text{for } -1 \leq x < 0 \\ 1 & \text{for } 0 \leq x \leq 1 \end{cases}$ Let $a = 0$, so $f(a) = 1$. As $|f(x)| = 1\ \forall x \in [-1,1]$, f is continuous at $a = 0$. But if $x_n = \dfrac{-1}{n}$ then $x_n \to 0$ but $f(x_n) = -1 \nrightarrow f(a) = 1$

Proposition 6.1.9 *Composition of two continuous functions is continuous.i.e. if $f : A \to Y$, $B \subseteq Y$, $g : B \to Z$ ((Z, d'') is another m.s.), and if f is continuous at a, $f(A) \subseteq B$ and g is continuous at $b = f(a)$, then $g \circ f : A \to Z$ is continuous at a.*

Proof 6.1.10 *Let $x_n \in A$, $x_n \to a$, then*

$$g \circ f\,(x_n) = \underbrace{g\,(\underbrace{f\,(x_n)}_{b_n})}_{b = f(a)} \rightarrow g(f(a)).$$

Hence $g \circ f$ is continuous.

Example 6.1.11 *On $\mathbb{R}$, $f(x) = \dfrac{e^{\tan(x^2+1)}}{\sin(x^2+1)+2}$.*
$\Phi(x) = x$ is continuous, so $\Phi^2(x) = x^2$ is continuous. $\Psi(x) = 1$ is continuous, hence $\Phi^2(x) + \Psi(x)$ is continuous.
$\tan x$ is continuous, so $\tan(x^2+1)$ is continuous. $\sin x$ is continuous, and $\sin(x^2+1)+2 \neq 0$. Thus f is continuous on $\mathbb{R}$.

6.1.1 Continuity and Compactness

Let $f :]0,1] \to \mathbb{R}$, $f(x) = \dfrac{1}{x}$. Then f is continuous on $]0,1]$; $]0,1]$ is a bounded set, but $f(]0,1]) = [1,+\infty[$ is an unbounded set.

Next, let $f : \left[0, \dfrac{\pi}{2}\right[\to \mathbb{R}$, $f(x) = \sin x$. Then $\sup_{x\in[0,\frac{\pi}{2}[} f(x) = 1$, but there is no $x_0 \in \left[0, \dfrac{\pi}{2}\right[$ such that $f(x_0) = 1$. i.e. f is bounded but does not attain its supremum on $\left[0, \dfrac{\pi}{2}\right[$.

Theorem 6.1.12 *Let $K \subseteq X$ be a compact set and $f : K \to Y$ a continuous function. Then $f(K)$ is also compact.*

Proof 6.1.13 *Let us see that,*

1. *$f(K) = \tilde{K}$ is closed in Y.*
2. *Any sequence $(y_n)_{n\in\mathbb{N}}$ in $f(K) = \tilde{K}$ has a cluster point $y \in \tilde{K}$.*

1. *To show that $\tilde{K}$ is closed in Y, let $(y_n)_{n\in\mathbb{N}}$ be a sequence in $\tilde{K}$ that converges to some $y \in Y$. We have to show that $y \in \tilde{K}$. (i.e. $y = f(x)$ for some $x \in K$). Since $y_n \in \tilde{K}$, $y_n = f(x_n)$ for some $x_n \in K$. As K is compact, a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in K$. As f is continuous at x, $f(x_{n_k}) \to f(x)$. So $y_{n_k} \to f(x)$. As $y_n \to y$, $y_{n_k} \to y$ too. So, $y = f(x) \in \tilde{K}$. Hence $\tilde{K}$ is closed in Y.*
2. *Now, let $(y_n)_{n\in\mathbb{N}}$ be any sequence in $\tilde{K}$. So $y_n = f(x_n)$. As $x_n \in K$, K is compact, $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to some $x \in K$. Then $f(x_{n_k}) \to f(x)$. So $y_{n_k} \to f(x)$. i.e. $(y_n)_{n\in N}$ has a cluster point, namely $y = f(x)$. Hence $\tilde{K}$ is compact.*

Corollary 6.1.14 *Let $K \subseteq X$ be a compact set and $f : K \to \mathbb{R}$ be a continuous function. Then,*

1. *$f(K)$ is a closed and bounded set.*
2. *There exist $x_0, y_0 \in K$ such that $\sup_{x\in K} f(x) = f(x_0)$ and $\inf_{x\in K} f(x) = f(y_0)$ i.e. f attains its* max *and* min *on K.*

Proof 6.1.15 *By the Theorem 6.1.12, $\tilde{K} = f(K)$ is compact in $\mathbb{R}$. In $\mathbb{R}$, compact sets are exactly closed bounded sets. So $\tilde{K}$ is closed and bounded. Hence $\sup \tilde{K} = \alpha$ and $\inf \tilde{K} = \beta$ exist and $\alpha, \beta \in \tilde{K}$. So, $\alpha = f(x_0)$ and $\beta = f(y_0)$ for some $x_0, y_0 \in K$.*

i.e. $\sup_{x\in K} f(x) = f(x_0)$, $\inf_{x\in K} f(x) = f(y_0)$.

6.2 Global Characterization of the Continuity

Let $(X, d), (Y, d')$ be two m.s. Let $f : X \to Y$ be a continuous function. We have seen that for any compact set $K \subseteq X$, $f(K)$ is compact. But under f the images of an open/closed set need not to be open/closed.

Indeed, let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$. Then f is continuous on $\mathbb{R}$ and $O =]-2\pi, 2\pi[$ is open but $f(O) = [-1, 1]$, which is closed.

If $f(x) = \dfrac{x^2}{1 + x^2}$, then f is continuous on $\mathbb{R}$. The set $F = [0, +\infty[$ is closed however $f(F) = [0, 1[$ is not closed.

Theorem 6.2.1 *For any function $f : X \to Y$, the following assertions are equivalent:*

1. *f is continuous on X.*
2. *For any open set $O' \subseteq Y$, $f^{-1}(O')$ is open in X.*
3. *For any closed set $F' \subseteq Y$, $f^{-1}(F')$ is closed in X.*
4. *$\forall A \in X$, $f(\overline{A}) \subseteq \overline{f(A)}$.*

Proof 6.2.2

- $1 \to 2$: *Suppose f is continuous on X. Let $O' \subseteq Y$ be an open set and $O = f^{-1}(O')$. We have to show that O is open. Let $x_0 \in O$ be a point. Then $f(x_0) \in O'$. As O' is open, $\exists\, \varepsilon > 0 :\ B_\varepsilon(f(x)) \subseteq O'$. As f is continuous at x_0, there is $\eta > 0$ such that for $x \in B_\eta(x_0)$, $f(x) \in B_\varepsilon(f(x_0))$ i.e $f(B_\eta(x_0)) \subseteq B_\varepsilon(f(x_0)) \subseteq O'$. This implies that $B_\eta(x_0) \subseteq f^{-1}(O')$ i.e. $B_\eta(x_0) \subseteq O$. Hence O is open in X.*

- $2 \to 3$: *Trivial since $f^{-1}(B^c) = f^{-1}(B)^c\, \forall\ B \subseteq Y$.*

- $3 \to 4$: *Suppose that for any closed set $F' \subseteq Y$, $f^{-1}(F')$ is closed in X. We have to show that, for any $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$. So, let $A \subseteq X$ be any set. Let $F' = \overline{f(A)}$ and put $F = f^{-1}(F')$. By hypothesis, F is closed. As $F' \supseteq f(A)$, $F = f^{-1}(F') \supseteq f^{-1}(f(A)) \supseteq A$. As F is closed in X, $\bar{A} \subseteq F$, i.e. $\bar{A} \subseteq f^{-1}(F')$. Then $f(\bar{A}) \subseteq f(f^{-1}(F')) \subseteq F' = \overline{f(A)}$.*

- $4 \to 3$: *To prove 3, let $F' \subseteq Y$ be a closed set. Take $A = f^{-1}(F')$. 4 says that $f(\bar{A}) = F'$. Hence $\bar{A} \subseteq f^{-1}(F') = A$. So A is closed.*

- $2 \to 1$: *Let $x_0 \in X$ be a point and $\varepsilon > 0$ an arbitrary number. Let $O' = B_\varepsilon(f(x_0))$. As O' is open by 2, $f^{-1}(O')$ is open in X. Moreover, $x_0 \in f^{-1}(O')$. As $f^{-1}(O')$ is open, then there is $\eta > 0$ such that $B_\eta(x_0) \subseteq f^{-1}(O')$. Hence $f(B_\eta(x_0)) \subseteq B_\varepsilon(f(x_0))$ i.e. $d(x, x_0) < \eta \to d'(f(x_1), f(x_0)) < \varepsilon$. So f is continuous at x_0.*

6.2.1 Open Mapping, Closed Mapping, Homeomorphism

Definition 6.2.3 *Let (X, d), (Y, d') be two m.s. and $f : X \to Y$ be a mapping.*

1. *If for each $O \subseteq X$ open, $f(O)$ is open in Y then we say that f is an* ***open mapping****.*
2. *If for each $F \subseteq X$ closed, $f(F)$ is closed in Y, then f is* ***closed mapping****.*
3. *If f is bijective and both f and f^{-1} are continuous then we say that f is a* ***homeomorphism****.*

Example 6.2.4 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous, strictly increasing function. Then $f(]a, b[) =]f(a), f(b)[$ is open. Now, every open set $O \subseteq \mathbb{R}$ is a union of open intervals $O = \cup_{k\in\mathbb{N}}]a_n, b_n[$. Then $f(O) = \cup_{n\in\mathbb{N}}]f(a_n).f(b_n)[$ is open. Let $f(x) = e^x$ for any $O \subseteq]0, \infty[$ open $f(O)$ is open.*

2. *If (X, d) is a* ***compact*** *m.s., then any continuous function $f : X \to Y$ is a closed mapping.*
3. *Let $X = \mathbb{R}, Y =]0, \infty[$, and $f : X \to Y$, $f(x) = e^x$. So, f is a bijection. As $f^{-1}(x) = \ln x :]0, \infty[\to \mathbb{R}$, f is continuous, so $f : \mathbb{R} \to]0, \infty[$ is a homeomorphism.*
4. *$f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ is a homeomorphism.*
5. *$f : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to \mathbb{R}, f(x) = \tan x$ is a homeomorphism.*
6. *$f : \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to]-1, 1[\,,\ f(x) = \sin x$ is a homeomorphism.*
7. *$f : [0, 1] \to [a, b]\,,\ f :]0, 1[\to]a, b[\,,\ f(t) = (1 - t)a + tb$ is a homeomorphism.*
8. *$f : \mathbb{R} \to]-1, 1[\,.\ f(x) = \dfrac{1}{1 + |x|}$ is a continuous bijection, and $f^{-1}(x) = \dfrac{x}{1 - |z|}$ is also continuous bijection. So f is a homeomorphism.*

Proposition 6.2.5 *Let $f : X \to Y$ be a bijection. Then,*

1. *f is open $\Leftrightarrow f^{-1} : Y \to X$ is continuous.*
2. *f is open $\Leftrightarrow$ f is closed.*

Proof 6.2.6 1. *Let $O \subseteq X$ open. Then $f^{-1}(f^{-1})(O) = f(O)$, from this the result follows.*

2. *As f is bijective, $\forall\, A \subseteq X$, $f(A)^c = f(A)^c$, from this the result follows.*

Corollary 6.2.7 *Let $f : X \to Y$ be a bijection. Then f is a homeomorphism $\Leftrightarrow$ f is continuous and open $\Leftrightarrow$ f is continuous and closed.*

Definition 6.2.8 *Two m.s.* (X, d), (Y, d') *are said to be* ***"homeomorphic"*** *if there exists a homeomorphism between them.*

Example 6.2.9

1. $\mathbb{R}$ *and* $]-1, 1[$ *are homeomorphic.*
2. $[a, b]$ *and* $[0, 1]$ *are homeomorphic.*
3. $]a, b[$ *and* $]0, 1[$ *are homeomorphic.*
4. $\mathbb{R}$ *and* $]0, \infty[$ *are homeomorphic.*
5. $\mathbb{R}$ *and* $]a, b[$ *are homeomorphic.*

Proposition 6.2.10 *If* (X, d) *is a compact m.s., every continuous bijection* $f : X \to Y$ *is a homeomorphism (i.e.* f^{-1} *is automatically continuous).*

Proof 6.2.11 *Indeed* f *is a closed mapping!*

Example 6.2.12 *Let* $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]\,,\ f(x) = \sin x$*. Then* f *is continuous and bijective. Then* $f^{-1}(x) = arc\sin x$ *is continuous.*

Example 6.2.13 *Let* $f : [a, b] \to [f(a), f(b)]$ *be a strictly increasing continuous function. Then* f^{-1} *is continuous i.e.* $f(x) = x^2$ $f : [0, 20] \to [0, 400]\,,\ f^{-1}(x) = \sqrt{x}$ *is continuous.*

6.2.2 Exercises I

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for all x, y in $\mathbb{R}$, $f(x+y) = f(x) + f(y)$. Show that f is continuous iff $f(x) = cx$ for some $c \in \mathbb{R}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that, for all $x, y \in \mathbb{R}$, $f(x+y) = f(x)f(y)$. Prove that

   (a) $f(x) \geq 0$ for all $x \in \mathbb{R}$ and that if $f(x) = 0$ for one $x \in \mathbb{R}$, then f is identically zero on $\mathbb{R}$.

   (b) If f is continuous at zero, then f is continuous at every $x \in \mathbb{R}$.

   (c) The only continuous function satisfying the above equality is $f(x) = a^x$ with $a = f(1)$.

3. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Define $g : [a, b] \to \mathbb{R}$ as follows: $g(a) = f(a)$ and for $x \in \,]a, b]$, $g(x) = \sup\{f(y) : y \in [a, x]\}$. Prove that g is monotone increasing and continuous on $[a, b]$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Show that f is continuous on $\mathbb{R}$.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases}$. Show that f is continuous only at $x = 0$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. If $f(x) = 0$ for each $x \in \mathbb{Q}$, show that then $f \equiv 0$ on $\mathbb{R}$.

7. Find a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous everywhere except for the set of positive integers.

6.3 Uniform Continuity, Lipschitzean Mappings

Let $(X,d),(Y,d')$ be 2 m.s. $A \subseteq X$ a set and $f : A \to Y$ be a function. To say that f is continuous on A means this:

$$\forall\, a \in A,\ \forall\, \varepsilon > 0,\ \exists\, \eta = \eta(\varepsilon, a) > 0 : \{\forall\, x \in A,\, d(x,a) < \eta,\, \Rightarrow d'(f(x), f(a)) < \varepsilon \quad \text{Eq 7.1}$$

Definition 6.3.1 *If in Eq 7.1 it is possible to choose η independent from $a \in A$, we get:* $\forall\, \varepsilon > 0,\ \exists\, \eta = \eta(\varepsilon) > 0 : \{\forall\, x \in A, \forall\, y \in A,\, d(x,y) < \eta \Longrightarrow d'(f(x), f(y)) < \varepsilon$

*If this last condition holds, then we say that f is **uniformly continuous** on A.*

Example 6.3.2 *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. Then f is continuous on $\mathbb{R}$, but f is not uniformly continuous on $\mathbb{R}$. Indeed for a contradiction assume that f is uniformly continuous on $\mathbb{R}$. So we have:*

$$\forall\, \varepsilon > 0,\ \exists\, \eta > 0,\ \{\forall\, x,y \in \mathbb{R}\ |x-y| < \eta \Rightarrow |f(x) - f(y)| < \varepsilon \qquad \text{Eq 7.2}$$

Let $n \geq 1$ be such that $\frac{1}{n} < \eta$. So if we take $x = n$ and $y = n + \frac{1}{n}$, then we see that $|x-y| = \frac{1}{n} < \eta$, but $|f(x) - f(y)| = |x^2 - y^2| = |n^2 - (n + \frac{1}{n})^2| = 2 + \frac{1}{n^2}$. Hence if we choose $0 < \varepsilon < 2$, then Eq 7.2 cannot hold.

Question: Why we need uniform continuity?

Let $f :]0,1] \to \mathbb{R}$, $f(x) = \frac{1}{x}$. Let $x_n = \frac{1}{x}$. Then $x_n \in\,]0,1]$ and $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and f is continuous on $]0,1]$. But $f(x_n) = n$ is not a Cauchy sequence. So a continuous function does not send in general Cauchy sequences to Cauchy sequences.

Proposition 6.3.3 *If $f : A \to Y$ is uniformly continuous on A, then for any Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ in A, $(f(x_n))_{n\in\mathbb{N}}$ is also Cauchy.*

Proof 6.3.4 *Let us write what we have:*

f is uniformly continuous on A: $\forall\, \varepsilon > 0\ \exists\, \eta > 0 \{\forall\, x \in A, \forall\, y \in A,\, d(x,y) < \eta \Rightarrow d'(f(x), f(y)) < \varepsilon$

$(x_n)_{n\in N}$ is Cauchy: $\forall\, \varepsilon' > 0\ \exists\, N \in \mathbb{N}\ \forall\, n \geq N,\ \forall\, m \geq N,\, d(x_n, x_m) < \varepsilon'$

Hence for $\varepsilon' = \eta$ we see that $d(x_n, x_m) < \eta$ for $n,m \geq N$ so that $d'(f(x_n), f(x_m)) < \varepsilon$ for $n,m \geq N$.

This shows that $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy.

Example 6.3.5 *Let $A =]0,1]$, $f : A \to \mathbb{R}$, $f(x) = \frac{1}{x}$. The Proposition 6.3.3 and the last example shows that f is not uniformly continuous on A.*

Theorem 6.3.6 ***(Heine)*** *If A is compact, then every continuous function $f : A \to Y$ is uniformly continuous on A.*

Proof 6.3.7 *Since f is continuous on A we have:* $\forall\ a \in A,\ \forall\ \varepsilon > 0,\ \exists\ \eta = \eta(\varepsilon, a) > 0,$

$\{\forall\ x \in A,\ d(x, a) \leq \eta, \Rightarrow d'(f(x), f(a)) < \varepsilon$

Fix $\varepsilon > 0$ *and let* $\eta_a = \eta(\varepsilon, a)$. *Then* $A \subseteq \cup_{a \in A} B_{\eta_{\frac{a}{2}}}(a)$. *As A is compact,*

$A \subseteq B_{\eta_{\frac{a_1}{2}}}(a_1) \cup \ldots \cup B_{\eta_{\frac{a_n}{2}}}(a_n)$ *for some* $a_1, \ldots, a_n \in A$. *Let* $\eta = \min\{\eta_{\frac{a_1}{2}}, \ldots, \eta_{\frac{a_n}{2}}\}$. *Then* $\eta > 0$. *Let now* $x, y \in A$ *be any 2 points such that* $d(x, y) < \eta$.

As $A \subseteq B_{\eta_{\frac{a_1}{2}}}(a_1) \cup \ldots \cup B_{\eta_{\frac{a_n}{2}}}(a_n)$, *$x$ is one of these balls, say* $x \in B_{\eta_{\frac{a_1}{2}}}(a_1)$. *Then both* $x, y \in B_{\eta_{a_1}}(a_1)$. *Then,*

$$d'(f(x), f(y)) < \underbrace{d'(f(x), f(a_1))}_{< \varepsilon} + \underbrace{d'(f(a_1), f(y))}_{< \varepsilon} < 2\varepsilon$$

This shows that f is uniformly continuous on A.

6.3.1 Exercises II

1. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as $f(x)=\begin{cases}0 & \text{if } x\in\mathbb{R}\backslash\mathbb{Q}\\ 1 & \text{if } x=0\\ \dfrac{1}{q} & \text{if } x\in\dfrac{p}{q} \text{ and } (p,q)=1\end{cases}$

   Show that f is continuous at every irrational number and discontinuous at every rational number.

2. Is there a function $f:\mathbb{R}\to\mathbb{R}$ which is continuous at every rational number and discontinuous at every irrational number?

3. Let (X,d) be a m.s., $f:X\to\mathbb{R}$ be a function and $g:X\to\mathbb{R}$ is given by

   $g(x)=\dfrac{f(x)}{1+|f(x)|}$.

   Let $x_0\in X$ be a point. Show that g is continuous at x_0 iff f is continuous at x_0.

4. Let X be any set and $f:X\to\mathbb{R}$ be a one-to-one function. Put, for $x,y\in X$,

   $d(x,y)=|f(x)-f(y)|$. Show that d is a metric on X and f is continuous for this metric.

5. A function $f:\mathbb{R}\to\mathbb{R}$ is said to be periodic if there is a number $p>0$ such that $f(x+p)=f(x)$ for all $x\in\mathbb{R}$. Show that every periodic continuous function is uniformly continuous on $\mathbb{R}$ and is bounded on $\mathbb{R}$.

6. Let (X,d) be a metric space, $A\subseteq X$ a set and $f:A\to\mathbb{R}$ be a function. Assume that f is continuous at a point $x_0\in A$.

   Show that there are $\delta>0$ and $M>0$ such that $|f(x)|\le M$ for every $x\in B_\delta(x_0)\cap A$.

   Thus every continuous function is locally bounded.

7. Let $f:]0,1[\to\mathbb{R}$ be defined by $f(x)=\begin{cases}0 & \text{if } x \text{ is irrational}\\ n & \text{if } x=\dfrac{n}{m} \text{ and } (m,n)=1\end{cases}$

   Prove that f is unbounded on every open interval $I\subseteq\,]0,1[$.

   Deduce from 6 that f is discontinuous at every $x\in\,]0,1[$

8. Prove that the function $f(x)=\sqrt{x}$ is uniformly continuous on $[0,\infty[$ and Lipschitzean on $[a,+\infty[$ for each $a>0$.

9. Let (X,d) be a m.s., $A\subseteq X$ a set and $f:A\to\mathbb{R}$ be a function which is uniformly continuous on A.

   Show that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A, then $(f(x_n))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$.

6.4 Uniform Extension Theorem

Let $f : A \to Y$ be a continuous function and $A \subseteq B \subseteq X$ be a set.

Problem: Can we extend f continuously to B i.e. given $f : A \to Y$ continuous, is there a continuous function $\tilde{f} : B \to Y$ such that $\tilde{f} = f$ on A?

Example 6.4.1 *Let $A = \mathbb{R}\backslash\{0\}$, $f(x) = \sin\frac{1}{x}$, $f : A \to \mathbb{R}$. Then f is continuous, and $\forall\, x \in A$, $|f(x)| \leq 1$, but as we have seen that f has no continuous extension to $\mathbb{R}$.*

Theorem 6.4.2 ***(Uniform Extension Theorem)*** *If (Y, d') is complete and $f : A \to Y$ is uniformly continuous on A, then there exists a unique uniformly continuous function. $\tilde{f} : \bar{A} \to Y$ that extends f.*

Proof 6.4.3 *Let $x \in \bar{A}$ be a point. We want to extend f continuously to x. How to define $\tilde{f} : \bar{A} \to \mathbb{R}$ at x?*

Let $x_n \in A$, $x_n \to x$. Since $x_n \to x$, $(x_n)_{n\in N}$ is Cauchy. As f is uniformly continuous on A, $(f(x_n))_{n\in N}$ is Cauchy. As (Y, d') is complete, $(f(x_n))_{n\in N}$ converges to some point $\alpha_x \in Y$.

Let us see that α_x does not depend on the sequence $(x_n)_{n\in N}$ that converges to x. To see this, let $(y_n)_{n\in\mathbb{N}}$ be another sequence in A that converges to x. Then for the same reasons as above, $(f(y_n))_{n\in N}$ also converges in Y to same $\beta_x \in Y$.

Is $\alpha_x = \beta_x$?

Let $(z_n)_{n\in N}$ be the mixture of $''(x_n)_{n\in N}$ and $(y_n)''_{n\in\mathbb{N}}$ i.e.

$$\begin{array}{ccccc} z_n = & x_0\ , & y_0, & x_1, & y_1, & \ldots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \\ & z_0 & z_1 & z_2 & z_3 & \end{array}$$

Then $z_n \in A$ and $z_n \to x$. Then $(f(z_n))_{n\in\mathbb{N}}$ is Cauchy, so $f(z_n) \to \gamma_x$ for some $\gamma_x \in Y$. As $(f(x_n))_{n\in N}$ and $(f(y_n))_{n\in\mathbb{N}}$ are subsequences of $(f(z_n))_{n\in N}$ $\alpha_x = \gamma_x$ and $\beta_x = \gamma_x$. So $\alpha_x = \gamma_x$. In particular, if $x \in A$, and if we take $x_0 = x_1 = \ldots = x_n = \ldots = x$, $\alpha_x = f(x) = \lim_{n\to\infty} f(x_n)$.

So we define $\tilde{f} : \bar{A} \to \mathbb{R}$ as $\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$ for any sequence $(x_n)_{n\in\mathbb{N}}$ in A that converges to $x \in \bar{A}$.

What we did above shows that $\tilde{f}$ is well-defined on $\bar{A}$. Moreover $\tilde{f}(x) = f(x)$, $\forall\, x \in A$. Hence $\tilde{f}$ is an extension of f to $\bar{A}$. Let us see that $\tilde{f}$ is uniformly continuous on $\bar{A}$. As f is uniformly continuous on A, we have:

$$\forall\, \varepsilon > 0, \exists\, \eta > 0, \forall\, x, y \in A,\, d(x, y) \leq \eta \Longrightarrow d'(f(x), f(y)) \leq \varepsilon \qquad \text{Eq 7.3}$$

Let us see that the same η works for $\tilde{f}$.

To see this, let $x, y \in \bar{A}$ be such that $d(x, y) < \eta$. Let x_n, $y_n \in A :\ x_n \to x$ and $y_n \to y$. Then, there is an $N \in \mathbb{N}$ such that for $n \geq N$, $d(x_n, y_n) < \eta$, since $d(x_n, y_n) \to d(x, y)$. Then by Eq 7.3, $d'(f(x_n), f(y_n)) < \varepsilon$. As $d(\grave{f}(x), \tilde{f}(y)) = \lim_{n\to\infty} d(f(x_n), f(y_n)) \leq \varepsilon$. So that $\tilde{f}$ is uniformly continuous on $\bar{A}$.

Uniqueness: *If $g : \bar{A} \to Y$ is another uniformly continuous function extending f, then $g(x) = \tilde{f}(x) = f(x)$, $\forall\, x \in A$. Hence if $x \in \bar{A}$ and $x_n \in A$, with $x_n \to x$, then $g(x) = \lim_{n\to\infty} f(x_n) = \tilde{f}(x)$. So $\tilde{f} = g$ on $\bar{A}$.*

Example 6.4.4 *Every uniformly continuous $f : \mathbb{Q} \to \mathbb{R}$ is the restriction of an uniformly continuous $\tilde{f} : \mathbb{R} \to \mathbb{R}$.*

Example 6.4.5 *The function $f(x) = \sin \frac{1}{x}$ is not uniformly continuous on $\mathbb{R} \setminus \{0\}$.*

Definition 6.4.6 • *Let $f : A \to Y$ be a function. If there is a number $k > 0$ such that $\forall x, y \in A$, $d'(f(x), f(y)) \leq kd(x, y)$, then we say that f is a **Lipschitzean function** on A.*

- *If $0 < k < 1$ then f is said to be a **contraction**.*
- *If $d'(f(x), f(y)) = d(x, y)$ $\forall x, y \in A$, then f is said to be an **isometry**.*

Example 6.4.7 *Let $f(x) = \sin x$, $f : \mathbb{R} \to \mathbb{R}$. Then $\sin(x) - \sin(y) = (x - y)\cos(c)$ for some c between x and y.*

So, $\sin(x) - \sin(y)| \leq |x - y|$. Hence $f(x) = \sin x$ is a Lipschitzean function.

More generally, if $f : [a, b] \to \mathbb{R}$ is differentiable and f' is bounded on $[a, b]$, then f is Lipschitzean.

Example 6.4.8 *Let $f :]0, \infty[\to]0, \infty[$, $f(x) = x + \dfrac{1}{x}$, then $|f(x) - f(y)| < |x - y|$ for $x \neq y$*

Example 6.4.9 *If $f : A \to Y$ is Lipschitzean, then f is uniformly continuous. Indeed, if for some $k > 0$ $d'(f(x), f(y)) \leq kd(x, y)$, then $\forall \varepsilon > 0$, $\exists\, \eta = \dfrac{\varepsilon}{k} : \forall x, y \in A$, $d(x, y) < \eta \Rightarrow d'(f(x), f(y)) \leq kd(x, y) \leq \varepsilon$*

The converse is false. The function $f : [o, \infty[\to [0, \infty[, f(x) = \sqrt{x}$ f is uniformly continuous but not Lipschitzean.

Theorem 6.4.10 ***(Banach Fixed Point Theorem):*** *Let (X, d) be a complete m.s. and $f : X \to X$ be a contraction. Then $\angle\exists! x \in X$ such that $f(x) = x$.*

Proof 6.4.11 *Let $x_0 \in X$ be any point. Then put $x_1 = f(x_0), x_2 = f(x_1), ..., x_n = f(x_n)$*

In this way, we obtain a sequence x_n. Let us see that this sequence is Cauchy.

First, $d(x_n, x_{n-1}) = d(f(x_{n-1}), f(x_{n-2})) \leq kd(x_{n-1}, x_{n-2})$, where $0 < k < 1$ is a constant independent from x_n's.

Hence $d(x_n, x_{n-1}) \leq k^{n-2} d(x_1, x_0)$. Then,

$$\begin{aligned} d(x_{n+p}, x_n) &\leq d(x_{n+p}, x_{n+p-1}) + ... + d(x_{n+1}, x_n) \\ &\leq (k^{n+p-2} + ... + k^{n-1}) d(x_1, x_0) = k^{n-1} d(x_1, x_0)[1 + k + ... + k^{p-1}] = k^{n-1} d(x_1, x_0) \frac{1 - k^p}{1 - k} \\ &\leq \frac{k^{n-1} d(x_1, x_0)}{1 - k} \end{aligned}$$

As $0 < k < 1$, $k^{n-1} \to 0$ as $n \to \infty$. Hence $d(x_{n+p}, x_n) \to 0$ for any $p \in N$ as $n \to \infty$. This shows that x_n is Cauchy. As our m.s. (X, d) is complete, x_n converges to some $x \in X$.

Now, $f(x_{n-1}) = x_n$ and since f is Lipschitzean, f is continuous on X. As $x_n = f(x_{n-1})$, letting $n \to \infty, x = f(x)$.

Uniqueness: *Suppose that, for some $y \in X, y = f(y)$ too. If $x \neq y$, then $f(x) \neq f(y)$. So $d(x,y) = d(f(x), f(y)) \leq kd(x,y)$. If $x \neq y$, then $d(x,y) \neq 0$, so $1 \leq k$. Contradiction, since $0 < k < 1$.*

Example 6.4.12 *$f(x) \sin x$, $f : [0, 2\pi] \to [0, \pi]$, $f(0) = 0$. So, $x = 0$ is a fixed point of f.*

Example 6.4.13 *$f(x) = e^x$, $f : \mathbb{R} \to \mathbb{R}$, then the equation $e^x = x$ has no solution. So, f has no fixed point.*

Application: Let $X = C[0,1] =$ the space of the continuous functions, $f : [0,1] \to \mathbb{R}$, with the supremum metric. $d(f,g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|$. The m.s. (X,d) is complete. Show that $C[0,1]$ is closed in $\mathbb{B}[0,1]$. Let $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $|F(t,x) - F(t,y)| \leq k|x-y| \; \forall x, y \in \mathbb{R}$ since $0 < k < 1$ independent from x, y and t.

Consider the differential equation:

$$\begin{cases} y'(t) & = F(t, y(t)) \; , \\ y(0) & = 0 \; . \end{cases}$$

Question: Is there a solution?

Theorem 6.4.14 *The equation system above has a solution.*

Proof 6.4.15 *Let $T : C[0,1] \to C[0,1]$ be the mapping defined by $T(y)(t) = \int_0^t F(s, y(s)) d_s$. Then, for $y, z \in C[0,1]$,*

$$\begin{aligned} |T(y)(t) - T(z(t))| & = |\textstyle\int_0^t F(s,y(s)) - F(s, z(s) d_s| \leq \int_0^t |F(s,y(s)) - F(s,z(s))| d_s \\ & \leq k \textstyle\int_0^t |y(s) - z(s)| d_s \leq \int_0^1 |y(s) - z(s)| d_s \\ & \leq k \sup_{0 \leq s \leq 1} |y(s) - z(s)| = kd(y,z) \end{aligned}$$

Then, $d(T(y), T(z)) \leq kd(y,z)$

Hence by the Banach Fixed Point Theorem, T has a unique fixed point y, $T(y)(t) = y(t)$. So $y(t) = \int_0^t F(s, y(s)) d_s$, hence $y'(t) = F(t, y(t))$ and $y(0) = 0$

Theorem 6.4.16 ***(Brewer Fixed Point Theorem):*** *Let $B = \{x \in R^n : ||x||_2 \leq 1\}$ be the closed unit ball of $\mathbb{R}^n$. Then every continuous f from B to B has a fixed point.*

6.4.1 The distance function

Let (X,d) be a m.s. $A \subseteq X$ a nonempty set and for $x \in X$. Let $d(x,y) = \inf_{y \in A} d(x,y)$. This is by definition, the distance of the point x to the set A. In this way, we define a function $f : X \to \mathbb{R}$, $f(x) = d(x,A)$. We want to study the properties of the function f.

Proposition 6.4.17 *Let $x \in X$ be given. Then $d(x,A) = 0 \Leftrightarrow x \in \bar{A}$*

(so that if $x \notin \bar{A}$, $d(x,A) > 0$)

Proof 6.4.18 *Suppose first that $x \in \bar{A}$, then $\exists\, x_n \in A : x_n \to x$. Now*
$d(x, A) = \inf_{y \in A}(x, y) \leq d(x, x_n)\ \forall n \in N$. *Hence* $d(x, A) = 0$ *(let $n \to \infty$)*

Conversely, suppose $\inf_{y \in A} d(x, y) = 0$*, so* $\forall\, n \geq 1$, $\exists\, y_n \in A$, $\ni\ d(x, y_n) < \dfrac{1}{n}$*. Hence* $y_n \to x$*. So,* $x \in \bar{A}$*.*

Proposition 6.4.19 $\forall x_1, x_2 \in X$, $|d(x_1, A) - d(x_2, A)| \leq d(x_1, x_2)$ *(so $f(x) = d(x, A)$ is a Lipschitzean function on X.)*

Proof 6.4.20 *For any $y \in A$, $d(x_1, y) \leq d(x_1, x_2) + d(x_2, y)$ so that*
$\inf_{y \in A} d(x, y) \leq d(x_1, x_2) + \inf d(y_2, x_2) \Longrightarrow d(x_1, A) \leq d(x_1, x_2) + d(x_2, A)$.
Hence, $d(x_1, A) - d(x_2, A) \leq d(x_1, x_2)$.
Changing x_1 and x_2, we get $d(x_2, A) - d(x_1, A) \leq d(x_1, x_2)$.

Proposition 6.4.21 *If A is compact, then $\forall x \in A$, there is a $y_0 \in A, \ni d(x, A) = d(x, y_0)$.*

Proof 6.4.22 *Let $f : A \to \mathbb{R}$, $f(y) = d(x, y)$, f is continuous on X. Hence if A is compact, f attains its maximum on A, i.e. $\exists\, y_0 \in A : f(y_0) = \inf_{y \in A} d(x, y)$, i.e. $d(x, y_0) = d(x, A)$.*

6.4.2 Distance Between Two Points

Let (X, d) be a m.s. A, B be 2 nonempty sets. We define the distance between two sets $d(A, B)$ as $d(A, B) = \inf_{x \in B} d(x, A) = \inf_{y \in A,\, x \in B} d(x, y)$

Example 6.4.23 *Even if A, B are closed and disjoint, we may have $d(A, B) = 0$.*
Let $X = \mathbb{R}^2$, $d =$ the Euclidean metric, $A = \{(x, \frac{1}{x}) : x > 0\}$ and $B = \mathbb{R} \setminus \{0\}$. Then A, B are closed and disjoint and $d(A, B) = \inf\{\sqrt{(x-a)^2 + (\frac{1}{x} - 0)^2} : x > 0, a \in \mathbb{R}\}$. Then $d(A, B) \leq d((n, \frac{1}{n}), (n, 0)) = \dfrac{1}{n}$, $\forall\, n > 0$. This being true for each n, we get $d(A, B) = 0$.

However,

Theorem 6.4.24 *If A is compact, B is closed and $A \cap B = \emptyset$, then $d(A, B) > 0$.*

Proof 6.4.25 *Let $f : A \to \mathbb{R}$, $f(x) = d(x, B)$. This is a continuous function. By the Proposition 6.4.21, $d(x, B) > 0$, $\forall x \in A$. As A is compact f attains its minimum: $\exists\, x_0 \in A \ni f(x_0) = \inf_{x \in A} d(x, B) = d(A, B)$. So, $d(A, B) = f(x_0) = d(x_0, B) > 0$.*

Example 6.4.26 *We use this proposition in the following form:*
Let $O \subseteq X$ be an open set and $K \subseteq O$ be a compact set. Then $d(K, \delta O) > 0$.
e.g. Let $X = \mathbb{R}^2$, let $O = B_1(0)$, $K \subseteq O$ be any compact set. Show that for some $0 < r < 1$, $K \subseteq B_r(0)$.

Example 6.4.27 *$f : \mathbb{R} \to \mathbb{R}$ be the Dirichlet function* $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{otherwise.} \end{cases}$

f is continuous iff $\forall\, x_0 \in \mathbb{R}, \forall x_n \in \mathbb{R},\ x_n \to x_0 \Rightarrow f(x_n \to f(x_0))$.
So this function is not continuous as $f|_{\mathbb{Q}} = 1$ and $f|_{\mathbb{R}\setminus\mathbb{Q}} = 0$

Lemma 6.4.28 ***(Urysohn Lemma):*** *Let (X, d) be any m.s., A, B 2 nonempty disjoint sets. Then $\exists$ a function $f : X \to \mathbb{R}$, continuous such that;*

1. *$\forall x \in X,\ 0 \leq f(x) \leq 1$.*
2. *$\forall x \in A,\ f(x) = 0$.*
3. *$\forall x \in B,\ f(x) = 1$.*

Proof 6.4.29 *For $x \in X$, let* $f(x) = \dfrac{d(x, A)}{d(x, A) + d(x, B)}$

As $A \cap B = \emptyset$ and A, B are closed $d(x, A) + d(x, B) \neq 0, \forall x \in X$. Hence f is continuous on $\mathbb{R}$, as the quotient of two continuous functions is continuous provided that the denominator is not zero.
It is clear that $0 \leq f(x) \leq 1$, for $x \in A, f(x) = 0$ and for $x \in B, f(x) = 1$.

6.4.3 F_σ-sets, G_δ-sets in $\mathbb{R}$

Consider the sets $[a, b], (a, b)$, one is closed and the other is open set. And we can write them as:
$[a, b] = \cap_{n\geq1}]a - \frac{1}{n}, b + \frac{1}{n}[$, and $(a, b) = \cup_{n\geq1} [a + \frac{1}{n}, b - \frac{1}{n}]$

Questions:

1. Can we write every closed set as a union of countable open sets?
2. Can we write every open set as a union of countable closed sets?

Definition 6.4.30 *Let (X, d) be a m.s. and $E \subseteq X$ a set. We say that;*

1. *E is a G_δ**-set**, if it is possible to write E as the intersection of countably many open sets. (i.e. $E = \cap_{n\in\mathbb{N}} O_n$, O_n open)*
2. *E is said to be a F_σ**-set**, if it is possible to write E as the union of countably many closed sets. (i.e. $E = \cup_{n\in\mathbb{N}} F_n$, F_n closed.)*

Obviously,

1. E is a G_δ-set $\Leftrightarrow$ E^c is an F_δ-set
2. Every open set O is a G_δ-set

3. Every closed set F is a F_δ-set

4. Every countable subset $E \subseteq X$ is a F_δ-set. So E^c is a G_δ-set.

Notation: Let (X,d) be a m.s. $A \subseteq X\ (\neq \emptyset)$ any set and $\varepsilon > 0$. Let
$B_\varepsilon(A) = \{x \in X : d(x,A) < \varepsilon\}$
$B'_\varepsilon(A) = \{x \in X : d(x,A) \leq \varepsilon\}$

As the function $f : X \to \mathbb{R}$, $f(x) = d(x,A)$ is continuous, $B_\varepsilon(A) = f^{-1}(]-\infty, \varepsilon[)$ is open and $B'_\varepsilon(A) = f^{-1}(]-\infty, \varepsilon])$ is closed. Also $A \subseteq B_\varepsilon(A) \subseteq B'_\varepsilon(A)$. If, in $\mathbb{R}$, we take, $A = [a,b]$ and $\varepsilon = \frac{1}{n}$, then $B_{\frac{1}{n}}(A) = \{x \in R : d(x,A) < \frac{1}{n}\} =]a - \frac{1}{n}, b + \frac{1}{n}[$.

Theorem 6.4.31 ***a)*** *If $A \subseteq X$ is closed, then $A = \cap_{n\geq 1} B_{\frac{1}{n}}(A)$*
b) *If $A \subseteq X$ is open, then $A = \cup_{n\geq 1}(B_{\frac{1}{n}}(A^c))^c$*

Proof 6.4.32 *It is enough to prove a.*
(a)($\Rightarrow$) *The inclusion $A \subseteq \cap_{n\geq 1} B_{\frac{1}{n}}(A)$ is clear.*
($\Leftarrow$) *Let $x \notin A$. As A is closed, there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \cap A = \emptyset$. In particular, $d(x,A) \geq \varepsilon$. If $\frac{1}{n} < \varepsilon$, then $x \notin B_{\frac{1}{n}}(A)$. Hence $x \notin \cap_{n\geq 1} B_{\frac{1}{n}}(A)$. Hence $A = \cap_{n\geq 1} B_{\frac{1}{n}}(A)$.*

Conclusion: In any m.s. (X,d)
a) Every closed set can be written as an intersection of countably many open sets.(G_δ-set)
b) Every open set can be written as an union of countably many closed sets.(F_σ-set)

Question: In $\mathbb{R}$, we have seen that $\mathbb{Q}$ is an F_σ-set and $\mathbb{R} \setminus \mathbb{Q}$ is G_δ-set. Is $\mathbb{Q}$ a G_δ-set? i.e. is it possible to write $\mathbb{Q}$ as $\mathbb{Q} = \cap_{n\geq 1} O_n$, $O_n \subseteq \mathbb{R}$ open.
This is **not** the case.

Remark: Let f be any function. ($f : \mathbb{R} \to \mathbb{R}$). Let $C_f = \{x \in \mathbb{R} : f \text{ is continuous at } x\}$.
If $f(x) = \sin x$, then $C_f = \mathbb{R}$.
If $f(x) = \chi_{\mathbb{Q}}$, then $C_f = \emptyset$.
If $f(x) = \sin \frac{1}{x}$, then $C_f = \mathbb{R} \setminus \{0\}$.

Now let $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 0 & \text{if } x = 0, \\ \frac{1}{m} & \text{if } x = \frac{n}{m} \text{ and } x \in \mathbb{Q}\ (n,m) = 1 \end{cases}$

Let $x_0 \in \mathbb{R}$ be a given point. Then f is continuous at $x_0 \Leftrightarrow x_0 \in \mathbb{R} \setminus \mathbb{Q}$ i.e. $C_f = \mathbb{R} \setminus \mathbb{Q}$

Question: Is there a function $f : \mathbb{R} \to \mathbb{R}$ such that $C_f = \mathbb{Q}$?

Theorem 6.4.33 *Let $A \subseteq \mathbb{R}$ be a set ($\neq \emptyset$). Then there is a function $f : \mathbb{R} \to \mathbb{R}$ such that $C_f = A$ iff A is a G_δ-set.*

So, since $\mathbb{Q}$ is not a G_δ-set, there is no function $f : \mathbb{R} \to \mathbb{R}$ which is continuous at every rational x_0, and discontinuous at every irrational x_0.

6.5 Completion of a m.s. (X, d)

Question: Given any non-complete m.s. (X, d), is there a complete m.s. $(\tilde{X}, \tilde{d})$ such that:
1) $X \subseteq \tilde{X}$ and $\overline{X} = \tilde{X}$
2) For $x, y \in X$, $\tilde{d}(x, y) = d(x, y)$

Theorem 6.5.1 *Let $E \neq \emptyset$ be any set and $\mathbb{B}(E) = \{\varphi : E \to \mathbb{R} : \varphi$ is bounded $\}$ be the space of all the bounded functions $\varphi : E \to \mathbb{R}$ with the metric $d_\infty(\varphi, \psi) = \sup |\varphi(x) - \psi(x)|$. Then the m.s. $(\mathbb{B}(E), d_\infty)$ is complete.*

Proof 6.5.2 *Let $(\varphi_n)_{n\in\mathbb{N}}$ be a Cauchy sequence, i.e.*
$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N,\ d_\infty(\varphi_n, \varphi_m) = \sup |\varphi_n - \varphi_m| < \varepsilon$ *(*)*
We have to show that, for some $\varphi \in \mathbb{B}(E)$, $d_\infty(\varphi_n, \varphi) \to 0$ as $n \to \infty$.

From (*) above we see that for each $x \in E$, the sequence $(\varphi_n(x))_{n\in\mathbb{N}}$ is Cauchy in $\mathbb{R}$. As $(\mathbb{R}, d)$ is complete, this sequence converges to some number $\alpha_x \in \mathbb{R}$.

Let $\varphi : X \to \mathbb{R}$ be the function $\varphi(x) = \alpha_x$. Let us see that;
1) φ is bounded on $\mathbb{R}$
2) $d_\infty(\varphi_n, \varphi) \to 0$ as $n \to \infty$.

In (*) fix $n \geq N$ and let $m \to \infty$, then we get;
$\sup_{x\in E} |\varphi_n(x) - \varphi(x)| \leq \varepsilon \Rightarrow ||\varphi_n(x)| - |\varphi(x)|| \leq |\varphi_n(x) - \varphi(x)| \leq \varepsilon.$

Hence $\sup_{x\in E} |\varphi(x)| \leq \varepsilon + \sup_{x\in E} |\varphi_n(x)| < \varepsilon + M$, where $M = \sup_{x\in E} |\varphi_n(x)|$. So that φ is bounded. Hence $\varphi \in \mathbb{B}(E)$.

Let $x \in E$ be any point, since $\varphi_n(x) \to \varphi(x)$, i.e.
$\forall\varepsilon > 0\, \exists M \in \mathbb{N},\ \forall m \geq M,\ |\varphi_m(x) - \varphi(x)| \leq \varepsilon$ *(**)*

Let N be as in (*), let $n \geq N$, then choose an $m \ni\ m = max\{N, M\}$. Then;
$|\varphi_n(x) - \varphi(x)| \leq |\varphi_n(x) - \varphi_m(x)| + |\varphi_m(x) - \varphi(x)| < 2\varepsilon.$

As N is independent from $x \in E$, $\sup_{x\in E} |\varphi_n(x) - \varphi(x)| \leq 2\varepsilon$.
That is $d_\infty|\varphi_n, \varphi| \to 0$, as $n \to \infty$. Hence $(\mathbb{B}(E), d_\infty)$ is complete.

Remark: Let (X, d) be any metric space $K \subseteq X$ be a compact set and
$C(K) = \{\varphi : K \to \mathbb{R} : \varphi$ is continuous$\}$.
Since any $\varphi \in C(K)$ is bounded (since $\varphi(K)$ is compact in $\mathbb{R}$), we see that $C(K) \subseteq \mathbb{B}(K)$.
Hence to prove that the m.s. $(C(K), d_\infty)$ is complete, it is enough to show that $C(K)$ is closed in $\mathbb{B}(K)$.

Definition 6.5.3 *Let (X, d) be a m.s. A metric space $(Y, d')$is said to be a* ***completion of*** *(X, d) if:*

1. *(Y, d') is complete*
2. *There is an isometry $h : X \to Y$ such that $\overline{h(X)} = Y$*

Example 6.5.4 *Consider* $X = \mathbb{Q}$, $d(x,y) = |x-y|$. *Then* $j : \mathbb{Q} \to \mathbb{R}$, $j(x) = x$ $\hat{X} = \mathbb{R}$, $\hat{d}(x,y) = |x-y|$ *and* $\overline{j(\mathbb{Q})} = \mathbb{R}$.

$(\mathbb{Q}, d)$ *is not complete but* $(\tilde{\mathbb{Q}}, \tilde{d})$, *i.e.* $(\mathbb{R}, d)$ *is complete.*

Let (X,d) be any m.s. We are going to show that there is a complete m.s. $(\hat{X}, \hat{d})$ and an isometry $\jmath : X \to \hat{X}$ such that $\overline{j(X)} = \hat{X}$

Theorem 6.5.5 *Given any m.s.* (X,d), *there is a complete m.s.* $(\hat{X}, \hat{d})$ *and an isometry* $j : X \to \hat{X}$ *such that* $j(X) = \hat{X}$. *The space* $(\hat{X}, \hat{d})$ *is unique up to an isometry.*

Proof 6.5.6 *Let* $\mathbb{B}(X) = \{f : X \to \mathbb{R} : f$ *is bounded*$\}$ *be the space of bounded functions with the metric* $d_\infty(f,g) = \sup_{x\in X}|f(x) - g(x)|$. *We know that* $(\mathbb{B}(X), d_\infty)$ *is a complete m.s.*

Now, fix a point $a \in X$. *For any* $y \in X$, *let* $f_y : X \to \mathbb{R}$ *be the function defined by* $f_y(x) = d(a,x) - d(x,y)$. *Then* $|f_y(x)| = |d(a,x) - d(x,y)| \le d(a,y)$. *So,* f_y *is bounded on* X. *So,* $f_y \in \mathbb{B}(X)$.

Let $j : X \to \mathbb{B}(X)$ *be defined* $j(y) = f_y$. j *is an isometry i.e.* $d_\infty(f_y, f_{y'}) = d(y,y')$ *i.e.* $\sup_{x\in X}|f_y(x) - f_{y'}(x)| = d(y,y')$, *since*

$$|f_y(x) - f_{y'}(x)| = |d(a,x) - d(x,y) - d(a,x) + d(x,y')| = |d(x,y) - d(x,y')| \le d(y,y').$$

Hence $\sup |f_y(x) - f_{y'}(x)| \le d(y,y')$. *Then for* $y = y'$, $|f_y(x) - f_{y'}(x)| = d(y,y')$. *Hence* $\sup |f_y(x) - f_{y'}(x)| = d(y,y')$. *So,* $j : X \to \mathbb{B}(X)$ *is an isometry. Let* $\hat{X} = \overline{j(X)}$ *and* $\hat{d} = d'$ *on* $\hat{X}$. *Then* $(\hat{X}, \hat{d})$ *is a complete m.s. and* $\overline{j(X)} = \hat{X}$.

Note that: *In any complete m.s.* (Y,d), *if* $M \subseteq Y$ *closed, then* (M,d) *is complete.*

Uniqueness: *Let* $(\tilde{Y}, \tilde{d})$ *be another complete metric space and* $i : X \to \tilde{Y}$ *an isometry with* $\overline{i(X)} = \tilde{Y}$. *In* $i : X \to \tilde{Y}$ *consider* X *as a subspace of* $\hat{X}$. *In* $j : X \to \hat{X}$ *consider* X *as a subspace of* $\tilde{Y}$. *Then* i *has a uniformly continuous extension* $i^* : \hat{X} \to \tilde{Y}$. *And* j *has a uniformly continuous extension* $j^* : \tilde{Y} \to \tilde{X}$.

Both i^* *and* j^* *are isometries:* $i^* \circ j^* : \tilde{Y} \to \tilde{Y}$ *is an identity on* $\tilde{Y}$.

Example 6.5.7 *Let* $X = C_{00}$ *with the metric* $d_\infty(x,y) = \sup_{n\in\mathbb{N}}|x_n - y_n|$ *where the space is:*

$$\begin{aligned} C_{00} &= \{\varphi : \mathbb{N} \to \mathbb{R} : \varphi \text{ is almost finite}\} \\ &= \{\varphi : \varphi(n) = x_n = 0 \text{ for all but finitely many } n \in \mathbb{N}\} \end{aligned}$$

Let $\hat{X} = C_0 = \{\varphi : \mathbb{N} \to \mathbb{R} : \lim_{n\to\infty} \varphi(n) = 0\}$ *with the supremum metric. Then:*

(C_0, d_∞) *is complete,* $C_{00} \subseteq C_0$ *and* $\overline{C_{00}} = C_0$ *as* $j : X \to \hat{X}$, $j(x) = x$ *is an isometry.*

6.5.1 Equivalence of Metrics

Let (X, d_1) be a m.s. and d_2 be another metric on X.

Question: If we place d_1 by d_2, what we gain, what we loose?

Example 6.5.8 *Let on define the metrics;* $d(x,y)=|x-y|,\ d_1(x,y)=|x^3-y^2|$

$d_2(x,y)=|Arc\tan x - Arc\tan y|,\ d_3(x,y)=\dfrac{|x-y|}{1+|x-y|}$

For all these metrics, $|x_n-x|\to 0 \Leftrightarrow d_i(x_n,x)\to 0$ *is the same.*

But being bounded change;

Let $x_n=n$ *then* $d_2(x_n,x_m)=|Arc\tan x_n - Arc\tan x_m|\to 0$ *as* $n,m\to\infty$*, so it is Cauchy for this metric, but not Cauchy for* d *for example.*

Let $f:(X,d_1)\to(X,d_2)$ be identity mapping $f(x)=x$. Then;

1) f is continuous iff whenever $d_1(x_n,x)\to 0$ we have $d_2(x_n,x)\to 0$.

2) f is a homeomorphism iff $d_1(x_n,x)\to 0 \Leftrightarrow d_1(x_n,x)\to 0$

Definition 6.5.9 *A property (p) on a m.s.* (X,d) *is said to be* ***topological property*** *if it is definable in terms of open sets e.g. to be compact, separable,closed sets, convergent sequences...*

Definition 6.5.10 *There are 3 types of equivalences:*

a) d_1 *and* d_2 *are* ***topologically equivalent*** *if* f *is a homeomorphism. (i.e.* f *and* f^{-1} *are both continuous) In this, for any sequence* $x_n\in X$ *and* $x\in X$ $d_1(x_n,x)\to 0\Leftrightarrow d_2(x_n,x)\to 0$.

So, in this case, the spaces $(X,d_1),(X,d_2)$ *have the same topological properties. Thus, if we replace* d *by a topological equivalent metric* d_2*, then we do not lose any topological property, but we may lose non-topological properties such as completeness.*

b) d_1 *and* d_2 *are* ***uniformly equivalent*** *if both* f *and* f^{-1} *are uniformly continuous. If* d_1,d_2 *are uniformly equivalent then they are topologically equivalent. If we replace a metric* d_1 *by a uniformly equivalent metric* d_2*, then we do not lose any topological properties, we do not lose completeness, but it may happen that a set* $A\subseteq X$ *which is bounded for* d_1*, is not bounded for* d_2*, vice versa.*

c) d_1 *and* d_2 *are said to be* ***equivalent*** *if both* f *and* f^{-1} *are Lipschitzean. That is* d_1 *and* d_2 *are equivalent* $\Leftrightarrow \exists\ \alpha>0,\ \beta>0:\alpha d_1(x,y)\le d_2(x,y)\le\beta d_1(x,y)\ \forall\ x,y\in X$*. If we replace a metric* d_1 *by an equivalent metric* d_2 *we lose* <u>*almost*</u> *nothing. However a function* $f:X\to X$ *may be a* <u>*contraction*</u> *with respect to one of these metric but not w.r.t. other metric.*

Example 6.5.11 *Let* $X=\mathbb{R},\ d_1(x,y)=|x-y|,\ d_2(x,y)=|Arc\tan x - Arc\tan y|$*. Then for any* $x_n\in X$ *and* $x\in X,\ |x_n-x|\to 0\Leftrightarrow|Arc\tan x - Arc\tan y|\to 0$*. So that* d_1,d_2 *are topologically equivalent. But* $(\mathbb{R},d_1)$ *is complete,* $(\mathbb{R},d_2)$ *is not complete.*

Example 6.5.12 *Let* $X=\mathbb{R},\ d_1(x,y)=|x-y|,\ d_2(x,y)=\min\{1,|x-y|\}\le 1$*. Then for* $x_n\in\mathbb{R},\ x\in\mathbb{R}.\ d_1(x_n,x)\to 0\Leftrightarrow d_2(x_n,x)\to 0$*. Hence* d_1,d_2 *are topologically equivalent.* $\mathbb{R}$ *is not bounded for* d_1*, but* $\mathbb{R}$ *is bounded for* d_2*.*

Example 6.5.13 *Let* $X=\mathbb{R},\ d_1(x,y)=|x-y|,\ d_2(x,y)=\min\{1,|x-y|\}$*. Then* d_1 *and* d_2 *are uniformly equivalent. A sequence* $(x_n)_{n\in\mathbb{N}}$ *in* $\mathbb{R}$ *which is Cauchy w.r.t.* d_1 *iff it is Cauchy w.r.t.* d_2*. But the spaces* $(\mathbb{R},d_1)$ *and* $(\mathbb{R},d_2)$ *do not have the same bounded sets.*

Example 6.5.14 *Let* $X = \mathbb{R}^n$*, then any 2 of the metrics* $(d_p)_{1 \leq p \leq \infty}$ *are equivalent.*
But $B_1(0)$ *for* d_∞, $B_1(0)$ *for* d_2 *and* $B_1(0)$ *for* d_1 *are different geometrically.*

Chapter 7

Limit

1. **Definition and Existence of Limit**
2. **Limit from the left, from the right**
3. **Continuity of Monotone functions**
4. **Functions of Bounded Variation**
5. **Absolutely Continuous Functions**

In this chapter, we will work in the following setting:

$(X, d), (Y, d')$ are 2 m.s., $A \subseteq X$ a set, $a \in \bar{A}$ a point, and $f : A \to Y$ a function. The point a may or may not be in A and f need not to be defined at the point a.

7.1 Definition and Existence of Limit

Definition 7.1.1 *We say that* $\lim_{x\to a} f(x)$ *exists if;*

$\exists\ L \in Y$ *such that* $\forall\ \varepsilon > 0,\ \exists\ \eta > 0, \forall x \in A \cap B_\eta(a) :\ d'(f(x), L) < \varepsilon.$

In this case we write $L = \lim_{x\to a,\, x\in A} f(x)$ *(=limit of* $f(x)$ *as* x *goes to* a *while remaining in the set* A*). Then, the set* A *"determines" the way* x *goes to* a*.*

Example 7.1.2 *Let* $X = Y = \mathbb{R},\ A =]-1, 0[\cup]0, 1[,\ a = 0$ *(So,* $a \in \bar{A}$*, but also* $0 \in \overline{]-1, 0[}$ *and* $0 \in \overline{]0, 1[}$*)*

$$\text{Let } f : A \to \mathbb{R},\ f(x) = \begin{cases} -1 & \text{if } x \in]-1, 0[, \\ 1 & \text{if } x \in]0, 1[. \end{cases}$$

Then, $\lim_{x\to 0,\, x\in A} f(x)$ *does not exist. Indeed, otherwise we would have some* $L \in \mathbb{R}$ *satisfying the definition of limit:* $\forall\ \varepsilon > 0, \exists\ \eta > 0 : \forall\ x \in]-\eta, \eta\,[\cap A \Longrightarrow |f(x) - L| < \varepsilon$

Let $x \in]-\eta, \eta\,[\cap A,\ x > 0$*. Then* $f(x) = 1$*, so* $|1 - L| < \varepsilon$*. Then, let* $x \in]-\eta, \eta\,[\cap A,\ x < 0$*. Then* $f(x) = -1$*, so* $|-1 - L| < \varepsilon$*. So, for* $\varepsilon = \frac{1}{2}$*, this is not possible.*

Now, if we take $A =]0, 1[$ *then* $\lim_{x\to 0,\, x\in A} f(x) = 1$*. So, existence depends on the set.*

Example 7.1.3 *Let* $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$, $f(x,y) = \dfrac{xy}{x^2+y^2}$. *Here* $A = \mathbb{R}^2 \setminus \{(0,0)\}$ *and* $a = (0,0)$. *Does* $\lim_{x,y\to(0,0),\, x,y\neq(0,0)} f(x,y)$ *exists?*

- *Let us go to* $(0,0)$ *following the diagonal* $y = x$. *Then on this line* $f(x,y) = \dfrac{x^2}{x^2+y^2} = \dfrac{1}{2}$.

So that $\lim_{(x,y)\to(0,0),\, x=y} f(x,y) = \dfrac{1}{2}$

- *Now, let us go to* $(0,0)$ *following the* x*-axis. On* x*-axis,* $y = 0$. *So* $f(x,y) = \dfrac{x \times 0}{x^2+0} = 0$.

So that $\lim_{(x,y)\to(0,0),\, (x,y)\in\mathbb{R}} f(x,y) = 0$

- *If* $L = \{y = kx : (k > 0),\ k \in \mathbb{Z}\}$ *and if we go to* $(0,0)$ *following* L*, then,*

$$\lim_{(x,y)\to(0,0),\, (x,y)\in L} f(x,y) = \lim_{(x,y)\to(0,0),\, (x,y)\in L} f(x,y) = \frac{kx^2}{kx^2+x^2} = \frac{k}{k^2+1}$$

Thus, the limit depends on the way we go to $(0,0)$.

The next proposition says that $\lim_{(x,y)\to(0,0),\, (x,y)\neq(0,0)} f(x,y)$ does not exist!

Proposition 7.1.4 *Let* $A \subseteq (X,d)$ *be any set,* $a \in A$ *be a point and* $f : A \to \mathbb{R}$ *be a function. Then,*

1. *If* $\lim_{x\to a,\, x\in A} f(x)$ *exists, it is unique.*
2. *If* $B \subseteq A$, $a \in \bar{B}$ *also, then if* $\lim_{x\to a,\, x\in A} f(x) = L$ *then* $\lim_{x\to a,\, x\in B} f(x) = L$ *too.*

Proof 7.1.5

1. *Suppose* $f(x) \to L$ *and* $f(x) \to S$*, as* $x \in A$, $x \to a$ *and* $L \neq S$.

   Let $\varepsilon = \dfrac{d(L,S)}{2}$. *Then* $B_\varepsilon(L) \cap B_\varepsilon(S) = \emptyset$.

   Since $f(x) \to L$, $\exists\ \eta_1 > 0 :\ f(B_{\eta_1}(a) \cap A) \subseteq B_\varepsilon(L)$

   And as $f(x) \to S$, $\exists\ \eta_2 > 0 : f(B_{\eta_2}(a) \cap A) \subseteq B_\varepsilon(S)$

   Let $\eta = \inf\{\eta_1, \eta_2\}$. *Then,* $(B_\eta(a) \cap A) \subseteq B_\varepsilon(L) f(B_\eta(a) \cap A) \subseteq B_\varepsilon(S)$

   But $a \in \bar{A} \Rightarrow f(B_\eta(a) \cap A) \neq \emptyset \Rightarrow B_\varepsilon(L) \cap B_\varepsilon(S) \neq \emptyset$*, contradiction.*

2. *If* $\lim_{x\to a,\, x\in A} f(x) = L$. *Then we have:* $\forall\ \varepsilon > 0\ \exists\ \eta > 0 : f(B_\eta(a) \cap A) \subseteq B_\varepsilon(L)$. *Then, a fortiori,* $f(B_\eta(a) \cap B) \subseteq B_\varepsilon(L)$. *So* $\lim_{x\to a,\, x\in B} f(x) = L$

Theorem 7.1.6 ***(Existence of Limit):*** $\lim_{x\to a,\, x\in A} f(x)$ *exists iff for any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* A *converging to* a, $\lim_{n\to\infty} f(x_n)$ *exists.*

Proof 7.1.7 $(\Rightarrow)$ *Suppose* $\lim_{x\to a,\, x\in A} f(x) = L$ *exists. So, we have;*
$\forall\ \varepsilon\ \exists\ \eta > 0 : \forall\, x \in B_\eta(x) \cap A \Longrightarrow d'(f(x), L) < \varepsilon$.
Now, let $x_n \in A$ *be any sequence in* A *that converges to* a. *So we have* $\forall\ \varepsilon' > 0$ *(take* $\varepsilon' = \eta$*)* $\exists\ N \in \mathbb{N}$, $\forall\ n \geq N$, $x_n \in B_\eta(a)$. *Here* $d'(f(x_n), L) < \varepsilon\ \forall n \geq N$ *i.e.* $(f(x_n)) \to L$*, as* $n \to \infty$.

($\Leftarrow$) *Conversely, suppose that, for any sequence* $(x_n)_{n\in\mathbb{N}}$ *in* A *converging to* a $\lim_{n\to\infty} f(x_n)$ *exists.*

Let us first see that if $x_n \in A$*, with* $x_n \to a$*; and* $y_n \in A$ *with* $y_n \to a$*, then we have* $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} f(y_n)$*. To see this let* $L_1 = \lim_{n\to\infty} f(x_n)$*,* $L_2 = \lim_{n\to\infty} f(y_n)$*. Then mix up the sequences* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *to get a sequence* $(z_n)_{n\in\mathbb{N}}$ *as follows:*

$x_0, y_0, x_1, y_1, x_2, y_2, \ldots, x_n, y_n, \ldots$

Then $z_n \in A$ *and* $z_n \to a$ *too. So,* $L_3 = \lim_{n\to\infty} f(z_n)$ *exists. As* $(f(x_n))_{n\in\mathbb{N}}$ *and* $(f(y_n))_{n\in\mathbb{N}}$ *are subsequences of* $(f(z_n))_{n\in\mathbb{N}}$ $L_1 = L_2 = L_3$*. Hence* $\lim_{n\to\infty} f(x_n)$ *does not depend on the sequence* $(x_n)_{n\in\mathbb{N}}$ *chosen.*

Now, let $(x_n)_{n\in\mathbb{N}}$ *be any sequence in* A *that converges to* a*. Then let* $L = \lim_{n\to\infty} f(x_n)$*. Let us see that* $\lim_{x\to a,\, x\in A} f(x) = L$*. If not, we would have:* $\exists\ \varepsilon > 0\ \forall\ \eta > 0 : \exists\ x_\eta \in B_\eta(a) \cap A : d'(f(x_\eta), L) \geq \varepsilon$*. Let* $\eta = \frac{1}{n}$ *and denote by* x_n *the corresponding* x_η*. Then* $x_n \in B_{\frac{1}{n}}(a) \cap A$*. So* $x_n \in A$ *and* $x_n \to a$*.*

Hence $\lim_{n\to\infty} f(x_n) = L$ *but* $d'(f(x_n), L) \geq \varepsilon\ \forall n \geq 1$*, contradiction.*

So $\lim_{x\to a,\, x\in A} f(x) = L$*.*

Proposition 7.1.8 *Let* $f, g : A \to \mathbb{R}$ *be two functions. Suppose that* $\lim_{x\to x_0,\, x\in A} f(x)$ *and* $\lim_{x\to x_0,\, x\in A} g(x)$ *exists. Then;*

1. $\lim_{x\to x_0,\, x\in A}(f(x) + g(x))$ *exists and is equal to* $\lim_{x\to x_0,\, x\in A} f(x) + \lim_{x\to x_0,\, x\in A} g(x)$
2. $\lim_{x\to x_0,\, x\in A}(f(x)g(x))$ *exists and is equal to* $\lim_{x\to x_0,\, x\in A} f(x) \lim_{x\to x_0,\, x\in A} g(x)$
3. *If* $g(x) \neq 0$ *and* $\lim_{x\to x_0,\, x\in A} g(x) \neq 0$*, then* $\lim_{x\to x_0,\, x\in A} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to x_0,\, x\in A} f(x)}{\lim_{x\to x_0,\, x\in A} g(x)}$

7.1.1 Cauchy Condition For Limit

Let $f : A \to Y$ be a function. We say that f satisfies the **Cauchy Condition** at a if we have: $\forall\ \varepsilon > 0\ \exists\ \eta > 0\ \forall\ x, y \in B_\eta(a) \cap A,\ d'(f(x), f(y)) < \varepsilon$

Theorem 7.1.9 *Suppose that* (Y, d') *is complete. Then* $\lim_{x\to a,\, x\in A} f(x) = L$ *exists iff* f *satisfies the Cauchy condition at* a*.*

Proof 7.1.10 ($\Rightarrow$) *Suppose* $\lim_{x\to a,\, x\in A} f(x) = L$ *exists. So, we have:* $\forall\ \varepsilon > 0\ \exists\ \eta > 0 : \forall\ x \in B_\eta(a) \cap A \Rightarrow d'(f(x), L) < \dfrac{\varepsilon}{2}$*. Then,* $\forall\ x, y \in B_\eta(a)$*,*

$d'(f(x), f(y)) \leq d'(f(x), L) + d'(f(y), L) < \varepsilon$*.*

($\Leftarrow$) *Conversely, suppose* f *satisfies the Cauchy condition at* a*. Then,*

$\forall \varepsilon > 0\ \exists\ \eta > 0\ \forall\ x, y \in B_\eta(a) \cap A : d'(f(x), f(y)) < \varepsilon$

Now, let $(x_n)_{n\in\mathbb{N}}$ *be a sequence in* A *that converges to* a*. Then (for* $\varepsilon = \eta$*) there is* $N \in \mathbb{N}$ *such that* $\forall\ n \geq N$*,* $d(x_n, a) < \eta$*. So,* $\forall\ n \geq N$*,* $\forall\ m \geq N$*,* $x_n, x_m \in B_\eta(a) \cap A$*. Hence* $d'(f(x_n), f(x_m)) < \varepsilon$*. So,* $(f(x_n))_{n\in\mathbb{N}}$ *is Cauchy. As* (Y, d') *is complete,* $\lim_{n\to\infty} f(x_n)$ *exists.*

Hence by the Theorem 7.1.6, $\lim_{x\to a,\, x\in A} f(x) = L$ *exists.*

Example 7.1.11 *Let $Y = \mathbb{R}$, $f : A \to \mathbb{R}$ be a function and $M > 0$ a number such that $\forall x, y \in A$, $|f(x) - f(y)| \leq Md(x, y)$. For $x_0 \in A$ show that $\lim_{x \to x_0, x \in A} f(x)$ exists. Let us see that Cauchy condition is satisfied:*

Let $\varepsilon > 0$ be any given number, let $0 < \eta \leq \frac{\varepsilon}{2M}$, then $\forall x, y \in A \cap B_\eta(x_0)$,

$|f(x) - f(y)| \leq Md(x, y) \leq M[d(x, x_0) + d(y, x_0)] \leq M(\frac{\varepsilon}{2M} + \frac{\varepsilon}{2M}) = \varepsilon$

So f satisfies Cauchy condition at x_0, hence the limit exists.

Here we can even determine the limit. Indeed, the condition given for the function above implies that f is uniformly continuous on A. Hence by the 'Extension by the Uniform Continuity theorem' we can extend f continuously to a function $f^* : \overline{A} \to \mathbb{R}$. As f^* is continuous on $\overline{A}$ and $x_0 \in \overline{A}$, for any sequence $x_n \in A$ converging to x_0, $\lim_{n\to\infty} f(x_n) = f^*(x_0)$. Hence $L = f^*(x_0)$ is the limit of f as $x \to x_0$, $x \in A$.

7.1.2 Limit and continuity

Theorem 7.1.12 *Suppose that $a \in A$ (so that f is defined at a). Then $f : A \to Y$ is continuous at a iff $\lim_{x\to a, x\in A} f(x) = f(a)$.*

Proof 7.1.13 ($\Rightarrow$) *Suppose f is continuous at a. Then, for any sequence (x_n) that converging to a, $f(x_n) \to f(a)$. So, by the theorem 7.1.6, $\lim_{x\to a, x\in A} f(x) = f(a)$.*

($\Leftarrow$) *Suppose that $\lim_{x\to a, x\in A} f(x) = f(a)$. Then, again by the same theorem, for any sequence $(x_n)_{n\in\mathbb{N}}$ in A converging to a, $f(x_n) \to f(a)$. So f is continuous at a.*

7.2 Limit From the Left, From the Right

First, a general result:

Lemma 7.2.1 *Suppose that $A = B \cup C$ for $B, C \subseteq X$ and $x_0 \in \overline{B}$, $x_0 \in \overline{C}$. Then $\lim_{x\to x_0, x\in A} f(x)$ exists iff both $\lim_{x\to x_0, x\in B} f(x)$ and $\lim_{x\to x_0, x\in C} f(x)$ exist and they are the same.*

Proof 7.2.2 ($\Rightarrow$) *Suppose that $\lim_{x\to a, x\in A} f(x)$ exists and is L. So, we have:*

$\forall\ \varepsilon > 0,\ \exists\ \eta > 0,\ \forall\ x \in B_\eta(a) \cap A$: $d'(f(x), L) < \varepsilon$.

Then, obviously $\forall\ x \in B_\eta(a) \cap B, d'(f(x), L) < \varepsilon$, so $\lim_{x\to a, x\in B} f(x) = L$ and similarly $\forall\ x \in B_\eta(a) \cap B, d'(f(x), L) < \varepsilon$, so $\lim_{x\to a, x\in C} f(x) = L$.

($\Leftarrow$) *Conversely, suppose that this two limits exist and they are the same. Let L be the common value of these. So we have:*

$\forall\varepsilon > 0, \exists\eta_1 > 0 : \forall x \in B \cap B_{\eta_1}(x_0),\ d'(f(x), L) < \varepsilon$

$\forall\varepsilon > 0, \exists\eta_2 > 0 : \forall x \in C \cap B_{\eta_2}(x_0),\ d'(f(x), L) < \varepsilon$

Let $\eta = \min\{\eta_1, \eta_2\}$ then $\forall x \in A \cap B_\eta(X_0)$, $d'(f(x), L) < \varepsilon$.

Hence $\lim_{x\to x_0, x\in A} f(x) = L$.

Now, in this section our $X = \mathbb{R}$, d=usual metric. And let $A = [b, a[\cup]a, c]$ with $B = [b, a[$ and $C =]a, c]$. We see that $a \in \bar{A}$, $a \in \bar{B}$ and $a \in \bar{C}$.
So, for $f : A \to Y$, $\lim_{x \to a,\, x \in A} f(x)$, $\lim_{x \to a,\, x \in B} f(x)$, and $\lim_{x \to a} x \in Cf(x)$ are meaningful.
The Lemma 7.2.1 says that:

$\lim_{x \to a,\, x \in A} f(x)$ exists $\Leftrightarrow$ $\lim_{x \to a,\, x \in B} f(x)$ and $\lim_{x \to a,\, x \in C} f(x)$ exists and are equal.

$\lim_{x \to a,\, x \in B} f(x)$ is said to be <u>the limit from the left</u> and is denoted as $\lim_{x \to a,\, x < a} f(x)$.

$\lim_{x \to a,\, x \in C} f(x)$ is said to be <u>the limit from the right</u> and is denoted as $\lim_{x \to a,\, x > a} f(x)$.

Mathematically:
$\lim_{x \to a,\, x < a} f(x) = L \Leftrightarrow \forall\, \varepsilon > 0\ \exists\, \eta > 0 : \forall\, x \in]a - \eta, a[\to d'(f(x), L) < \varepsilon$
$\lim_{x \to a,\, x > a} f(x) = L \Leftrightarrow \forall\, \varepsilon > 0\ \exists\, \eta > 0 : \forall\, x \in]a, a + \eta[\to d'(f(x), L) < \varepsilon$

Example 7.2.3 $f : [-1, 0[\cup]0, 1[\to \mathbb{R}$, $f(x) = \begin{cases} 1 & 0 \le x \le 1, \\ -1 & -1 \le x \le 0 \end{cases}$

$\lim_{x \to 0,\, x < 0} f(x) = -1$ *and* $\lim_{x \to 0,\, x > 0} f(x) = 1$

Notation: Now on we shall denote $\lim_{x \to x_0,\, x < x_0} f(x)$ as $f(x_0^-)$ whenever this limit exists. Similarly, we denote $\lim_{x \to x_0,\, x > x_0} f(x)$ as $f(x_0^+)$ whenever this limit exists.

Example 7.2.4 *Let* $f : [-\pi, 0[\cup]0, \pi] \to \mathbb{R}$, $f(x) = \sin \frac{1}{x}$. *Then neither of the limits* $\lim_{x \to x_0,\, x < x_0} f(x)$, $\lim_{x \to x_0,\, x > x_0} f(x)$ *exist.*

- *If we take* $x_n = \dfrac{1}{\pi/2 + n\pi}$ *then* $x_n \in]0, \pi]$ *but* $\lim_{n \to \infty} f(x_n)$ *does not exist.*
- *If we take* $x_n = \dfrac{-1}{\pi/2 + n\pi}$ *then* $x_n \in [-\pi, 0[$ *but* $\lim_{n \to \infty} f(x_n)$ *does not exist.*

Continuity From the Left, Continuity From the Right

Definition 7.2.5 *Let* $b < a < c$, $f : [b, c] \to Y$ *be a function (So* f *is defined at* a*).*

- *If* $\lim_{x \to a,\, x < a} f(x) = f(a)$, *then we say* f ***is continuous at*** a ***from left***.
- *If* $\lim_{x \to a} x > af(x) = f(a)$, *then we say* f ***is continuous at*** a ***from right***.

Example 7.2.6 *Let* $f_1, f_2, f_3 : [-1, 1] \to \mathbb{R}$

1. $f_1(x) = \begin{cases} -1 & -1 \le x \le 0, \\ 1 & 0 \le x \le 1 \end{cases}$ *As* $\lim_{x \to 0,\, x < 0} f_1(x) = -1 = f_1(0)$, f *is continuous at* 0.

2. $f_2(x) = \begin{cases} -1 & -1 \le x \le 0, \\ 1 & 0 \le x \le 1 \end{cases}$ *As* $\lim_{x \to 0,\, x > 0} f_2(x) = 1 = f_2(0)$, f *is continuous at* 0.

3. $f_3(x) = \begin{cases} -1 & -1 \le x \le 0, \\ 1 & 0 \le x \le 1 \end{cases}$ $\lim_{x \to 0,\, x < 0} f_3(x) = -1 \ne f_3(0) \ne \lim_{x \to 0,\, x > 0} f_3(x)$, f *is not continuous neither from left, nor from right.*

Example 7.2.7 *Let* $f : [\frac{-\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}, \;\; f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0. \end{cases}$

Then, $\lim_{x\to 0,\, x<0} f(x) = 1 = \lim_{x\to 0,\, x>0} f(x)$

$\lim_{x\to 0,\, x\in[-\frac{\pi}{2},\frac{\pi}{2}]} f(x)$ *exists, but* f *is not continuous at zero.*

From all these we say that, f is continuous at $a \Leftrightarrow f(a_+) = f(a) = f(a_-)$

7.3 Continuity of Monotone Functions

Theorem 7.3.1 *Let* $F : [a, b] \to \mathbb{R}$ *be an increasing function and* $a < x_0 < b$ *be a point. Then,*

1. $f(x_0^-)$ *and* $f(x_0^+)$ *exist.* $f(x_0^-) = \sup\{f(y) : \; a \leq y \leq x_0\}$
   $f(x_0^+) = \inf\{f(y) : x_0 \leq y \leq b\}$
2. $f(x_0^-) \leq f(x_0) \leq f(x_0^+)$
3. *If* $a < x_0 < y_0 < b$ *then* $f(x_0^+) \leq f(x_0^-)$
4. *The set* $D_f = \{x_0 \in [a, b] : f$ *is discontinuous at* $x_0\}$ *is at most countable.*

Proof 7.3.2

1. *Let us prove that* $f(x_0^-) = \sup\{f(y) : a \leq y \leq x_0\}$*. As* f *is increasing, for* $a \leq y \leq x_0$, $f(y) \leq f(x_0)$*. So the set* $E = \{f(y) : a \leq y \leq x_0\}$ *is bounded from above. Hence its supremum exists. Let* $\alpha = \sup E$*. So we have:*

   $$\begin{cases} 1^\circ) & \forall\, y \in [a, x_0[, \; f(y) \leq \alpha \\ 2^\circ) & \forall\, \varepsilon > 0, \; \exists \; y_\varepsilon \in [a, x_0[\colon f(y_\varepsilon) > \alpha - \varepsilon \end{cases}$$

   Then for $y_\varepsilon \leq y < x_0$ *(i.e.* $\eta = x_0 - y_\varepsilon$*), as* f *is increasing* $f(y) \geq f(y_\varepsilon) > \alpha - \varepsilon$*.*

   As $f(y) \leq \alpha < \alpha + \varepsilon$*, we have that* $|f(y) - \alpha| < \varepsilon, \; \forall\, x \in]x_0 - \eta, x_0[$*. Then,*

   $\alpha = \lim_{y\to x_0,\, y_0<x_0} f(y) = f(x_0^-)$

2. *As* $f(x_0^-) = \sup\{f(y) : a \leq y \leq x_0\}$ *we see that* $f(x_0^-) \leq f(x_0)$*.*

   Since $f(x_0^+) = \inf\{f(y) : x_0 < y \leq b\}$ *we see that* $f(x_0^+) \geq f(x_0$*.*

   Thus $f(x_0^-) \leq f(x_0) \leq f(x_0^+)$

3. *By 1 above,*

   $$\begin{aligned} f(x_0^+) &= \inf\{f(y) : x_0 < y \leq b\} = \inf\{f(y) : x_0 < y < y_0\} \\ &\leq \sup\{f(y) : x_0 < y < y_0\} = f(y_0^-) \end{aligned}$$

4. *By 2 above, f is discontinuous at a point $x_0 \in]a,b[\Leftrightarrow f(x_0^-) < f(x_0^+)$.*

   Now, let $x_0 < y_0,\ x_0, y_0 \in D_f$. So, $f(x_0^-) < f(x_0^+) \leq f(y_0^-) < f(y_0^+)$. Let $r_{x_0}, r_{y_0} \in \mathbb{Q}$ be such that $f(x_0^-) < r_{x_0} < f(x_0^+)$ and $f(y_0^-) < r_{y_0} < f(y_0^+)$. Then $x_0 < y_0 \Rightarrow r_{x_0} < r_{y_0}$. Then we can define a one-to-one mapping from D_f into $\mathbb{Q}$. As $\mathbb{Q}$ is countable, we can conclude that D_f is countable.

7.4 Functions of Bounded Variation

Definition 7.4.1 *Consider an interval $[a,b]$. A finite subset $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ is said to be a **partition** of $[a,b]$ if $a = x_0 < x_1 < ... < x_n = b$.*

Definition 7.4.2 *Let $f : [a,b] \to \mathbb{R}$ be any function. The quantity;*

$$V(f,P) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

*is said to be the **variation of f relative to the partition** P.*

Example 7.4.3 *Suppose that f is an increasing function. Then,*

$$V(f,p) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = f(b) - f(a).$$

Definition 7.4.4 *Let $f : [a,b] \to \mathbb{R}$ be any function and $\mathbb{P}[a,b]$ be the set of all partitions of $[a,b]$. Put $V_a^b(f) = \sup_{P \in \mathbb{P}[a,b]} V(f,P)$. If $V_a^b(f) < \infty$ we say that f is a **function of bounded variation**. ($V_a^b(f)$ is the **total variation of f on** $[a,b]$.)*

Example 7.4.5 *If $f : [a,b] \to \mathbb{R}$ is increasing then $V_a^b(f) \leq f(b) - f(a) < \infty$. Hence any increasing function $f : [a,b] \to \mathbb{R}$ is of bounded variation.*

Example 7.4.6 *Even if a function $f : [a,b] \to \mathbb{R}$ is continuous, it need not to be of bounded variation. Let $f : [0,1] \to \mathbb{R}$, $f(x) = \begin{cases} x \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$*

Then f is continuous on $[0,1]$.

Now, let $n \geq 1$ any number and P be a partition of $[0,1]$.

$$x_0 = 0 < x_1 = \frac{1}{\pi + \pi/2} < x_2 = \frac{1}{2\pi + \pi/2} < ... < \frac{1}{n\pi + \pi/2} < x_{n+1} = 1.$$

$V(f,p) = \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})|$ *Now,*

$$\begin{aligned}|f(x_i)-f(x_{i-1})| &= |\frac{1}{i\pi+\pi/2}\sin(i\pi+\pi/2)-\frac{1}{(i-1)\pi+\pi/2}\sin((i-1)\pi+\pi/2)| \\ &= |\frac{1}{i\pi+\pi/2}(-1)^i-\frac{1}{(i-1)\pi+\pi/2}(-1)i-1| \\ &= \frac{1}{i\pi+\pi/2}+\frac{1}{(i-1)\pi+\pi/2} \\ &\geq \frac{1}{i\pi+\pi/2}\geq\frac{1}{\pi(1+i)}\end{aligned}$$

$V(f,p)\geq\frac{1}{\pi}\sum_{i=1}^{n}\frac{1}{1+i}\to\infty$, *as* $n\to\infty$. *Hence* $\sup_{p\in\mathbb{P}[a,b]}V(f,p)=\infty$.

Remark: Every function of bounded variation, $f:[a,b]\to\mathbb{R}$ is bounded.

Indeed, let $x\in]a,b[$, let $P=\{a,x,b\}$, then
$V(f,P)=|f(a)-f(x)|+|f(x)-f(b)|\leq V(f,[a,b])<\infty$.
Hence $|f(x)|-|f(a)|+|f(x)|-|f(b)|\leq V(f,[a,b])$
$\Rightarrow\ |f(x)|\leq\frac{1}{2}[|f(a)|+|f(b)|+V(f,[a,b])]$, so that f is bounded.

Example 7.4.7 *If f and g are of bounded variation then, $f\pm g$ and cf are of bounded variation. Thus the space $BV[a,b]$ of the functions $f:[a,b]\to\mathbb{R}$ of bounded variation is a vector space.*

Lemma 7.4.8 *Let $f:[a,b]\to\mathbb{R}$ be a function of bounded variation and $a<c<b$ be numbers. Then $V(f,[a,b])=V(f,[a,c])+V(f,[c,b])$*

Proof 7.4.9 *Let $P=\{x_0,x_1,...,x_n\}$ be any partition of $[a,b]$, then $\tilde{P}=P\cup\{c\}$ is also a partition of $[a,b]$.*

Moreover, $V(f,P)\leq V(f,\tilde{P})$ since $|f(x_i)-f(x_{i+1})|\leq|f(x_i)-f(c)|+|f(c)-f(x_{i+1})|$
Let $P_1=[a,c]\cap\tilde{P}$, $P_2=[c,b]\cap\tilde{P}$, then P_1 is a partition of $[a,c]$, P_2 is a partition of $[c,b]$.
Now $V(f,\tilde{P})=V(f,P_1\cup P_2)=V(f,P_1)+V(f,P_2)\leq V(f,[a,c])+V(f,[c,b])$
Hence $V(f,P)\leq V(f,[a,c])+V(f,[c,b])$

$V(f,[a,b])=\sup_{P\in\mathbb{P}[a,b]}V(f,P)\leq V(f,[a,c])+V(f,[c,b])$

To prove that the reverse inequality, let $\varepsilon>0$ arbitrary. Then there are partitions P_3,P_4 of $[a,c],[c,b]$ respectively, such that;

$V(f,P_3)>V(f,[a,c])-\varepsilon/2$ *and* $V(f,P_4)>V(f,[c,b])-\varepsilon/2$

Let $P'=P_3\cup P_4$ then P' is a partition of $[a,b]$, so then

$V(f,[a,b])\geq V(f,P')=V(f,P_3)+V(f,P_4)\geq V(f,[a,c])+V(f,[c,b])-\varepsilon$

Hence $V(f,[a,b])=V(f,[a,c])+V(f,[c,b])$

Theorem 7.4.10 ***(Characterization of the functions of bounded variation):*** *Let $f:[a,b]\to\mathbb{R}$ be a given function. Then f is of bounded variation iff f is the difference of two increasing functions, i.e. $f=f_1-f_2$, where f_1,f_2 are increasing functions.*

Proof 7.4.11 *($\Leftarrow$) If $f = f_1 - f_2$, then for any partition P of $[a,b]$, $V(f,p) \leq V(f_1,p) + V(f_2,p) \leq f_1(b) - f_2(a)$. So that $V_a^b(f) < \infty$.*

($\Rightarrow$) Conversely, suppose $V_a^b(f) < \infty$. Let, for $a \leq x \leq b$, $g(x) = V_a^x(f)$. It is clear that g is increasing. Now let $h(x) = V_a^x(f) - f(x)$. Let us see that h is increasing:

Let $0 \leq x \leq y$ be two points. Then $h(y) - h(x) = V_a^y(f) - V_a^x(f) - [f(y) - f(x)]$. Now, $V_a^y(f) - V_a^x(f) \geq |f(y) - f(x)|$. Hence $h(y) - h(x) \geq 0$.

So h is increasing and $f(x) = g(x) - h(x)$.

Example 7.4.12 *If $f : [a,b] \to \mathbb{R}$ is differentiable and f' is bounded on $[a,b]$ then $V_a^b(f) < \infty$. Indeed, let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[a,b]$. Since by the intermediate value theorem, $f(x_i) - f(x_{i-1}) = (x_i - x_{i-1})f'(c_i)$ for some $x_{i-1} < c_i < x_i$.*

If $|f'(x)| < M \quad \forall x \in [a,b]$, then;

$$V(f,p) = \sum_{i=1}^{n+1} |f(x_i) - f(x_{i-1})| \leq \sum_{i=1}^{n+1} (x_i - x_{i-1})|f'(c_i)| \leq M \sum_{i=1}^{n+1} (x_i - x_{i-1}) = M(b-a).$$

So, $V_a^b(f) = \sup_{p \in \mathbb{P}[a,b]} V(f,P) \leq M(b-a) < \infty$.

Theorem 7.4.13 *(Dirichlet, 1829) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Suppose that f has only finitely many local max and min. Then the Fourier series of f converges at every $x \in [a,b]$ to $f(x)$.*

Theorem 7.4.14 *(Raymond, 1863) There exists a continuous function $f : [0, 2\pi] \to \mathbb{R}$ such that the Fourier series of f diverges at infinitely many points in $[0, 2\pi]$.*

Theorem 7.4.15 *(Jordan, 1867) For any function f of bounded variation, the Fourier series of f converges for each $x \in]0, 2\pi[$, the series converges to $\dfrac{f(x^+) + f(x^-)}{2}$. Hence, if f is continuous at x then the Fourier series of f at x converges to $f(x)$.*

7.5 Absolutely Continuous Functions

Let $[a,b]$ be an interval. Let (a_i, b_i) for $i = 1, 2, ..., n$ be subintervals of $[a,b]$. (a_i, b_i) stands for any kind of intervals (open, closed, half open). We say that the intervals (a_i, b_i) are **non-overlapping** if, for $i \neq j$, the intersection $(a_i, b_i) \cap (a_j, b_j)$ has no interior point.

Now, let $f : [a,b] \to \mathbb{R}$ be a bounded (say $|f(x)| \leq M$, $\forall x \in [a,b]$). Put $F(x) = \int_a^x f(t)dt$. Let $(a_1, b_1), ..., (a_n b_n)$ be non-overlapping subintervals of $[a,b]$.

Let us look at the quantity: $\displaystyle\sum_{i=1}^{n} |F(b_i) - F(a_i)|$.

As, $F(b_i) = \int_a^{b_i} f(t)dt$ and $F(a_i) = \int_a^{a_i} f(t)dt$, we see that;

$|F(b_i) - F(a_i)| \leq \int_{a_i}^{b_i} f(t)dt \leq M(b_i - a_i)$.

Hence $\sum_{i=1}^{n} |F(b_i) - F(a_i)| \leq M \sum_{i=1}^{n} (b_i - a_i)$. **(1)**

Definition 7.5.1 *A function $f : [a, b] \to \mathbb{R}$ is said to be* ***absolutely continuous****, if given any $\varepsilon > 0$, $\exists \eta > 0$ such that for any non-overlapping subintervals (a_i, b_i), for $i = 1, 2, ..., n$, of $[a, b]$ satisfying $\sum_{i=1}^{n} (b_i - a_i) < \eta$, we have $\sum_{i=1}^{n} |g(b_i) - g(a_i)| < \varepsilon$.*

Example 7.5.2 *Let $F(x) = \int_a^x f(t)dt$. Then* ***(1)*** *shows that F is absolutely continuous.*

Example 7.5.3 *If $g : [a, b] \to \mathbb{R}$ is differentiable and $g'(x)$ is bounded, then g is absolutely continuous. Indeed, let $(a_1, b_1), ..., (a_n b_n)$ be non-overlapping subintervals of $[a, b]$. As, $g(b_i) - g(a_i) = (b_i - a_i)g'(c_i)$, with $a_i < c_i < b_i$, we see that $\sum_{i=1}^{n} |g(b_i) - g(a_i)| \leq \sum_{i=1}^{n} |b_i - a_i||g'(c_i)| \leq M \sum_{i=1}^{n} (b_i - a_i) \leq M(b - a)$. So that given any $\varepsilon > 0$ if we choose $\eta = \frac{\varepsilon}{M}$, then we see that, whenever $\sum_{i=1}^{n} |b_i - a_i| < \eta$, we have $\sum_{i=1}^{n} |g(b_i) - g(a_i)| < M \sum_{i=1}^{n} (b_i - a_i) < \varepsilon$.*

Remark: From the definition, it is clear that every absolutely continuous function is uniformly continuous. The converse is false. e.g. the function $f(x) = \begin{cases} x \sin 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ is continuous. So, uniformly continuous on $[0, 1]$, but it is not absolutely continuous.

Proposition 7.5.4 *Every absolutely continuous function is of bounded variation.*

Proof 7.5.5 *Let f be absolutely continuous on $[a, b]$. So we have:*

$\forall \varepsilon > 0$, $\exists \eta > 0$ such that $\forall (a_i, b_i)_{1 \leq i \leq N}$, non-overlapping subintervals of $[a, b]$ satisfying $\sum_{i=1}^{n} (b_i - a_i) < \eta \Rightarrow \sum_{i=1}^{n} |g(b_i) - g(a_i)| < \varepsilon$.

This definition shows that we take any subinterval $[c, d]$ of $[a, b]$ with $d - c < \eta$ then $V(f, [c, d]) < \varepsilon$

Now, we can cover $[a, b]$ by finitely many, say N, subintervals $[c_i, d_i]$ with $d_i - c_i < \eta$, $\forall i$.

Then $V(f, [a, b]) \leq \sum_{i=1}^{N} V(f, [c_i, d_i]) \leq N\varepsilon < \infty$

7.6 Exercises

1. For $x \in \mathbb{R}$, let $[x]$ be the largest integer (positive or negative) smaller than x.

   (e.g. $[2.1] = 2$). Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = [x]$.

   • For each $x_0 \in \mathbb{R}$, determine $\lim_{x \to x_0,\, x < x_0} f(x)$ and $\lim_{x \to x_0,\, x > x_0} f(x)$.

   • For each $x_0 \in \mathbb{R}$, determine $\lim_{x \to x_0,\, x < x_0} g(x)$ and $\lim_{x \to x_0,\, x > x_0} g(x)$, for $g(x) = x - [x]$.

2. Let $g : \mathbb{R} \to \mathbb{R}$, $g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ -x & \text{if } x \notin \mathbb{Q} \end{cases}$.

   For $c \in \mathbb{R}$, study the existence of the limit: $\lim_{x \to c} g(x)$

3. Let $f : [a, b] \to \mathbb{R}$ be a function. Assume that, for each $x_0 \in [a, b]$, $\lim_{x \to x_0,\, x \in [a,b]} f(x)$ exists. Show that f is bounded.

4. Study the existence of the limits: $\lim_{x \to x_0,\, x < x_0} \sin \frac{1}{x}$, and $\lim_{x \to x_0,\, x > x_0} \sin \frac{1}{x}$

5. Let $f : [a, b] \to \mathbb{R}$ be a monotone increasing function and $x_0, x_1, ..., x_n, ...$ be the points in $]a, b[$ at which f is discontinuous. Put $c_n = f(x_n^+) - f(x_n^-)$. Show that, for each $n \in \mathbb{N}$, $c_0 + c_1 + ... + c_n \leq f(b) - f(a)$.

6. Let $c_n \in \mathbb{R}$, $c_n > 0$ be arbitrary numbers such that $\sum_{n=0}^{\infty} c_n < \infty$. Let $x_0, ..., x_n, ...$ be arbitrary points in $]a, b[$. For $x \in [a, b]$, let $g(x) = \sum_{x_n < x} c_n$ (if there is no $x_n < x$, then we put $g(x) = 0$). Show that

   (a) $g : [a, b] \to \mathbb{R}$ is monotone increasing.

   (b) For x_{n_0} fixed, $g(x_{n_0}) = g(x_{n_0}^+) = \lim_{\varepsilon \to 0} g(x_{n_0} - \varepsilon)$
   $g(x_{n_0}^+) = \lim_{\varepsilon \to 0} g(x_{n_0} + \varepsilon) = \sum_{x_n \leq x_{n_0}} c_n$

   (c) $g(x_{n_0}^+) - g(x_{n_0}^-) = c_{n_\varepsilon}$.

   (d) g is continuous on $]a, b[\setminus \{x_0, x_1, ..., x_n, ...\}$

7. Let $f : [a, b] \to \mathbb{R}$ be a left continuous increasing function, $x_0, ..., x_n, ...$ be the points in $]a, b[$ at which f is discontinuous. Let $c_n = f(x_n+) - f(x_n-)$ and g be as in the question 6 with this choice of c_n's. Show that $h = f - g$ is continuous on $[a, b]$ so that $f = g + h$. Then every increasing function f is the sum of a continuous increasing function and a "jump function" g.

8. Let $f :]a, \infty[\to \mathbb{R}$ be a function. Define $g :]0, \frac{1}{a}[\to \mathbb{R}$ by $g(x) = f(\frac{1}{x})$. Show that $\lim_{x\to\infty} g(x) = L$ ($L \in \mathbb{R}$).
   Show that $\lim_{x\to\infty}(f(x+1) - f(x)) = 0$.
   Deduce that if $(x_n)_{n\in\mathbb{N}}$ is a convergent sequences, then $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Show that TFAE:
   (a) $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$
   (b) $\lim_{|x|\to\infty} f(x) = 0$
   (c) $\forall \varepsilon > 0, \ \exists \ M > 0 : \ |x| > M \to |f(x)| < \varepsilon$

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{|x|\to\infty} f(x) = 0$. Show that f is bounded and uniformly continuous on $\mathbb{R}$.

11. Show that the function $f(x) = \dfrac{x}{1+x^2}$ is bounded and uniformly continuous on $\mathbb{R}$.

12. Let L be the space of the bounded function $f : \mathbb{R} \to \mathbb{R}$ and C_0 be the space of the bounded function $g : \mathbb{R} \to \mathbb{R}$ such that $\lim_{|x|\to\infty} g(x) = 0$. Show that L is a commutative unital ring and C_0 is an ideal of it.

Chapter 8

Connectedness

1. **Definition and properties**
2. **Connected components of a set**
3. **Pointwise connected sets**
4. **Applications**

8.1 Introduction: The Role of *Interval* in Analysis

Theorem 8.1.1 ***Intermediate Value Theorem:*** *If I is an interval, $f : I \to \mathbb{R}$ is continuous and for some $x_0, y_0 \in I$, $f(x_0) < 0$ and $f(y_0) > 0$, then there is $c \in (x_0, y_0)$ such that $f(c) = 0$.*

1. Let $A =]-1,0[\cup]0,1[$, $f : A \to \mathbb{R}$ $f(x) = \begin{cases} -1 & \text{if } x \in]-1,0[\\ +1 & \text{if } x \in]0,1[\end{cases}$

   Let $x_0 \in A$ be any point. Say $x_0 \in]0,1[$. So $0 < x_0 < 1$. Hence, if $|h|$ is small, $x_0 + h$ is also in $]0,1[$. Hence, $\lim_{h\to 0,\, h\neq 0} \dfrac{f(x_0+h) - f(x_0)}{h} = 0$. Hence, $\forall x \in A$, $f'(x) = 0$ but f is not constant on A.

2. Also, this f is *continuous* on A. $f(1/2) > 0$, $f(-1/2) < 0$ but there is no $x_0 \in A$ such that $f(x_0) = 0$.

3. Let $f : \mathbb{Q} \to \mathbb{R}$, $f(x) = x^2 - 2$. Then f is continuous; $f(2) > 0$, $f(1) < 0$ but there is no $x \in \mathbb{Q}$ such that $f(x) = 0$.

Definition 8.1.2 *Let $A \subseteq \mathbb{R}$, then A is an <u>interval</u> iff for each $a, b \in A$ with $a < b$, every $x \in \mathbb{R}$ with $a < x < b$ is in A; i.e. for any $O_1, O_2 \subseteq \mathbb{R}$ open disjoint sets with $A \subseteq O_1 \cup O_2$ either $A \subseteq O_1$ or $A \subseteq O_2$.*

Basic Question: What is the analogue of the "interval" in an abstract m. s. (X, d)? These are connected sets.

8.2 Definition and Properties

Definition 8.2.1 *Let (X,d) be a m.s. A subset $A \subseteq X$ is said to be* ***disconnected*** *if there are two nonempty, open, disjoint subsets O_1, O_2 of X such that:*

1. $A \subseteq O_1 \cup O_2$
2. $A \cap O_1 \neq \emptyset$ *and* $A \cap O_2 \neq \emptyset$.

• *If A is not connected, then it is said to be* ***connected****. Thus A is connected if whenever we have $A \subseteq O_1 \cup O_2$, where O_1, O_2 are nonempty, open, disjoint subsets of X we have either $A \subseteq O_1$ or $A \subseteq O_2$.*

Example 8.2.2 *Let $X = \mathbb{R}$, $d(x,y) = |x-y|$.*

1. *$A = \mathbb{N}$ is disconnected: Indeed, let $O_1 =]-1, 5/2[$, $O_2 =]5/2, \infty[$, then $\mathbb{N} \subseteq O_1 \cup O_2$, $\mathbb{N} \cap O_1 \neq \emptyset$, $\mathbb{N} \cap O_2 \neq \emptyset$*
2. *Similarly, $A = \mathbb{Z}$ is disconnected.*
3. *$A = \mathbb{Q}$ is disconnected. Indeed, let $O_1 =]-\infty, \sqrt{5}[$, $O_2 =]\sqrt{5}, \infty[$ then $\mathbb{Q} \subseteq O_1 \cup O_2$, $\mathbb{Q} \cap O_1 \neq \emptyset$, $\mathbb{Q} \cap O_2 \neq \emptyset$*
4. *In any m.s. (X,d) any finite set $A = \{x_1, ..., x_n\}$ is disconnected.*
5. *In any m.s. (X,d) let A_1, A_2 be two disjoint, nonempty, closed sets. Then $A = A_1 \cup A_2$ is disconnected.*

   Indeed, ***"Urysohn Lemma"*** *says that we have a continuous function $f : X \to [0,1]$ such that $f(A_1) = 0$, $f(A_2) = 1$.*

   Let $O_1 = f^{-1}(]-1, 1/2[)$, $O_2 = f^{-1}(]1/2, 2[)$. Then O_1, O_2 are open and disjoint, $A_1 \subseteq O_1$, $A_2 \subseteq O_2$. So that, $A \subseteq O_1 \cup O_2$, $A \cap O_1 \neq \emptyset$, $A \cap O_2 \neq \emptyset$.

Theorem 8.2.3 $(\mathbb{R}, d)$ *is a connected m.s. i.e., we cannot write $\mathbb{R}$ as the union of two nonempty, open, disjoint sets.*

Proof 8.2.4 *For a contradiction suppose that $\mathbb{R} = O_1 \cup O_2$, where O_1 and O_2 are nonempty, open, disjoint sets. Then O_1 and O_2 are closed, too since $O_1^C = O_2$, $O_2^C = O_1$.*

Let $A = O_1$. Then $\emptyset \neq A \neq \mathbb{R}$ and A is both open and closed. Since $\emptyset \neq A \neq \mathbb{R}$, let $\gamma \in \mathbb{R}$ be such that $\gamma \notin A$. Then $A \subseteq]-\infty, \gamma[\cup]\gamma, \infty[$. So, $A \cap]-\infty, \gamma[\neq \emptyset$ or $A \cap]\gamma, \infty[\neq \emptyset$. Say, $A \cap]\gamma, \infty[\neq \emptyset$.

Put $B = A \cap]\gamma, \infty[$. B is open since A and $]\gamma, \infty[$ are open. As $\gamma \notin A$, $A \cap [\gamma, \infty[= A \cap]\gamma, \infty[$.

Hence, $B = A \cap [\gamma, \infty[$ is closed. Then B is both open and closed and $\neq \emptyset$. $B \subseteq [\gamma, \infty[$.

Hence $\beta = \inf B$ exists. Now, as B is closed, $\beta \in B$, as B is open $\beta \notin B$. Contradiction.

Hence, $\mathbb{R}$ is connected.

Proposition 8.2.5 *Let (X,d), (Y,d') be two m.s. $A \subseteq X$ and $f : A \to Y$ be a continuous function. If A is connected then $f(A)$ is also connected.*

Proof 8.2.6 *For a contradiction suppose that $f(A)$ is disconnected. Then there exists two disjoint, open sets O_1, O_2 such that; $f(A) \subseteq O_1 \cup O_2$, $f(A) \cap O_1 \neq \emptyset$, $f(A) \cap O_2 \neq \emptyset$*
This implies that $A \subseteq f^{-1}(O_1 \cup O_2) = f^{-1}(O_1) \cup f^{-1}(O_2)$, $A \cap f^{-1}(O_1) \neq \emptyset$, $A \cap f^{-1}(O_2) \neq \emptyset$
As f is continuous on A, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are open. So that A should be disconnected. Contradiction.

Example 8.2.7 *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \arctan x$. We know that $\mathbb{R}$ is connected and f is continuous. So $f(R) =]-\pi/2, \pi/2[$ is connected. Since any open interval $]a,b[$ is homeomorphic to $]-\pi/2, \pi/2[$, we see that $]a,b[$ is connected.*

Corollary 8.2.8 *Every open interval in $\mathbb{R}$ is connected.*

Proof 8.2.9 *Let $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = \dfrac{x}{1+|x|}$ then φ is continuous and $\varphi(\mathbb{R}) =]-1,1[$, so $]-1,1[$ is connected.*

If we take $f :]-1,1[\to]0,1[$, $f(x) = \dfrac{x+1}{2}$ then f is continuous and $f(]-1,1[) =]0,1[$ is connected.

If we take $g :]0,1[\to]a,b[$, $(a<b)$, where $g(x) = a(1-x) + bx$, then g is continuous and onto so that $]a,b[$ is connected.

Remarks:

1. Every set is either connected or disconnected.
2. "Connectedness" is, as "compactness", an absolute notion. That is, if $A \subseteq X \subseteq X' \subseteq X'' \subseteq ...$ are metric spaces with continuous injections (i.e. $i : X \to X'$, $i(x) = x$ is continuous).
3. As the natural injection $i : A \longrightarrow X$ and $i^* : X \longrightarrow X'$ are continuous, if A is connected in X then $i^* \circ i(A) = A$ is connected in X'. Also, (A,d) is connected as a m. s. of its own iff A is connected as a subset of X.
4. Let (X,d) be a m.s. From the definition of connected sets, the following is clear:

$$\begin{aligned} X \text{ is connected} \quad &\Leftrightarrow \text{ the only open and closed subsets of} X \text{ are } X \text{ and } \emptyset \\ &\Leftrightarrow \forall A \subseteq X, \emptyset \neq A \neq X, \partial A \neq \emptyset \\ &\Leftrightarrow \forall A \subseteq X, \emptyset \neq A \neq X, \chi_A : X \to \mathbb{R} \text{ is not continuous} \end{aligned}$$

Notation: Let $D = \{0,1\}$. On D we put the discrete metric. So that the open sets of D are $\emptyset, \{1\}, \{0\}, D$. Now, if A is any set and $\varphi : A \to D$ is a mapping, then to say that "φ is not constant" is equivalent to say that "φ is onto". That is, φ is constant on A iff $\varphi(A) = \{0\}$ or $\varphi(A) = \{1\}$.

Theorem 8.2.10 *Let (X, d) be a m.s., $A \subseteq X$ a nonempty set. Then A is connected $\Leftrightarrow$ every continuous function $\varphi : A \to D$ is constant.*

Proof 8.2.11 ($\Rightarrow$) *Suppose A is connected. Let $\varphi : A \to D$ be a continuous mapping. Let us see that φ is constant. If not, we had $\varphi(A) = \{0, 1\}$, then the sets $O_1 = \varphi^{-1}(\{0\})$ and $O_2 = \varphi^{-1}(\{1\})$ would be nonempty, open, disjoint and $A \subseteq O_1 \cup O_2$, $A \cap O_1 \neq \emptyset$, $A \cap O_2 \neq \emptyset$, contradicting connectedness of A.*
($\Leftarrow$) *Conversely suppose that every continuous $\varphi : A \to D$ is constant.*

Let us see that A is connected. If not, we would have two nonempty, disjoint, open sets O_1, O_2 such that $A \subseteq O_1 \cup O_2$, $A \cap O_1 \neq \emptyset$, $A \cap O_2 \neq \emptyset$

Now, define a mapping $\varphi : O_1 \cup O_2 \to D$ such that $\varphi(x) = 0 if\ x \in O_1$, $\varphi(x) = 1 if\ x \in O_2$ Then φ is continuous since $\varphi^{-1}(0), \varphi^{-1}(1), \varphi^{-1}(\emptyset), \varphi^{-1}(D)$ are open. φ, as a mapping from A to D is also continuous and $\varphi(A) = \{0, 1\}$. Contradiction.

Proposition 8.2.12 *Let (X, d) be a m.s., $A \subseteq X$ a set and $A \subseteq B \subseteq \bar{A}$. If A is connected then B is also connected. In particular, if A is connected then $\bar{A}$ is connected.*

Proof 8.2.13 *Suppose A is connected. To prove that B is connected by Theorem 8.2.10, we have to show that any continuous $\varphi : B \to D$ is constant.*

Let $\varphi : B \to D$ be a continuous mapping. As φ is also continuous on A and A is connected, i.e. $\varphi(A) = 0$ or $\varphi(A) = 1$ Say $\varphi(A) = 0$. Let $x \in B$ be any point. As $B \subseteq \bar{A}$, there is a sequence x_n in A that converges to x. As φ is continuous on A, $\varphi(x_n)$ converges to $\varphi(x)$. Since $\varphi(x_n) = 0$, we conclude that $\varphi(x) = 0$.

Hence B is connected.

Warning: Converse of this result is false!
In $\mathbb{R}$, $A = \mathbb{Q}$ is disconnected, but $\bar{\mathbb{Q}} = \mathbb{R}$ is connected.

Proposition 8.2.14 *In $\mathbb{R}$ a set A is connected $\Leftrightarrow$ A is an interval.*

Proof 8.2.15 ($\Rightarrow$) *Let $A \subseteq \mathbb{R}$ be a connected set. If $A = \emptyset$ or $card(A) = 1$, then A is a degenerated interval. Suppose that $card(A) \geq 2$. If A was not an interval, then we would have: $a, b \in A$, $a < b$ and x with $a < x < b$ such that $x \notin A$. Then taking $O_1 =]-\infty, x[$, $O_2 =]x, \infty[$ we would have: $A \subseteq O_1 \cup O_2$, $A \cap O_1 \neq \emptyset$, $A \cap O_2 \neq \emptyset$, a contradiction.*

($\Leftarrow$) *For any $a, b \in \mathbb{R}$, $a < b$, by the corollary 8.2.8, we know that $]a, b[$ is connected, then by proposition 8.2.12, $[a, b]$ is also connected.*

Example 8.2.16 *Let $\varphi : [0, 2\pi] \to \mathbb{R}^2$, $\varphi(x) = (\cos x, \sin x)$. Since $[0, 2\pi]$ is connected and φ is continuous, $\varphi([0, 2\pi])$ is connected. $\varphi([0, 2\pi])$ is the unit circle.*

Example 8.2.17 *Let A_1, A_2 be two circles in the complex plane which are intersecting in two points, i.e. $A_1 \cap A_2 = \{z_1, z_2\}$. We know that these circles are connected and finite sets are disconnected. So, the intersection of two connected sets need not to be connected.*

Proposition 8.2.18 *Let (X, d) be a m.s. and $(A_\alpha)_{\alpha\in I}$ be a family of connected subsets of X. Suppose that;*

- $\cap_{\alpha\in I} A_\alpha \neq \emptyset$ *or*
- $\exists \alpha_0 \in I$ *such that* $A_{\alpha_0} \cap A_\alpha \neq \emptyset \ \forall \alpha \in I$.

Then the union $\cup_{\alpha\in I} A_\alpha$ is connected.

Proof 8.2.19 *To prove that A is connected it is enough to show that every continuous $\varphi : A \to D$ is constant, i.e. $\forall x, y \in A$, $\varphi(x) = \varphi(y)$*

Let $x, y \in A$. Then $x \in A_\alpha$, $y \in A_\gamma$ for some $\alpha, \gamma \in I$. Let $x' \in A_\alpha \cap A_{\alpha_0}$ and $y' \in A_\gamma \cap A_{\alpha_0}$. As $\varphi : A_\alpha \to D$ and $\varphi : A_\gamma \to D$ are continuous and $A_\alpha, A_\gamma, A_{\alpha_0}$ are connected, $\varphi(x) = \varphi(x') = \varphi(y') = \varphi(y)$. Hence A is connected.

8.3 Connected Components of a Set

Let (X, d) be a m.s., $A \subseteq X$ any set. $(A \neq \emptyset)$. Let $x \in A$ be any point. Let $A_x = \{B \subseteq A : B$ is connected and $x \in B\}$. Observe that, $A \neq \emptyset$ since $B = \{x\} \in A_x$.

By proposition 8.2.18 $\cup_{B\in A_x} B$ is a connected set. Let $C_x = \cup_{B\in A_x} B$. Then C is the largest connected subset of A that contains x.

We call this C_x **the connected component of** A **containing** x. By the properties below, connected components of a set form a partition.

Proposition 8.3.1 *Properties of connected components of a set:*

1. *For $x \neq y$, $x, y \in A$ Either $C_x = C_y$ or $C_x \cap C_y = \emptyset$.*
2. $\cup_{x\in A} C_x = A$.

Proof 8.3.2
1. *If $C_x \cap C_y \neq \emptyset$ then $C_x \cup C_y$ is connected. As $x \in C_x \cup C_y$, by maximality of C_x, $C_x \cup C_y \subseteq C_x$. So $C_x = C_x \cup C_y$. Similarly $C_y = C_x \cup C_y$. Hence $C_x = C_y$.*
2. *Trivial.*

Example 8.3.3 *In $(\mathbb{R}, d)$,*

1. *If $A = \mathbb{N}$ or $A = \mathbb{Z}$ then $\forall x \in A$, $C_x = \{x\}$.*
2. *If $A = \mathbb{Q}$ or $A = \mathbb{R} \setminus \mathbb{Q}$ then $\forall x \in A$, $C_x = \{x\}$.*
3. *If $A =]-2, -1[\cup \{0\} \cup [1, 2[\cup [3, 7]$, then all these four sets are connected components of A.*

Definition 8.3.4 *A set $A \subseteq X$ is said to be* ***totally disconnected****, if $\forall x \in A$, $C_x = \{x\}$.*

Example 8.3.5 *1. In $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R} \setminus \mathbb{Q}$ are totally disconnected.*

2. Any discrete m.s. (X, d) is totally disconnected. (Converse is false)

3. Any continuous mapping f from a connected set A into a totally disconnected set Y must be constant. (any continuous $f : [a, b] \to \mathbb{Q}$ is constant)

Proposition 8.3.6 *If $A \subseteq X$ is closed then each of its components C_x is also closed.*

Proof 8.3.7 *Let C_x be one of the components of A, then since A is closed, $\overline{C_x} \subseteq A$. As $x \in C_x \subseteq \overline{C_x} \subseteq A$, and $\overline{C_x}$ is connected, by the maximality of C_x, $C_x = \overline{C_x}$*

8.4 Pointwise Connected Sets

Let (X, d) be a m.s. $A \subseteq X$ a set. A **path (curve)** in A is a continuous mapping $\varphi : [0, 1] \to A$. The point $\varphi(0)$ is said to be the beginning point of φ. $\varphi(1)$ is said to be the end point of φ.

Example 8.4.1 *Let $X = \mathbb{R}^2$. $\varphi : [0, 1] \to \mathbb{R}^2$ $\varphi(t) = (\cos 2\pi t, \sin 2\pi t)$ is a path such that $\varphi(0) = \varphi(1)$.*

Example 8.4.2 *If $a, b \in \mathbb{R}^2$ then $\varphi : [0, 1] \to \mathbb{R}^2$, $\varphi(t) = a(1 - t) + bt$ is the equation of a line that joins a to b.*

Warning: There exist continuous functions $\varphi : [0, 1] \to \mathbb{R}^2$ such that $\varphi([0, 1]) = [0, 1] \times [0, 1]$. ("space filling curves")

Joining two Paths: Let $A \subseteq X$ be a set, and $\varphi_1 : [0, 1] \to A$ and $\varphi_2 : [0, 1] \to A$ are two paths such that $\varphi_1(1) = \varphi_2(0)$.

Let $\varphi : [0, 1] \to A$ be defined by $\varphi(t) = \begin{cases} \varphi_1(2t) & \text{if } 0 \leq t \leq 1/2 \\ \varphi_2(1 - 2t) & \text{if } 1/2 \leq t \leq 1 \end{cases}$.

Then $\varphi([0, 1]) = \varphi_1([0, 1]) \cup \varphi_2([0, 1])$.

We shall denote this φ as $\varphi_1 * \varphi_2$

Definition 8.4.3 ***Convex Sets:*** *A subset A of $\mathbb{R}^n$ is said to be **convex**, if for any two points $a, b \in A$ the line $a(1 - t) + bt$, $(0 \leq t \leq 1)$ joining a to b lies in A.*

Example 8.4.4 *For instance any ball $B_r(x)$ is convex, but the sphere $S_r(x)$ is not convex.*

Definition 8.4.5 *A subset A of a m.s. (X, d) is said to be **pathwise connected (or pointwise)**, if it is possible to join any two points a, b of A by a path $\varphi : [0, 1] \to A$.*

Example 8.4.6 • *Let* $X = \mathbb{R}^2$, $A = \mathbb{R}^2 \setminus \{(0,0)\}$. *This set is pointwise connected. But if* $X = \mathbb{R}$, $A = \mathbb{R} \setminus \{0\}$ *is not pointwise connected.*

- *In* $\mathbb{R}^n$ *any convex set is pointwise connected.*
- $\mathbb{R}^2 \setminus \mathbb{Q}^2$ *is pointwise connected.*

Proposition 8.4.7 *Any pointwise connected set in any m.s. is connected.*

Proof 8.4.8 *Suppose* A *is pointwise connected. So given any two points* $a, b \in A$*, there is a continuous function* $f : [0,1] \to A$ *such that* $f(0) = a$ *and* $f(1) = b$. *To see that* A *is connected, we have to show that every continuous* $\varphi : A \to D$ *is constant, i.e.* $\forall a, b \in A$, $\varphi(a) = \varphi(b)$.

Let $a, b \in A$ *be any points. Let* $f : [0,1] \to A$ *be a path that joins* a *to* b *and let* $\varphi : A \to D$ *any continuous mapping. Then* $\varphi \circ f : [0,1] \to D$ *is continuous. As* $[0,1]$ *is connected this mapping must be constant on* $[0,1]$, *say* $\varphi \circ f(t) = 0, \ \forall t \in [0,1]$. *Then* $\varphi(f(0)) = \varphi(f(1)) = 0$, *i.e.* $\varphi(a) = \varphi(b)$. *So* $\varphi(A) = \{0\}$.

Hence A *is connected.*

Example 8.4.9 *Not every connected set is pointwise connected. Let* $\varphi :]0,1] \to \mathbb{R}^2$, *be such that* $\varphi(x) = (x, \sin 1/x)$. *Then* φ *is a continuous function. So the set* $A = \varphi(]0,1])$ *is a connected set in* $\mathbb{R}^2$. *Then* $\bar{A} = (\{0\}x\,[-1,1]) \cup A$. *As* A *is connected, so is* $\bar{A}$. *Let us see that* $\bar{A}$ *is not pointwise connected. Let* $a = (0,0)$, $b = (1, \sin 1)$. *Then* $a, b \in \bar{A}$.
Let us see that there is no continuous $f : [0,1] \to \bar{A}$ *such that* $f(0) = a$ *and* $f(1) = b$. *If we had such an* f, *for* $t > 0$, *we would have* $f(1) \in A$ *(not* $\bar{A}$*), i.e.* $f(t) = (t, \sin 1/t)$. *As* f *is continuous at zero and* $\sin 1/t$ *is not (and cannot be extended continuously to zero) continuous at zero. such an* f *cannot exist. So* $\bar{A}$, *although connected, is not pointwise connected.*

Proposition 8.4.10 $A \subseteq \mathbb{R}^2$ *open connected* $\Longrightarrow A$ *is pointwise connected.*

Proof 8.4.11 *Fix a point* $a \in A$. *Let* $D = \{b \in A : \exists$ *a curve* $\varphi : [0,1] \to A$ *joining* a *to* $b\}$ *Then;*

1. $D \neq \emptyset$ *since* $a \in D$ *with constant function.*

2. D *is open in* A. *Indeed, for* $b \in D$, *as* A *is open, there is an* $\varepsilon > 0 \ \ni B_\varepsilon(b) \subseteq A$. *Let* $x \in B_\varepsilon(b)$, *as it is convex (so that pointwise connected), by a curve* φ_2 *we can join* x *to* b. *By definition of* D, *there is a curve* φ_1 *joining* a *to* b *in* A. *Then* $\varphi_1 * \varphi_2$ *joins* a *to* x. *Hence* $x \in D$, *i.e.* $B_\varepsilon(b) \subseteq D$. *So* D *is open.*

3. D *is closed in* A. *(i.e.* $\overline{D}^A = \overline{D} \cap A \subseteq D$*)*

   Let $x \in \overline{D} \cap A$, *and see that* $x \in D$. *Since* $x \in \overline{D}$, $\forall \varepsilon > 0$. $B_\varepsilon(x) \cap D \neq \emptyset$. *As* $x \in A$ *and* A *is open, for some* $\varepsilon_0 > 0$, $B_{\varepsilon_0}(x) \subseteq A$. *So* $B_{\varepsilon_0}(x) \cap D \neq \emptyset$ *and* $B_{\varepsilon_0}(x) \subseteq A$.

   Let $b \in B_{\varepsilon_0}(x) \cap D$, *then we can join* a *to* b *by a curve* φ_1. *Then by another curve* φ_2, *we can join* b *to* x *(since* $B_{\varepsilon_0}(x)$ *is pointwise connected). Hence* $\varphi_1 * \varphi_2$ *joins* a *to* x, *so that* $x \in D$. *Hence* D *is closed in* A.

Thus;

$D \neq \emptyset$, $D \subseteq A$ *open in* A, $D \subseteq A$ *closed in* A *and* A *is connected. So* $D = A$, *hence* A *is pointwise connected.*

8.5 Some Applications

Connectedness can be used to obtain different results:

8.5.1 To find "fixed point theorems"

Theorem 8.5.1 *Any continuous function* $f : [a, b] \to [a, b]$ *has at least one fixed point. (i.e.,* $\exists x_0 \in [a, b] : f(x_0) = x_0$*)*

Proof 8.5.2 *First assume that* $a = 0$ *and* $b = 1$. *So that* $f : [0, 1] \to [0, 1]$. *If* $f(0) = 0$ *or* $f(1) = 1$, *we are done. Otherwise* $f(0) > 0$, *and* $f(1) < 1$.

Let $g(t) = t - f(t)$. *Then* $g : [0, 1] \to \mathbb{R}$ *is continuous, and we have;*

$g(0) = -f(0) < 0$

$g(1) = 1 - f(1) > 0$

Since g *is continuous and* $[0, 1]$ *is connected* $g([0, 1])$ *is connected, so it is a compact interval* $[c, d]$. *Since* $[c, d]$ *contains both negative and positive numbers, zero must be in* $[c, d]$. *Hence* $g(x_0) = 0$ *for some* $x_0 \in [0, 1]$, *i.e.* $f(x_0) = x_0$.

To prove the general case , let $\varphi : [0, 1] \to [a, b]$ *be* $\varphi(t) = a(1 - t) + bt$. *Then;*

$\varphi^{-1} \circ f \circ \varphi : [0, 1] \to [0, 1]$. *So, by the first step there is a* $t_0 \in [0, 1]$, $\ni$ $\varphi^{-1} \circ f \circ \varphi(t_0) = t_0$.

Hence $f(\varphi(t_0)) = \varphi(t_0)$.

Theorem 8.5.3 *(Brewer, 1908) Given any compact, convex subset* K *of* $\mathbb{R}^n$, *every continuous function* $f : K \to K$ *has a fixed point.*

8.5.2 Existence of Real Roots of Polynomials

Theorem 8.5.4 *Every polynomial* $P(x) = a_n x^n + ... + a_1 x + a_0$ *of odd degree has at least one real root.*

Proof 8.5.5 *We can assume* $a_n > 0$. *Then* $\lim_{x\to\infty} P(x) = \infty$ *and* $\lim_{x\to-\infty} P(x) = -\infty$.

So for $b > 0$ *large enough,* $P(b) > 0$ *and* $a < 0$ *small enough* $P(a) < 0$. *So* $P : [a, b] \to \mathbb{R}$ *assures on the interval* $[a, b]$ *both positive and negative values. As* P *is continuous* $P([a, b])$ *is an interval. Hence* $0 \in P([a, b])$, *hence* $P(x_0) = 0$ *for some* $x_0 \in [a, b]$.

8.5.3 The Structure of Open Sets in $\mathbb{R}$

Theorem 8.5.6 *A subset A of $\mathbb{R}$ is open iff A is a union of countably many, pairwise disjoint, open intervals, $]a_n, b_n[$ $(n \in \mathbb{N})$*

Proof 8.5.7 *The implication $(\Leftarrow)$ is trivial since union of open sets is open.*

To prove $(\Rightarrow)$ suppose A is open. $\forall x \in A$, let C_x be the component of A that contains x. So C_x is a connected maximal set contained in A and containing x. As every connected set in $\mathbb{R}$ is an interval, C_x is an interval. Let us see that it is open. Let $y \in C_x$. As $C_x \subseteq A$ and A is open, there is some $\varepsilon > 0$ such that $]y-\varepsilon, y+\varepsilon[\, \subseteq A$. As $]y-\varepsilon, y+\varepsilon[\, \cap C_x \neq \emptyset$, the set $]y-\varepsilon, y+\varepsilon[\, \cup C_x$ is connected and is contained in A. Hence, by maximality of C_x, $]y-\varepsilon, y+\varepsilon[\cup C_x = C_x$. So C_x is open.

Hence, C_x is an open interval. As $A = \cup_{x\in A} C_x$ and for $x \neq y$ either $C_x = C_y$ or $C_x \cap C_y = \emptyset$. A is a union of some number of open, disjoint intervals. Since in every interval there is a rational number, the number of these intervals must be countable. So $A = \cup_{n\in\mathbb{N}}\,]a_n, b_n[$, and $]a_n, b_n[\cap]a_m, b_m[= \emptyset$, for $m \neq n$.

8.5.4 To find if given two metric spaces are homeomorphic or not

Theorem 8.5.8 *For $n \neq m$, there exists a homeomorphism between the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$.*

Proposition 8.5.9 *For $n > 1$, $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic.*

Proof 8.5.10 *Suppose that we have a continuous bijection $f : \mathbb{R}^n \to \mathbb{R}$. Let $a \in \mathbb{R}^n$ be such that $f(a) = 0$. Then $f(\mathbb{R}^n \setminus \{a\}) = \mathbb{R} \setminus \{0\}$. As $\mathbb{R}^n \setminus \{a\}$ is connected and $\mathbb{R} \setminus \{0\}$ is not connected, this equality is not possible. So there is no continuous bijection $f : \mathbb{R}^n \to \mathbb{R}$.*

8.6 Exercises

1. Let $f : [a.b] \to \mathbb{R}$ be a monotone increasing function. Show that f is continuous iff $f([a, b])$ is an interval.

2. Let I be an interval and $f : I \to \mathbb{R}$ be a strictly increasing continuous function. Let $J = f(I)$. Show that $f^{-1} : J \to \mathbb{R}$ is also continuous.

3. Let $f :]0, \infty[\to \mathbb{R}$, $f(x) = x^2$. Show that f is strictly increasing and continuous. Deduce that $f^{-1}(x) = \sqrt{x}$ is also continuous from $]0, \infty[= f(]0, \infty[)$ to $\mathbb{R}$.

4. Let $f : [0, 2] \to \mathbb{R}$ be a continuous function. Suppose that $f(0) = f(2)$. Show that there exists $c \in [0, 1]$ such that $f(c) = f(c + 1)$.

5. Let $f : [a, b] \to \mathbb{R}$ be a continuous function, $m = \inf_{a \le x \le b} f(x)$ and $M = \sup_{a \le x \le b} f(x)$. Show that $f([a, b]) = [m, M]$.

6. Let $S = \{(x, y) \in R^2 : x^2 + y^2 = 1\}$. Show that the sets S and $[0, 2\pi]$ are not homeomorphic. Show also that the mapping $\phi : [0, 2\pi] \to S$, $\phi(x) = (\cos x, \sin x)$ is continuous, closed and onto.

7. Let I be an open interval and $f : I \to \mathbb{R}$ be a strictly increasing continuous function. Show that, for each open subset U of I, $f(U)$ is also open.

8. Let I be an open interval and $f : I \to \mathbb{R}$ one-to-one continuous function. Show that f is strictly monotone on I.

9. Let (X, d) be m.s. and A, B be 2 closed subsets of X such that both the sets $A \cap B$ and $A \cup B$ are connected. Show that A and B are connected.

10. Let (X, d) be a m.s. and $K_1 \supseteq K_2 \supseteq ... \supseteq K_n \supseteq ...$ are nonempty compact and connected subsets of X. Show that the set $K = \cap_{n \ge 1} K_n$ is also connected.

11. Let in $\mathbb{R}^2$, $K_n = \{(x, y) \in R^2 : x \ne 0 \text{ and } 0 \le |y| < \frac{1}{n}\}$. Show that K_n is connected, $K_1 \supseteq .... \supseteq K_n \supseteq ...$, but $K = \cap_{n \ge 1} K_n$ is not connected.

12. The following result says that "at any time, on the surface of the earth, there exists two diametrically opposite points at which the temperature is the same." To prove this statement let $T : S \to \mathbb{R}$ be a continuous function, $S = \{z = (x, y) \in R^2,\ x^2 + y^2 = 1\}$. Show that there is $z_0 \in S$ such that $T(z_0) = T(-z_0)$.

Chapter 9

Numerical Series

1. **Generalities about series**
2. **Tests of convergence for positive series**
3. **Absolute and Unconditional Convergence**
4. **Abel and Dirichlet tests**
5. **Conditional Convergence and Riemann Theorem**
6. **Product of Absolutely Convergent Series**

9.1 Generalities about Series

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$. The "formal sum" $\sum_{n=0}^{\infty} x_n$ is said to be a **series** whose general term is $(x_n)_{n\in\mathbb{N}}$. As such, this sum has no meaning, what we want is to give a meaning to this sum. Put;

$$\begin{array}{ll} S_0 & = x_0, \\ S_1 & = x_0 + x_1, \\ \vdots & \vdots \\ S_n & = x_0 + ... + x_n \end{array}.$$

In this way we get a sequence $(S_n)_{n\in\mathbb{N}}$. This S_n is said to be a *partial sum* of the series $\sum_{n=0}^{\infty} x_n$. As with any sequence, this sequence $(S_n)_{n\in\mathbb{N}}$ either converges or diverges.

Definition 9.1.1 *We say that the series* $\sum_{n=0}^{\infty} x_n$ *converges, if the sequence* $(S_n)_{n\in\mathbb{N}}$ *converges.*

Or equivalently, $\sum_{n=0}^{\infty} x_n$ *converges iff* S_n *is Cauchy,*

i.e., $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall m > N, \ \forall n > N, \ |S_m - S_n| = |\sum_{k=n+1}^{m} x_k| < \varepsilon$, *(if* $m < n$*) (1)*

Equivalently, $\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall p \in \mathbb{N}, |\sum_{k=N+1}^{N+p} x_k| < \varepsilon$.

If the series $\sum_{n=0}^{\infty} x_n$ converges then the number $S = \lim_{n\to\infty} S_n$ is said to be the sum of the series, so that $S = \sum_{n=0}^{\infty} x_n$. Hence;

1. If $\sum_{n=0}^{\infty} x_n$ converges then $|S_n - S_{n-1}| \to 0$ as $n \to \infty$. Hence, since $S_n - S_{n-1} = x_n$, $x_n \to 0$ as $n \to \infty$. From this we get that:
   $\sum_{n=0}^{\infty} x_n$ converges $\Rightarrow x_n \to 0$.

   Hence by contrapositive, if $x_n \nrightarrow 0$, then the series $\sum_{n=0}^{\infty} x_n$ diverges.
2. As in any convergence problem, there are two different problems:
   i) To find whether a given series converges or not.
   ii) If it converges, find its sum.

Example 9.1.2 *1. The series* $\sum_{n=0}^{\infty} (-1)^n$ *diverges, since the sequence* $x_n = (-1)^n$ *does not converge to 0, by 1 above.*

2. Now, consider the series $\sum_{n=1}^{\infty} 1/n$. *Here* $x_n = 1/n \to 0$.

But $S_n = 1 + 1/2 + ... + 1/n \to \infty$. *Hence the series* $\sum_{n=0}^{\infty} x_n$ *diverges.*

3. If two series $\sum_{n=0}^{\infty} x_n$ *and* $\sum_{n=0}^{\infty} y_n$ *converge, then the series* $\sum_{n=0}^{\infty} (x_n + y_n)$ *converges and* $\sum_{n=0}^{\infty} c \cdot x_n$ *converges for any* $c \in \mathbb{R}$. *However,* $\sum_{n=0}^{\infty} (-1)^n + (-1)^{n+1}$ *converges, although*

$$\sum_{n=0}^{\infty}(-1)^n \text{ and } \sum_{n=0}^{\infty}(-1)^{n+1} \text{ diverge.}$$

Remark: Infinite sums are not commutative. For instance;

$1-1+1-...=0$!
$(1+1+...+1)-(1+1+...+1)=\infty-\infty$!
$1+1-1+1+1-1+...=\infty$!
$-1-1+1-1-1+1-...=-\infty$!

Example 9.1.3 *Study the convergence of the series* $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$.

Here $S_n=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{n\cdot(n+1)}$
As $\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$, $S_n=(1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+...+(\frac{1}{n}-\frac{1}{n+1})=1-\frac{1}{n+1}$
Hence, $\lim_{n\to\infty}S_n=1$.
So, $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}$ *converges and* $\sum_{n=1}^{\infty}\frac{1}{n(n+1)}=\lim_{n\to\infty}S_n=1$.

Example 9.1.4 *Let* $r\in\mathbb{R}$ *be a given number. Study the convergence of the series* $\sum_{n=1}^{\infty}r^n$ *(Geometric series)*

Here $S_n=1+r+...+r^n$.

- *If* $r=1$ *then* $S_n=n+1$ *and the series* $\sum_{n=1}^{\infty}r^n$ *diverges.*
- *If* $|r|>1$, $a_n=r^n\nrightarrow 0$ *so the series diverges.*
- *If* $|r|<1$. *Then* $S_n=1+r+r^2+...+r^n=\frac{1-r^{n+1}}{1-r}$.

As $|r|<1$, $\lim_{n\to\infty}r^{n+1}=0$. *Hence* $\lim_{n\to\infty}S_n=\frac{1}{1-r}$.

So the series $\sum_{n=0}^{\infty}r^n$ *converges iff* $|r|<1$ *and in this case,*

$$\sum_{n=0}^{\infty}r^n=\lim_{n\to\infty}S_n=\frac{1}{1-r}$$

Remark: In a series $\sum_{n=0}^{\infty}x_n$, if we drop finitely many terms then this does not affect the convergence or divergence of the series but changes the sum of the series. For instance;

$$\begin{aligned}\sum_{n=70}^{\infty}r^n &= r^{70}+r^{71}+...\\ &= r^{70}(1+r+r^2+...)=\frac{r^{70}}{1-r}\text{ for }|r|<1\end{aligned}$$

9.2 Tests of convergence for positive series

Definition 9.2.1 *A series* $\sum_{n=0}^{\infty} x_n$ *is said to be a* ***"positive series"*** *if* $x_n \geq 0$ *for all but finitely many* $n \in \mathbb{N}$.

Let, now $\sum_{n=0}^{\infty} x_n$ be a positive series and $S_n = x_0 + ... + x_n$ be its partial sums. Then $(S_n)_{n\in\mathbb{N}}$ is a monotone increasing sequence. Therefore, $(S_n)_{n\in\mathbb{N}}$ converges iff it is bounded from above.

Hence, $\sum_{n=0}^{\infty} x_n$ is convergent iff $\exists M > 0,\ \forall n \in \mathbb{N},\ \sum_{i=0}^{n} x_i \leq M$.

Example 9.2.2 *Study the convergence of the series* $\sum_{n=1}^{\infty} \frac{1}{n!}$

$S_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$

We know that $S_n \leq 3$, *for every* $n \geq 0$ *and that* S_n *converges to some number* $e \in \mathbb{R}$. *Hence* $\sum_{n=1}^{\infty} \frac{1}{n!} = e$.

Theorem 9.2.3 ***(The First Comparison Test)*** *Let* $\sum_{n=0}^{\infty} x_n$ *and* $\sum_{n=0}^{\infty} y_n$ *be positive series. Suppose that* $x_n \leq y_n$ *for all but finitely many* $n \in \mathbb{N}$. *Then;*

1. *If* $\sum_{n=0}^{\infty} y_n$ *converges, so does* $\sum_{n=0}^{\infty} x_n$.

2. *If* $\sum_{n=0}^{\infty} x_n$ *diverges, so does* $\sum_{n=0}^{\infty} y_n$.

Proof 9.2.4 1. *Suppose* y_n *converges. Let* $M = \sum_{n=0}^{\infty} y_n$. *Then* $\sum_{i=0}^{n} x_i \leq M, \forall n \in \mathbb{N}$.

Hence $\sum_{n=0}^{\infty} x_n$ *converges.*

2. *If* $\sum_{n=0}^{\infty} x_n$ *diverges, then the sum* $S_n = x_0 + ... + x_n$ *is unbounded.*

As $T_n = y_0 + ... + y_n \geq S_n$, T_n *is also unbounded. So* $\sum_{n=0}^{\infty} y_n$ *diverges.*

Example 9.2.5 *Show that the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges. Indeed, first observe that* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges iff the series* $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ *converges.*

Now $\frac{1}{(n+1)^2} \leq \frac{1}{n(n+1)}, \forall n > 1$. *Hence* $\sum_{k=1}^{n} \frac{1}{(k+1)^2} \leq \sum_{k=1}^{n} \frac{1}{k(k+1)} \leq 1$. *Hence,* $\sum_{k=1}^{\infty} \frac{1}{k^2} \leq 2, \forall n \geq 1$. *So, the series* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *converges.*

Hence, for every $p \geq 2$, *the series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges.*

For some $p > 1$, *we can find the sum but not for all* p. *For instance;* $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$

Example 9.2.6 *Let* $0 < x < 1$ *and* $(a_n)_{n \in \mathbb{N}}$ *be a bounded sequence* $a_n \geq 0$. *Show that the series* $\sum_{n=0}^{\infty} a_n x^n$ *converges.*

Let $M = \sup_{n \in \mathbb{N}} a_n$, *then* $a_n x^n \leq M x^n$.

As $\sum_{n=0}^{\infty} M x^n = M \sum_{n=0}^{\infty} x^n = M \frac{1}{1-x}$ *since* $0 < x < 1$. *So we can conclude that* $\sum_{n=0}^{\infty} a_n x^n$ *converges.*

Theorem 9.2.7 ***(The Second Comparison Test)*** *Let* $\sum_{n=0}^{\infty} x_n$, $\sum_{n=0}^{\infty} y_n$ *be two positive series. If* $\lim_{n\to\infty} \frac{x_n}{y_n} = L$ *and,*

1. *if* $L \neq 0$ *then both series are of the same nature. (both converge or both diverge)*
2. *if* $L = 0$ *and* $\sum_{n=0}^{\infty} y_n$ *converges, then* $\sum_{n=0}^{\infty} x_n$ *converges.*
3. *if* $L = \infty$ *and* $\sum_{n=0}^{\infty} x_n$ *converges, then* $\sum_{n=0}^{\infty} y_n$ *converges.*

Proof 9.2.8

1. *Suppose* $L \neq 0$. *As* $\frac{x_n}{y_n} \geq 0$, $L > 0$. *Let* $\varepsilon = L/2$. *Then, since* $\frac{x_n}{y_n} \to L$, $\exists\, N \in \mathbb{N}$ *such that* $\forall n \in \mathbb{N}$, $L/2 \leq \frac{x_n}{y_n} \leq 3L/2$. *(i.e.* $\frac{L}{2} \cdot y_n \leq x_n \leq \frac{3L}{2} \cdot y_n, \ \forall n \in \mathbb{N}$*)*

   Hence by the first comparison test, both series are of the same nature.

2. *If $L = 0$, then for $\varepsilon = 1$, $\exists N \in \mathbb{N}$, $\forall n \in \mathbb{N}$, $\dfrac{x_n}{y_n} \leq 1$. So $x_n \leq y_n$ $\forall n \in \mathbb{N}$. Apply again the first comparison test.*

3. *Consider $\dfrac{y_n}{x_n} \to 0$, then apply 2.*

Example 9.2.9 *Study the convergence of the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$. Compare this series with $\displaystyle\sum_{n=1}^{\infty} 1/n$. As $\lim_{n\to\infty} \dfrac{\frac{1}{n}}{\frac{1}{n^{1+\frac{1}{n}}}} = \lim_{n\to\infty} n^{\frac{1}{n}} = 1$. Hence, by the second comparison test $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ diverges.*

Recall: Let $(x_n)_{n\in\mathbb{N}}$ be a bounded sequence, $l = \liminf x_n$, and $L = \limsup x_n$. $x_n < l - \varepsilon$ and $x_n > L + \varepsilon$ are finitely many.

1. $\forall \varepsilon > 0$, $x_n < l - \varepsilon$ for at most finitely many $n \geq 0$.

2. $\forall \varepsilon > 0$, $x_n > L + \varepsilon$ for at most finitely many $n \geq 0$.

3. $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$ $l - \varepsilon < x_n < L + \varepsilon$

Theorem 9.2.10 ***(Root Test I)*** *Let $\displaystyle\sum_{n=0}^{\infty} a_n$ be a positive series. $L = \limsup \sqrt[n]{a_n}$ and $l = \liminf \sqrt[n]{a_n}$. Then,*

1. *If $L < 1$ then the series converges.*

2. *If $l > 1$ then the series diverges.*

3. *If $L = 1$ or $l = 1$ we cannot conclude.*

Proof 9.2.11 *1. Let $L < 1$. Choose an $\varepsilon > 0$ small enough to have $L + \varepsilon < 1$. Since $L = \limsup \sqrt[n]{a_n}$, there is $N \in \mathbb{N} : \forall n \geq N$, $\sqrt[n]{a_n} \leq L + \varepsilon$. Hence $a_n \leq (L + \varepsilon)^n$.*

Now, since $L + \varepsilon < 1$, the geometric series $\displaystyle\sum_{n=0}^{\infty} (L + \varepsilon)^n$ converges. Hence, by the first comparison test the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges.

2. Let $l > 1$. Let $\varepsilon > 0$ be small enough to still have $l - \varepsilon > 1$. Since $l = \liminf \sqrt[n]{a_n}$ there is $N \in \mathbb{N} : \forall n \geq N$, $\sqrt[n]{a_n} \geq l - \varepsilon$. Hence $a_n \geq (l - \varepsilon)^n$. As $l - \varepsilon > 1$, $(l - \varepsilon)^n \to \infty$.

This means that a_n does not go to zero. So, the series $\displaystyle\sum_{n=0}^{\infty} a_n$ diverges.

3. Consider the series;

1) $\sum_{n=1}^{\infty} \frac{1}{n}$. *Here* $a_n = \frac{1}{n}$. *So* $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$. *Hence* $l = L = 1$ *and the series diverges.*

2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$. *Here* $a_n = \frac{1}{n^2}$. *So* $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$.

Hence $l = L = 1$ *but this time the series converges.*

Theorem 9.2.12 ***(Root Test II)*** *Let* $\sum_{n=0}^{\infty} a_n$ *be a positive series, and* $L = \limsup \sqrt[n]{a_n}$. *Then,*

1. *If* $L < 1$, *then the series converges.*

2. *If* $L > 1$, *then the series diverges.*

3. *If* $L = 1$, *then we cannot conclude.*

Proof 9.2.13 *We have to prove just 2. So suppose* $L > 1$. *Choose an* $\varepsilon > 0$ *small enough to still have* $L - \varepsilon > 1$. *Since* $\limsup \sqrt[n]{a_n} = L > L - \varepsilon > 1$, *for infinitely many* $n \in \mathbb{N}$ *we must have* $\sqrt[n]{a_n} \geq L - \varepsilon$. *But, then* $a_n \geq (L-\varepsilon)^n$ *for this n's. As* $L - \varepsilon > 1$, $(L-\varepsilon)^n \to \infty$, *as* $n \to \infty$, *we see that* a_n *is not bounded. In particular* $a_n \nrightarrow 0$, *so our series diverges.*

Example 9.2.14 *Let* $a_n \geq 0$, $x \geq 0$ *and consider the series* $\sum_{n=0}^{\infty} a_n x^n$. *Let* $\rho = \frac{1}{\limsup \sqrt[n]{a_n}}$, *then* $\limsup \sqrt[n]{a_n x^n} = \frac{x}{\rho}$.

So this series; converges if $x < \rho$, *diverges if* $x > \rho$ *and we cannot conclude if* $x = \rho$.

Theorem 9.2.15 ***(Ratio Test)*** *Let* $\sum_{n=0}^{\infty} a_n$ *be a positive series,* $L = \limsup \frac{a_{n+1}}{a_n}$ *and* $l = \liminf \frac{a_{n+1}}{a_n}$. *Then,*

1. *If* $L < 1$ *then the series converges.*

2. *If* $l > 1$ *then the series diverges.*

3. *If* $L = 1$ *or* $l = 1$ *we cannot conclude.*

Proof 9.2.16 *1. Suppose* $L < 1$. *Choose an* $\varepsilon > 0$ *such that we still have* $L + \varepsilon < 1$. *As* $L = \limsup \frac{a_{n+1}}{a_n}$, $\exists N \in \mathbb{N}$ *such that* $\forall n \geq N$, $\frac{a_{n+1}}{a_n} \leq L + \varepsilon$. *Hence, for any given* $n \geq N$,

$$\begin{aligned} \frac{a_{N+1}}{a_N} &\le L+\varepsilon \\ \frac{a_{N+2}}{a_N} &\le L+\varepsilon \\ &\vdots \\ \frac{a_{n+1}}{a_n} &\le L+\varepsilon \end{aligned}.$$

Multiplying these inequalities we get: $\dfrac{a_n}{a_N} \le (L+\varepsilon)^{n-(N+1)}$.

Hence, $a_n \le \dfrac{a_N}{(L+\varepsilon)^{N+1}} \cdot (L+\varepsilon)^n$. *So* $a_n \le c(L+\varepsilon)^n,\ \forall n \ge N$.

Hence, $\displaystyle\sum_{n=0}^{\infty} a_n \le \sum_{n=0}^{N-1} a_n + c\sum_{n=N}^{\infty}(L+\varepsilon)^n \le M_1 + c(L+\varepsilon)^N \frac{1}{1-(L+\varepsilon)} \le M$. *So,* $\displaystyle\sum_{k=0}^{n} a_k \le M,\ \forall n \ge 0$. *Hence the series* $\displaystyle\sum_{n=0}^{\infty} a_n$ *converges.*

2. *Suppose* $l > 1$. *Let* $\varepsilon > 0$ *be such that* $l - \varepsilon > 1$. *As* $l = \liminf \dfrac{a_{n+1}}{a_n}$, *the there is* $N \in \mathbb{N}$ *such that for* $n \ge N$, $\dfrac{a_{n+1}}{a_n} \ge l - \varepsilon$. *As above we get* $a_n \ge c(l-\varepsilon)^n$ *for some number* c. *As* $(l-\varepsilon) > 1$, $(l-\varepsilon)^n \to \infty$, *as* $n \to \infty$. *So* $a_n \nrightarrow 0$. *So* $\displaystyle\sum_{n=0}^{\infty} a_n$ *diverges.*

3. *Consider again the series;*

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$

   b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

   For both series $\dfrac{a_{n+1}}{a_n} \to 1$ *but one is convergent , the other is divergent.*

Remark: Let $a_n \ge 0$ be a positive sequence. If $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ then $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L$, too. So that if the root test is inconclusive, the ratio test cannot conclude, too.

Example 9.2.17 *Let* $0 < a < b$ *be fixed numbers. Consider the series that goes as follows:* $a + ab + ab^2 + a^2b^2 + a^2b^3 + a^3b^3 + a^3b^4 + a^4b^4 + a^4b^5 + ...$

Then $\dfrac{a_{n+1}}{a_n} = \begin{cases} a & or \\ b \end{cases}$

Hence $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n}$ *does not exist. However,* $\limsup \dfrac{a_{n+1}}{a_n} = b$ *and* $\liminf \dfrac{a_{n+1}}{a_n} = a$.

Example 9.2.18 *Determine whether the series* $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ *converges or diverges.*

Here, $a_n = \frac{n!}{n^n}$, $\frac{a_{n+1}}{a_n} = \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} = (\frac{n}{n+1})^n = (\frac{1}{1+\frac{1}{n}})^n \to \frac{1}{e} < 1$. *So, this series converges.*

This also proves that $\frac{n!}{n^n} \to 0$, *as* $n \to \infty$.

Integral Test:

Observe that none of these tests we have seen so far applies to the series $\sum_{n=0}^{\infty} \frac{1}{n^p}$, $(p > 0)$, so we need another test, called integral test.

For this test we need some results about "improper integrals":

Let $a \geq 0$ and $f : [a, +\infty[\to [0, +\infty[$ be a continuous function.

For $b \geq a$ let $F(b) = \int_a^b f(x)dx$. If, $\lim_{b\to\infty} F(b)$ exists (and finite) then we say that the improper integral $\int_a^\infty f(x)dx$ converges.

Example 9.2.19 *Study the convergence of the improper integral* $\int_1^\infty \frac{dx}{x^p}$. *So we take a* $b \geq 1$ *we calculate the integral* $F(b) = \int_1^b \frac{dx}{x^p}$, *then we look for* $\lim_{b\to\infty} F(b)$

case 1: $p = 1$. *Then* $\int_1^b \frac{dx}{x} = \ln b$. *Hence* $\lim_{b\to\infty} F(b) = \lim_{b\to\infty} \ln b = \infty$.

case 2: $0 < p < 1$. *Then* $\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}|_1^b = \frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1} \to \infty$, *as* $b \to \infty$.

case 3: $p > 1$. *Then* $\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}|_1^b = \frac{b^{-p+1}}{-p+1} - \frac{1}{-p+1} \to \frac{1}{p-1}$, *as* $b \to \infty$.

Hence $\int_1^\infty \frac{dx}{x^p}$ *converges.*

Conclusion: $\int_1^\infty \frac{dx}{x^p}$ *converges* $\Leftrightarrow p > 1$.

Example 9.2.20 *Study the convergence of the improper integral* $\int_2^\infty \frac{dx}{x(\ln x)^q}$, $(q > 0)$.

Let $b > 2$. *Put* $u = \ln x$ *then* $du = \frac{dx}{x}$.

Hence $\int_2^b \frac{dx}{x(\ln x)^q} = \int_{\ln 2}^{\ln b} \frac{du}{(u)^q}$.

Hence $\int_2^\infty \frac{dx}{x(\ln x)^q}$ *converges iff* $q > 1$.

Theorem 9.2.21 ***(Integral Test)*** *Let* $f : [1, +\infty[\to [0, +\infty[$ *be a continuous, decreasing function. Then the series* $\sum_{n=1}^{\infty} f(n)$ *and the improper integral* $\int_1^\infty f(x)dx$ *are of the same nature.*

Proof 9.2.22 *Let* $a_n = f(n), \forall n \in \mathbb{N}$. *For each* $n \geq 1$, *and* $n \leq x \leq n+1$, *since* f *is decreasing,* $f(n+1) \leq f(x) \leq f(n)$, *i.e.* $a_{n+1} \leq f(x) \leq a_n$ *for* $x \in [n, n+1]$.

Integrating this inequalities on the interval $[n, n+1]$ *we get;*

$\int_n^{n+1} f(n+1)dx \leq \int_n^{n+1} f(x)dx \leq \int_n^{n+1} f(n)dx$

i.e. $a_{n+1} \leq \int_n^{n+1} f(x)dx \leq a_n$, $\forall n \geq 1$.

Hence, $\displaystyle\sum_{n=1}^{N} a_{n+1} \leq \int_1^{N+1} f(x)dx \leq \sum_{n=1}^{N} a_n$

and also $\displaystyle\sum_{n=1}^{N} f(n) \leq \int_1^{N+1} f(x)dx + f(1) \leq f(1) + \sum_{n=1}^{N} f(n)$

Put $S_n = a_1 + a_2 + ... + a_n$, *then above inequalities becomes;*

$S_{n+1} - a_1 \leq \int_1^{n+1} f(x)dx \leq S_n$

$S_N \leq \int_1^{N+1} f(x)dx + f(1) \leq a_1 + S_n$

Form this inequalities the conclusion follows.

Moreover, in the case where the convergence occurs, we have;

$S - a_1 \leq \int_1^{\infty} f(x)dx \leq S$ *where* $\displaystyle S = \sum_{n=1}^{\infty} a_n$

Example 9.2.23 1. *The series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges* $\Leftrightarrow p > 1$. *Take* $f(x) = \dfrac{1}{x^p}$,

$f : [1, +\infty[\to \mathbb{R}^+$, *and apply theorem 9.2.21.*

2. *The series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n \ln n}$ *diverges, since* $\int_2^{\infty} \dfrac{dx}{x \ln x}$ *diverges.*

9.3 Absolute and Unconditional Convergence

Let $\displaystyle\sum_{n=0}^{\infty} a_n$ be an arbitrary series. To such a series we cannot apply the above tests, but we can apply them to the series $\displaystyle\sum_{n=0}^{\infty} |a_n|$.

Question: What relation is there between the convergence of the series $\displaystyle\sum_{n=0}^{\infty} a_n$ and $\displaystyle\sum_{n=0}^{\infty} |a_n|$?

Definition 9.3.1 *A series* $\sum_{n=0}^{\infty} a_n$ *is said to be* ***absolutely convergent*** *if the positive series* $\sum_{n=0}^{\infty} |a_n|$ *converges.*

As we shall see later the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.
Actually $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$. But $\sum_{n=1}^{\infty} |\frac{(-1)^n}{n}| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. Hence, "convergence" and "absolute convergence" are not equivalent notions.

Theorem 9.3.2 *Every absolutely convergent series* $\sum_{n=0}^{\infty} a_n$ *converges.*

Proof 9.3.3 *Let* $S_n = a_0 + ... + a_n$, $T_n = |a_0| + ... + |a_n|$. *As the series* $\sum_{n=0}^{\infty} |a_n|$ *converges,* T_n *is Cauchy.*
So, we have:
$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N,\ \forall p \in N\ |T_{n+p} - T_n| = T_{n+p} - T_n = |a_{n+1}| + ... + |a_{n+p}| < \varepsilon.$
Hence, $\forall n \geq N\ \forall p \in N,\ |S_{n+p} - S_n| = |a_{n+1} + ... + a_{n+p}| \leq |a_{n+1}| + ... + |a_{n+p}| < \varepsilon.$
Thus, S_n *is Cauchy, so it converges. This means that the series* $\sum_{n=0}^{\infty} a_n$ *converges.*

Example 9.3.4 *If in* $\mathbb{R}$, $x_n \to x$, *then* $|x_n| \to |x|$.

So if $\sum_{n=0}^{\infty} a_n$ *converges absolutely,*

$$|\sum_{n=0}^{\infty} a_n| = \lim |\sum_{k=0}^{n} a_k| \leq \lim \sum_{k=0}^{n} |a_k| = \sum_{n=0}^{\infty} |a_n|, \text{ i.e. } |\sum_{n=0}^{\infty} a_n| \leq \sum_{n=0}^{\infty} |a_n|.$$

Example 9.3.5 *We have just seen that the series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges, although it is not absolutely convergent. Let us calculate the sum of the series.*
We know that $\ln(1+x)' = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$.

$\lim_{m\to\infty} \lim_{n\to\infty}(1+\frac{1}{n})^m = 1$, *and* $\lim_{n\to\infty} \lim_{m\to\infty}(1+\frac{1}{n})^m = \infty$.
Hence, integrating we get:

$$\int_0^x (\ln(1+x))' dx = \ln(1+x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}.$$

Hence, $\ln 2 = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ... + \frac{(-1)^{n+1}}{n} + ...$

Now, write this sum taking one positive term then two negative terms:

$(1 - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + ...$

Now, add another parenthesis:

$[(1 - \frac{1}{2}) - \frac{1}{4}] + [(\frac{1}{3} - \frac{1}{6}) - \frac{1}{8}] + ... = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + ... = \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...)$

Conclusion: $\frac{1}{2} \ln 2 = \ln 2$ *!*

This absurdity shows that in a series - even if it is convergent - we cannot change the order of the terms in an infinite sum.

Question: When can we change the order of the terms without obtaining divergent series or different sums?

Definition 9.3.6 *Let* $\sum_{n=0}^{\infty} a_n$ *be a series and* $\sigma : \mathbb{N} \to \mathbb{N}$ *be a bijection. Then the series* $\sum_{n=0}^{\infty} a_{\sigma(n)}$ *is said to be a* ***rearrangement*** *of the given series.*

Example 9.3.7 *Let* $\sum_{n=0}^{\infty} a_n$ *be a series. If* $\sigma(0) = 81$, $\sigma(1) = 92$, $\sigma(2) = 301, ...,$ *then* $\sum_{n=0}^{\infty} a_{\sigma(n)} = a_{81} + a_{92} + a_{301} + ...$

Definition 9.3.8 *A series* $\sum_{n=0}^{\infty} a_n$ *is said to be* ***unconditionally convergent*** *if every rearrangement of this series converges.*

Example 9.3.9 *It is easy to see that the series* $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ *is convergent but not unconditionally convergent.*

9.3.1 Absolutely Convergent vs. Unconditionally Convergent

Theorem 9.3.10 *If a series* $\sum_{n=0}^{\infty} a_n$ *is absolutely convergent, then it is unconditionally convergent and that for every bijection* $\sigma : \mathbb{N} \to \mathbb{N}$, $\sum_{n=0}^{\infty} a_{\sigma(n)} = \sum_{n=0}^{\infty} a_n$.

Proof 9.3.11 *Let* $M = \sum_{n=0}^{\infty} |a_n|$ *and let* $\sigma : \mathbb{N} \to \mathbb{N}$ *be any bijection. Then* $\forall n \in \mathbb{N}$, $\sum_{i=0}^{n} |a_{\sigma(i)}| \leq M$. *Hence the series* $\sum_{n=0}^{\infty} a_{\sigma(n)}$ *is absolutely convergent, so convergent.*

Let $S = \sum_{i=0}^{\infty} a_i$, *i.e.* $S = \lim_{n\to\infty} \sum_{i=0}^{n} a_i$

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} : \forall n \geq N\ |S - \sum_{i=0}^{n} a_i| = |\sum_{i=n+1}^{\infty} a_i| < \varepsilon.$$

In particular, for all $n \geq N$ *and* $m \geq N$, $|\sum_{i=n}^{m} a_i| < \varepsilon$. $(*)$

Now, let $\tilde{N}$ *be large enough to have:* $\{0, 1, ..., N\} \subseteq \{\sigma(0), ..., \sigma(\tilde{N})\}$. *Then, for* $n \geq \tilde{N}$, $|\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} a_{\sigma(i)}| = |$ *a sum of some terms* a_i *with* $i > N| < \varepsilon$ *by* $(*)$.

Now, since $\sum_{i=0}^{n} a_i \to S$, *we see that* $\sum_{i=0}^{n} a_{\sigma(i)} \to S$, *too, as* $n \to \infty$.

Hence $\sum_{i=0}^{\infty} a_{\sigma(i)} = \sum_{i=0}^{\infty} a_i$.

We have seen that every absolutely convergent series is unconditionally convergent. We are going to show that converse is also true.

Let $\sum_{n=0}^{\infty} a_n$ be a series in $\mathbb{R}$. Let $a_n^+ = \sup\{a_n, 0\}$ and $a_n^- = \sup\{-a_n, 0\}$

Then $a_n^+ \geq 0$, and $a_n^- \geq 0$.

Hence $a_n^+ + a_n^- = |a_n|$, and $a_n - a_n = a_n$. $(**)$

Also $a_n^+ \leq |a_n|$, and $a_n^- \leq |a_n|$.

Lemma 9.3.12 *If* $\sum_{n=0}^{\infty} a_n$ *converges but* $\sum_{n=0}^{\infty} |a_n|$ *diverges then both series* $\sum_{n=0}^{\infty} a_n^+$ *and* $\sum_{n=0}^{\infty} a_n^-$ *diverges to* $+\infty$.

Proof 9.3.13 *Suppose for a contradiction, the series* $\sum_{n=0}^{\infty} a_n^+$ *converges and* $\sum_{n=0}^{\infty} a_n^-$ *diverges to* ∞.

Since $-a_n^- = a_n - a_n^+$ *then* $\sum_{n=0}^{\infty}(a_n^+ - a_n) = \sum_{n=0}^{\infty} a_n^-$ *and since by hypothesis both* $\sum_{n=0}^{\infty} a_n^+$ *and* $\sum_{n=0}^{\infty} a_n$ *converges, the series* $\sum_{n=0}^{\infty}(a_n^+ - a_n)$ *converges, but then the series* $\sum_{n=0}^{\infty} a_n^-$ *converges, contradiction.*

Lemma 9.3.14 *The series* $\sum_{n=0}^{\infty} |a_n|$ *converges iff* $\sum_{n=0}^{\infty} a_n^+$ *and* $\sum_{n=0}^{\infty} a_n^-$ *converges.*

Proof 9.3.15 ($\Rightarrow$) *As* $a_n^+ \leq |a_n|$ *and* $a_n^- \leq |a_n|$ *the comparison test implies that if* $\sum_{n=0}^{\infty} |a_n|$ *converges then both converge, too.*
($\Leftarrow$) *As* $a_n^+ + a_n^- = |a_n|$, *if the series* $\sum_{n=0}^{\infty} a_n^+$ *and* $\sum_{n=0}^{\infty} a_n^-$ *converge, then* $\sum_{n=0}^{\infty} |a_n|$ *converges and* $\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} a_n^+ + \sum_{n=0}^{\infty} a_n^-$.

Example 9.3.16 $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ *converges but not absolutely. Let* $a_n = \frac{(-1)^{n+1}}{n}$. *Then both of the series* $\sum_{n=0}^{\infty} a_n^+ = \sum_{n=1}^{\infty} \frac{1}{2n}$, $\sum_{n=0}^{\infty} a_n^- = \sum_{n=1}^{\infty} \frac{1}{2n+1}$ *diverge.*

Theorem 9.3.17 ***(Riemann)*** *A series* $\sum_{n=0}^{\infty} a_n$ *is absolutely convergent iff it is unconditionally convergent. (i.e. if* $\sum_{n=0}^{\infty} a_n$ *converges but not absolutely, then a rearrangement of* $\sum_{n=0}^{\infty} a_n$ *diverges.)*

Proof 9.3.18 *Suppose that* $\sum_{n=0}^{\infty} a_n$ *converges but* $\sum_{n=0}^{\infty} |a_n|$ *diverges. So, both of the series* $\sum_{n=0}^{\infty} a_n^+$ *and* $\sum_{n=0}^{\infty} a_n^-$ *diverges to* $+\infty$. *Let* $b_n = a_n^+$ *and* $c_n = -a_n^-$
Let $A < B$ *be two arbitrary real numbers.*
Let n_1 *be the first integer such that* $b_0 + b_1 + \cdots + b_{n_1} > B$.
Then, let m_1 *be the first integer such that* $b_0 + \cdots + b_{n_1} + c_0 + \cdots + c_{m_1} < A$.
Let $n_2 > n_1$ *be the smallest integer such that*

$$b_0 + \cdots + b_{n_1} + b_{n_1+1} + \cdots + b_{n_2} + c_0 + \cdots + c_{m_1} + c_{m_1+1} + \cdots + c_{m_2} > B.$$

Then, let $m_2 > m_1$ be the smallest integer such that
$b_0 + \cdots + b_{n_1} + b_{n_1+1} + \cdots + b_{n_2} + c_0 + \cdots + c_{m_1} + c_{m_1+1} + \cdots + c_{m_2} < A.$
$\vdots$

In this way we produce a rearrangement of the series $\sum_{n=0}^{\infty} a_n$ *such that, if* T_n *is the partial sum of this rearrangement,* $T_n > B$ *for infinitely many* n *and* $T_n < A$ *for infinitely many* n. *Hence* $(T_n)_{n\in\mathbb{N}}$ *diverges. i.e., the rearranged series,* $b_0 + \cdots + b_{n_1} + c_0 + \cdots + c_{m_1} + b_{n_1+1} + \cdots$ *diverges.*

Definition 9.3.19 *If a series* $\sum_{i=0}^{\infty} a_i$ *converges but* $\sum_{i=0}^{\infty} |a_i|$ *diverges, then we say that* $\sum_{i=0}^{\infty} a_i$ *is* ***conditionally convergent****.*

9.4 Abel and Dirichlet Tests

None of the tests we have seen so far applies to a series to a series of the form $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$.
If we put $a_n = \sin nx$, $b_n = \frac{1}{n}$ then the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is a series of the form $\sum_{n=1}^{\infty}(a_n b_n)$ with b_n decreasing to 0.
Or put $a_n = (-1)^n$ and $b_n = \frac{1}{\sqrt{n}}$ for the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ with b_n decreasing to 0.

Abel Formula - First Form: Let $A_0 = 0$, $A_1 = a_1, ..., A_n = a_1 + ... + a_n$ so that $A_1 - A_0 = a_1$, $A_2 - A_1 = a_2, ..., A_n - A_{n-1} = a_n$. Then;

$$\begin{aligned} a_n b_1 + ... + a_n b_n &= (A_1 - A_0)b_1 + ... + (A_n - A_{n-1})b_n \\ &= A_1(b_1 - b_2) + A_2(b_2 - b_3) + ... + A_{n-1}(b_{n-1} - b_n) + A_n b_n. \end{aligned}$$

Thus, $\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} A_k(b_k - b_{k+1}) + A_n b_n$.

Abel Formula - II (Cauchy Form): Let, for $n \geq m$, $A_{m,n} = a_m + a_{m+1} + ... + a_n$ so that $A_{m,m} = a_m$, $A_{m+1,m} - A_{m,m} = a_{m+1}, ..., A_{m,n} - A_{m,n-1} = a_n$. Hence,

$$\begin{aligned} a_m b_m + a_{m+1} b_{m+1} + ... + a_n b_n &= A_{m,m} b_m + (A_{m+1,m} - A_{m,m}) b_{m+1} + ... + (A_{m,n} - A_{m,n-1}) b_n \\ &= A_{m,m}(b_m - b_{m+1}) + ... + A_{m,n-1}(b_{n-1} - b_n) + A_{m,n} b_n \end{aligned}$$

Thus, $\sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} A_{m,k}(b_k - b_{k+1}) + A_{m,n} b_n$.

Theorem 9.4.1 ***(Dirichlet Test)*** *Consider a series of the form* $\sum_{n=1}^{\infty} a_n b_n$ *with:*

1. $b_n \geq 0$, b_n *decreases to zero.*
2. $\exists M > 0 : \forall n \in \mathbb{N}, \ |a_1 + ... + a_n| \leq M$.

Then the series $\sum_{n=1}^{\infty} a_n b_n$ *converges.*

Proof 9.4.2 *Let* $S_n = \sum_{k=1}^{n} a_k b_k$. *Then for* $n \geq m$,

$$S_n - S_{m-1} = \sum_{k=m}^{n} a_k b_k = \sum_{k=m}^{n-1} A_{m,k}(b_k - b_{k+1}) + A_{m,n} b_n. \text{ Then;}$$

$$\begin{aligned} |S_n - S_{m-1}| &\leq \sum_{k=m}^{n-1} |A_{m,k}||b_k - b_{k+1}| + |A_{m,n}|\, b_n \\ &\leq \sum_{k=m}^{n-1} M(b_k - b_{k+1}) + M b_n \\ &= M[(b_m - b_{m+1}) + ... + (b_{n-1} - b_n) + b_n] \\ &= M b_m \to 0 \text{ as } m \to \infty \end{aligned}$$

Hence S_n *is Cauchy, so converges. Hence our series converges.*

Example 9.4.3 *Show that for all* $x \in \mathbb{R}$ *the series* $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$; $x \neq 2k\pi$, *converges.*

Here $a_n = \sin nx$, $b_n = \frac{1}{n}$ *so* $b_n \downarrow 0$. *Hence, it is enough to show that* $\exists M > 0$ *such that* $|\sin x + ... + \sin nx| \leq M$. *(M may depend on x but not on n)*
To calculate the sum $\sin x + ... + \sin nx$ *let* $A_n = \cos x + ... + \cos nx$, $B_n = \sin x + ... + \sin nx$. *Then;*

$$\begin{aligned} A_n + iB_n &= e^{ix} + e^{2ix} + ... + e^{inx} = e^{ix}[1 + e^{in} + ... + e^{i(n-1)}] \\ &= e^{ix}\tfrac{1-e^{inx}}{1-e^{ix}} = e^{ix}\frac{e^{\frac{in}{2}x}[e^{-\frac{in}{2}x} - e^{\frac{in}{2}x}]}{e^{\frac{i}{2}x}[e^{-\frac{ix}{2}} - e^{\frac{ix}{2}}]} \\ &= e^{i\frac{n+1}{2}x}\frac{\sin(\frac{n}{2}x)}{\sin\frac{x}{2}} \\ &= (\cos\tfrac{n+1}{2}x + i\sin(\tfrac{n+1}{2}x))\frac{\sin(\frac{n}{2}x)}{\sin\frac{x}{2}} \end{aligned}$$

Hence, $A_n = \cos x + ... + \cos nx = \cos(\frac{n+1}{2}x)\frac{\sin(\frac{n}{2}x)}{\sin\frac{x}{2}}$, $B_n = \sin x + ... + \sin nx = \sin(\frac{n+1}{2}x)\frac{\sin(\frac{n}{2}x)}{\sin\frac{x}{2}}$

Hence, $|\sin x + ... + \sin nx| \leq \frac{1}{|\sin\frac{x}{2}|}$ *for* $x \neq 2k\pi$, $k \in \mathbb{Z}$

Hence by Dirichlet test, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ *converges for every* $x \neq 2k\pi$, $(k \in \mathbb{Z})$. *In particular, for* $x \in [0, 2\pi]$ *this series converges.*

Remark also that if instead of $\frac{1}{n}$, we take $\frac{1}{\sqrt{n}}, \frac{1}{\sqrt[p]{n}}, \ldots$ or any $b_n \geq 0$, $b_n \downarrow 0$ the calculation will be the same.

Theorem 9.4.4 ***(Leibniz Test)*** *Let $b_n \geq 0$, $b_n \downarrow 0$. Then she alternating series $\sum_{n=1}^{\infty} (-1)^n b_n$ converges.*

Proof 9.4.5 *Let $a_n = (-1)^n$ So that $|a_1 + \ldots + a_n| \leq 1$. Hence Dirichlet test applies. So our series converges.*

Example 9.4.6 *The series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}, \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}, \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[10]{n}}$ converge.*

Theorem 9.4.7 ***(Abel Test)*** *Consider a series of the form $\sum_{n=1}^{\infty} a_n b_n$. Suppose that:*

1. *The series $\sum_{n=0}^{\infty} a_n$ is convergent.*
2. *$(b_n)_{n \in \mathbb{N}}$ is positive, monotone and bounded.*

Then the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

Proof 9.4.8 *We can assume that $(b_n)_{n \in \mathbb{N}}$ is increasing and since it is bounded, $b_n \to b$, for some $b \in \mathbb{R}$. Then $(b - b_n) \downarrow 0$. As $\sum_{n=0}^{\infty} a_n$ converges, $S_n = a_0 + \ldots + a_n$ converges. So, $\exists M > 0$ such that $|S_n| \leq M$, $\forall n \geq 1$. Hence by Dirichlet test the series $\sum_{n=1}^{\infty} a_n(b - b_n)$ converges. So, $\sum_{n=1}^{\infty} a_n b_n = -\sum_{n=1}^{\infty} a_n(b - b_n) + b \sum_{n=1}^{\infty} a_n$ converges.*

9.5 Product of Series

Let $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ be two series. If we multiply them "formally" then we get an infinite matrix:

$$\begin{bmatrix}
 & a_0 & a_1 & \dots & a_n & \dots & a_{2n} & \dots & \\
b_0 & a_0b_0 & a_1b_0 & \dots & a_nb_0 & \dots & a_{2n}b_0 & \dots & \xrightarrow{+} b_0\sum_{m=0}^{\infty} a_m \\
b_1 & a_0b_1 & a_1b_1 & \dots & a_nb_1 & \dots & a_{2n}b_1 & \dots & \xrightarrow{+} b_1\sum_{m=0}^{\infty} a_m \\
\vdots & & & & & & & & \\
b_n & a_0b_n & a_1b_n & \dots & a_nb_n & \dots & a_{2n}b_n & \dots & \xrightarrow{+} b_n\sum_{m=0}^{\infty} a_m \\
\vdots & & & & & & & & \\
b_{2n} & a_0b_{2n} & a_1b_{2n} & \dots & a_nb_{2n} & \dots & a_{2n}b_{2n} & \dots & \xrightarrow{+} b_{2n}\sum_{m=0}^{\infty} a_m \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
 & \downarrow + & \downarrow + & & \downarrow + & & \downarrow +b_n & & \\
 & a_0\sum_{m=0}^{\infty} b_m & a_1\sum_{m=0}^{\infty} b_m & & a_n\sum_{m=0}^{\infty} b_m & & a_n\sum_{m=0}^{\infty} b_m & &
\end{bmatrix}$$

Problem: In which order we should sum these a_nb_m's?

1. First sum the rows, and get the expressions in the right-hand side then sum these expressions to get: $\sum_{n=0}^{\infty} b_n(\sum_{m=0}^{\infty} a_m)$
2. Or first sum the columns and get the expressions the matrix and them sum them to get: $\sum_{n=0}^{\infty} a_n(\sum_{m=0}^{\infty} b_m)$
3. Or we can sum a_nb_m's in a random way.

Basic Problem: When we sum a_nb_m's in different ways do we get the same sum?

Example 9.5.1 *Consider the following example of sums. Let $\alpha_{i,j} \in \mathbb{R}$ be such that*

$$\alpha_{i,j} = \begin{cases} 1 & \text{if } i = j \\ -1 & \text{if } i = j+1 \\ 0 & \text{otherwise} \end{cases} \quad \text{Then } \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} \alpha_{i,j} \neq \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \alpha_{i,j}.$$

This is actually a matrix of the form:

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & \cdots \\ -1 & 1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & -1 & 1 & \vdots \\ 0 & \cdots & 0 & -1 & \vdots \end{bmatrix}$$

9.5.1 Cauchy Method of Sum

Now, let

$c_0 = a_0 b_0$
$c_1 = a_0 b_1 + a_1 b_0$
$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_0$
$\vdots$
$c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$
$\vdots$

Formally, we should have:

$$(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_n a_m = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_n a_m = \sum_{n=0}^{\infty} c_n.$$

Theorem 9.5.2 *Let* $\sum_{n=0}^{\infty} a_n$ *and* $\sum_{n=0}^{\infty} b_n$ *be two absolutely convergent series and define* $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. *Then the series* $\sum_{n=0}^{\infty} c_n$ *is also absolutely convergent and* $(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} b_n a_m = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} b_n a_m = \sum_{n=0}^{\infty} c_n$.

Proof 9.5.3 *Let* $S_n = a_0 + \cdots + a_n,\ T_n = b_0 + \cdots + b_n,\ W_n = c_0 + \cdots + c_n$

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$S_n T_n =$ *the sum of all* $a_i b_j$ *in the smaller square.*
$W_n =$ *the sum of all* $a_i b_j$ *in the triangle (I).*
$W_{2n} =$ *the sum of all* $a_i b_j$ *in the triangle (II).*

Suppose first that all $a_i \geq 0$, $b_n \geq 0$. *Then* $W_n \leq S_n T_n \leq W_{2n}$. *If* $S_n \to S$ *and* $T_n \to t$ *then* W_n *being bounded from above, converges. Hence* $W_n \to T \cdot S$.

As, our series are absolutely convergent, $|S_n T_n - W_n| \leq \tilde{S}_n \tilde{T}_n - \tilde{W}_n \to 0$.

Here $\tilde{S}_n = |a_0| + \cdots + |a_n|,\ \tilde{T}_n = |b_0| + \cdots + |b_n|,\ \tilde{W}_n = |a_0||b_0| + \cdots + |a_n||b_n|$

Hence $W_n \to T \cdot S$. *i.e.* $(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty} c_n$.

Example 9.5.4 *Consider the series* $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *and* $\sum_{n=0}^{\infty} \frac{y^n}{n!}$. *For every* $x, y \in \mathbb{R}$, *these series converge absolutely by ratio test.*

Let us determine the product of this series.

Let $a_n = \dfrac{x^n}{n!}$, $b_n = \dfrac{y^n}{n!}$ *and* $c_n = a_0 b_n + \cdots + a_n b_0$. *Then we know that:*

$$(\sum_{n=0}^{\infty} a_n)(\sum_{n=0}^{\infty} b_n) = \sum_{n=0}^{\infty} c_n.$$

Now let us calculate c_n :

$$\begin{aligned} c_n &= 1 \cdot \frac{y^n}{n!} + \frac{x}{1!} \cdot \frac{y^{n-1}}{(n-1)!} + \cdots + \frac{y}{1!} \cdot \frac{x^{n-1}}{(n-1)!} + 1 \cdot \frac{x^n}{n!} \\ &= \frac{1}{n!}\left[y^n + \frac{nx}{1!} y^{n-1} + \cdots + x^n\right] = \frac{1}{n!}(x+y)^n \end{aligned}$$

So that $\displaystyle\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{1}{n!}(x+y)^n$.

Let $S(x) = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $S(y) = \displaystyle\sum_{n=0}^{\infty} \frac{y^n}{n!}$, *then we see that:*

1. $S(x)S(y) = S(x+y)$.
2. $S(0) = 1$ *so that* $S(x-x) = S(x)S(-x) = S(0) = 1$, *so that* $S(-x) = \dfrac{1}{S(x)}$

   Hence $\dfrac{1}{\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}} = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n$.
3. $S(x) \neq 0$, $\forall x \in \mathbb{R}$, *since* $S(x)S(-x) = 1$.

In mathematics, instead of $S(x)$ *we write* e^x.

9.6 Exercises

1. Show that

   (a) $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

   (b) For any $m > 0$ and any $\alpha > 0$, $\lim_{n\to\infty} \dfrac{(\ln n)^m}{n^\alpha} = 0$.

   (c) For any $k > 0$, $\lim_{n\to\infty} n^k e^{-\frac{1}{n}} = 0$.

2. Test the following series for convergence or divergence.

   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^n}{n!}$

   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{(1+\frac{1}{n})}{e^n}$

   (c) $\displaystyle\sum_{n=1}^{\infty} \ln(1+\tfrac{1}{n})$

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \ln(1+\tfrac{1}{n})$

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n \ln(1+n)}$

   (f) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (1+n)^n}{n^{n+1}}$

3. Let $a_n \geq 0$. Show that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges $\Rightarrow \displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges.

4. Let $\infty \sum_{n=0} a_n$ be a convergent series and $(b_n)_{n\in\mathbb{N}}$ monotone bounded sequence. Show that the series $\displaystyle\sum_{n=0}^{\infty} a_n b_n$ converges.

5. Let $a_n > 0$ for all $n \in \mathbb{N}$. Show that the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges iff the series $\displaystyle\sum_{n=0}^{\infty} \frac{a_n}{1+a_n}$ converges.

6. Let $a_n \in \mathbb{R}$. Show that $\displaystyle\sum_{n=0}^{\infty} |a_n| < \infty \to \sum_{n=0}^{\infty} |a_n|^2 < \infty$

7. Let $a_n \in \mathbb{R}$, $b_n \in \mathbb{R}$. Show that;

   $$\sum_{n=0}^{\infty} |a_n|^2 < \infty \text{ and } \sum_{n=0}^{\infty} |b_n|^2 < \infty \Rightarrow \sum_{n=0}^{\infty} |a_n b_n|^2 < \infty.$$

8. Let $x \in \mathbb{R}$ be fixed. Show that the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ converges.

9. Let a, b be 2 constants with $0 < a < b$. Show that the series

   $1 + a + ab + a^2b + a^2b^2 + a^3b^3 + a^4b^3 + ...$

   - converges if

     (a) $b < 1$, or

     (b) $a < 1 < b$ and $ab > 1$.

   - diverges if $a < 1 < b$ and $ab > 1$.

10. Let $\displaystyle\sum_{n=0}^{\infty} a_n$, $(a_n \in \mathbb{R})$ be a series. Show that;

    (a) If for some $p > 1$, $\lim_{n\to\infty} n^p a_n = 0$, then the series $\displaystyle\sum_{n=0}^{\infty} a_n$ converges absolutely.

    (b) If $\lim_{n\to\infty} n a_n = A$ and $A \neq 0$, the series $\displaystyle\sum_{n=0}^{\infty} a_n$ diverges.

    (c) If $\lim_{n\to\infty} n a_n = 0$, we cannot conclude.

Chapter 10

Sequences and Series of Functions

1. **Sequences of functions: Pointwise and Uniform Convergence**
2. **Uniform Convergence and Continuity, Differentiation and Integration**
3. **Series of Functions: Pointwise and Uniform Convergence**
4. **Tests for Uniform Convergence: Weierstrass M-test, Abel and Dirichlet Tests**
5. **Differentiation and Integration of Series of Functions**
6. **Power Series and Taylor Expansion**
7. **Continuous but Nowhere Differentiable Functions**

10.1 Sequences of Functions

Let E be any set and (Y, d') be a m.s. Let for each $n \in \mathbb{N}$, $f_n : E \to (Y, d')$ be a function. Then we say that we have a sequence of functions $(f_n)_{n\in\mathbb{N}}$ from E to Y.

Example 10.1.1 $f_n : [0, 1] \to \mathbb{R}$ $f_n(x) = \sin nx$, $f_n(x) = x^n$, $f_n(x) = \dfrac{1}{1 + x^n}$

10.1.1 Pointwise Convergence

Definition 10.1.2 *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions from E to (Y, d'). If for each x in E the sequence $(f_n(x))_{n\in\mathbb{N}}$ converges in Y to some point $y_x \in Y$, then we say that $(f_n)_{n\in\mathbb{N}}$* ***converges pointwise on the set*** *E.*
Since, in any m.s. limit is unique, for each $x \in E$, there is only one $y_x \in Y$ such that $f_n(x) \to y_x$. So, if we define $f : E \to Y$ by $f(x) = y_x$ we get a new function. For this function we have: $\forall x \in E$, $f(x) = \lim_{n\to\infty} f_n(x)$, in Y. This function f is said to be the ***pointwise limit*** *of the sequence $(f_n)_{n\in\mathbb{N}}$ on the set E.*

We write this as "$f_n \to f$ pointwise on E".

$$\begin{aligned} f_n \to f \text{ pointwise on } E \quad & \Leftrightarrow \forall x \in E, \alpha_n(x) = |f_n(x) - f(x)| \to 0 \\ & \Leftrightarrow \forall x \in E,\ \forall \varepsilon > 0,\ \exists N = N(\varepsilon, x) \in \mathbb{N} : \forall n \geq N,\ |f_n - f(x)| < \varepsilon \\ & \Leftrightarrow \forall x \in E, \forall \varepsilon > 0,\ \exists N = N(\varepsilon, x) \in \mathbb{N} : \forall m, n \geq N,\ |f_n(x) - f_m(x)| < \varepsilon \end{aligned}$$

Example 10.1.3 1. *Let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$.*

Then $f_n(0) = 0 \to 0$, $f_n(1) = 1 \to 1$ and For $0 < x < 1$, $f_n(x) = x^n \to 0$.

Let $f(x) = \begin{cases} 0 \text{ if } 0 \leq x < 1 \\ 1 \text{ otherwise} \end{cases}$

Thus $f_n \to f$ pointwise on $[0,1]$. Observe that f_n is continuous on $[0,1]$ but f is not continuous at $x = 1$.

<u>Observe that</u> although each f_n is continuous on $[0,1]$, the limit function is not continuous.

2. *Let $f_n : [0,\infty[\to \mathbb{R}$, $f_n(x) = \dfrac{1}{1+x^n}$. Then $f_n(0) = 1 \to 1$.*

   For $0 < x < 1$, $x^n \to 0$, $f_n(x) \to 1$.

   For $x = 1$, $f_n(x) = \dfrac{1}{2} \to \dfrac{1}{2}$.

   For $x > 1$, $f_n(x) \to 0$.

   Hence the pointwise limit function of the sequence $(f_n)_{n\in\mathbb{N}}$ on the set $E = [0,\infty[$ is,
   $f(x) = \begin{cases} 1 \text{ for } 0 \leq x < 1 \\ \frac{1}{2} \text{ for } x = 1 \\ 0 \text{ for } x > 1 \end{cases}$

3. *Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{\sin nx}{n}$. Then $\forall x \in \mathbb{R}$, $f_n(x) \to 0$. Hence $f \equiv 0$ on $\mathbb{R}$ is the pointwise limit function of f_n in $\mathbb{R}$.*

   But $f_n'(x) = \cos nx$ and $f_n'(\pi) = (-1)^n$ diverges.

4. *Let $f_n : [0,1] \to \mathbb{R}$ be the following:*

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   $$f_n(x) = \begin{cases} 2n^2x & \text{if } 0 \leq x \leq \dfrac{1}{2n} \\ 2n - 2n^2x & \text{if } \dfrac{1}{2n} \leq x \leq \dfrac{1}{n} \\ 0 & \text{if } \dfrac{1}{n} \leq x \leq 1 \end{cases}$$

   Now, for $x = 0$, $f_n(0) = 0 \to 0$. For $x > 0$ for some $N \in \mathbb{N}$, $x > \dfrac{1}{N}$. Hence, for $n \geq N$ $f_n(x) = 0$. Hence, for any $x \in [0,1]$, $f_n(x) \to 0$.

Now $\int_0^1 f_n(x)dx =$ *Area of the triangle* $= 1$. *Hence* $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 1$.

On the other hand $\int_0^1 (\lim_{n\to\infty} f_n(x))dx = \int_0^1 0dx = 0$.

Thus, $\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 (\lim_{n\to\infty} f_n(x))dx$.

5. *Let* $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$. *Then* $\lim_{n\to\infty} \lim_{x\in[0,1],\, x\to 1} f_n(x) = 1$.

   But $\lim_{x\in[0,1[,\, x\to 1} \lim_{n\to\infty} f_n(x) = 0$.

   Hence, $\lim_{x\in[0,1[,\, x\to 1} \lim_{n\to\infty} f_n(x) \neq \lim_{n\to\infty} \lim_{x\in[0,1],\, x\to 1} f_n(x)$.

10.1.2 Uniform Convergence

Definition 10.1.4 *Let* $f_n : E \to Y$ *be a sequence of functions. We say that* $(f_n)_{n\in\mathbb{N}}$ ***converges uniformly on*** E *to a function* $f : E \to Y$ *if we have:*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N,\ \forall x \in E,\ d'(f_n(x), f(x)) < \varepsilon$$
$$\Leftrightarrow \quad \alpha_n = \sup_{x\in E} d'(f_n(x), f(x)) \to 0, \quad \text{as } n \to \infty$$
$$\Leftrightarrow \quad \forall \varepsilon > 0,\ \exists N \in \mathbb{N}, \forall n \geq N,\ \sup_{x\in E} |f_n(x) - f(x)| < \varepsilon$$

Remark: From the definition, clearly, uniform convergence implies pointwise convergence. Hence, in order to find out whether a given sequence $(f_n)_{n\in\mathbb{N}}$ converges uniformly on E, first we must determine its pointwise limit f and try to see if $\alpha_n = \sup_{x\in E} d'(f_n(x), f(x))$ goes or not to zero.

Example 10.1.5 1. *Let* $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$, *we know that* $f_n \to f$ *pointwise on* $[0,1]$, *where* f *is defined in example 10.1.3 and*

   $\alpha_n = \sup_{0\leq x\leq 1} |f_n(x) - f(x)| \geq |f_n(1 - \frac{1}{n}) - f(1 - \frac{1}{n})| = (1 - \frac{1}{n})^n \to \frac{1}{e}$. *So* $\alpha_n \nrightarrow 0$. *So the convergence is not uniform.*

2. *Now let* $0 < \beta < 1$ *is any fixed number and* $E = [0, \beta]$. $f_n : E \to \mathbb{R}$, $f_n(x) = x^n$. *Then* $f_n \to f \equiv 0$ *on* E *pointwise.*

   Moreover, $\alpha_n = \sup_{x\in E} |f_n(x) - f(x)| = \beta^n \to 0$. *Hence* $f_n \to f \equiv 0$ *uniformly on* E.

3. *Let* $f_n(x) = \dfrac{x}{n} e^{\frac{x}{n}}$ *for* $x \in [0, \infty[$. *For each* $x \in [0, \infty[$, $f_n(x) \to 0$. *So* $f_n \to f \equiv 0$ *on* $[0, \infty[$ *pointwise. But* $\beta_n = |f_n(n) - f(n)| = e$. *So convergence is not uniform on* $[0, \infty[$.

4. *Let* $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{\sin nx}{n}$ *then* $f_n \to f \equiv 0$ *pointwise on* $\mathbb{R}$. *Also*

   $\sup_{x\in\mathbb{R}} |f_n(x) - f(x)| = \sup_{x\in\mathbb{R}} \left|\dfrac{\sin nx}{n}\right| \leq \dfrac{1}{n} \to 0.$

5. *Let* $f_n, f : E \to Y$ *be some functions. If* $\sup_{x\in E} d'(f_n(x), f(x)) \leq \beta_n$ *and* $\beta_n \to 0$, *then* $f_n \to f$ *uniformly on* E.

Cauchy Condition for Uniform Convergence:

Definition 10.1.6 *Let $f_n : E \to Y$ be a sequence of functions. We say that $(f_n)_{n\in\mathbb{N}}$ is* ***uniformly Cauchy on*** *E if we have:*
$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \forall n, m \geq N, \forall x \in E,\ d'(f_n(x), f(x)) < \varepsilon$ *or* $\sup_{x\in E} d'(f_n(x), f(x)) < \varepsilon$.

Theorem 10.1.7 *Suppose that (Y, d') is complete. $f_n : E \to Y$ a sequence of functions converges uniformly on E iff it is uniformly Cauchy on E.*

Proof 10.1.8 ($\Rightarrow$) *Suppose that f_n converges uniformly on E. So we have:*
$\forall\varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N\ \sup_{x\in E} d'(f_n(x), f(x)) < \varepsilon.$
Then $\forall n, m \geq N,\ \sup_{x\in E} d'(f_n(x), f(x)) < 2\varepsilon.$

($\Leftarrow$) *Conversely suppose f_n is uniformly Cauchy, so we have:*
$\forall\varepsilon > 0,\ \exists N \in \mathbb{N}, \forall n, m \geq N\ \sup_{x\in E} d'(f_n(x), f_m(x)) < \varepsilon$ (1)
Hence, for each $x \in E$, $(f_n(x))_{n\in\mathbb{N}}$ is Cauchy in Y. As (Y, d') is complete, $(f_n)_{n\in\mathbb{N}}$ converges to some point $y_n \in Y$.
Let $f(x) = y_x$. Then $f_n \to f$ pointwise on E. Let us see that the convergence is uniform.
Let $x \in E$ be any point. Since $f_n \to f$, we have:
$\forall\varepsilon > 0, \exists M_x \in N : \forall m \geq M_x,\ d'(f_m(x), f(x)) < \varepsilon.$
Now, let N be as in (1) *(So N does not depend on x !)*
Let $n \geq N$, then for $m \geq \max\{N, M_x\}$,
$d'(f_n(x), f(x)) \leq d'(f_n(x), f_m(x)) + d'(f_m(x), f(x)) < 2\varepsilon$
So we have,
$\forall\varepsilon > 0,\ \exists N \in \mathbb{N}$ *(which does not depend on x),* $\forall n \geq N\ \forall x \in E\ d'(f_n(x), f(x)) < 2\varepsilon.$
So, $f_n \to f$ uniformly on E.

Remark : Let E be any set and $\mathbb{B}(E) = \{f : E \to \mathbb{R},\ f \text{ is bounded}\}$. On $\mathbb{B}(E)$ we put the "supremum metric", i.e. $d(f, g) = \sup_{x\in E} |f(x) - g(x)|$.
Now, let $f_n,\ f \in \mathbb{B}(E)$. It is clear that $f_n \to f$ uniformly on $E \Leftrightarrow d(f_n, f) \to 0$.

Theorem 10.1.9 *Let (X, d), (Y, d') be two m.s., $A \subseteq X$ a set, $a \in \bar{A}$ a point and $f_n : A \to Y$ be arbitrary functions. Suppose that (Y, d') is complete and*

1. *$(f_n)_{n\in\mathbb{N}}$ converges uniformly on A to some $f : A \to Y$.*
2. *$\forall n \in \mathbb{N}$, $\lim_{x\in A, x\to a} f_n(x) = L_n$ exists.*

Then $\lim_{n\to\infty} \lim_{x\in A,\ x\to a} f_n(x) = \lim_{x\in A,\ x\to a} \lim_{n\to\infty} f_n(x).$

Proof 10.1.10 *Since* $f(x) = \lim_{n\to\infty} f_n(x)$ *and* $L_n = \lim_{x\in A,\, x\to a} f_n(x)$ *it is enough to show that* $(L_n)_{n\in\mathbb{N}}$ *converges and* $\lim_{n\to\infty} L_n = \lim_{x\in A,\, x\to a} f(x)$. *Let us write what we have:*
$(f_n)_{n\in\mathbb{N}}$ *is uniformly Cauchy:* $\forall\varepsilon > 0, \exists N \in \mathbb{N} : \forall n, m \geq N, \forall x \in A,\ d'(f_n(x), f(x)) < \varepsilon\ (*)$
Since N *is independent of* x *in* A, *fixing* $n \geq N$ *and* $m \geq N$ *arbitrarily and letting* $x \to a$ $(x \in A)$, *we get :* $d'(L_n, L_m) \leq \varepsilon$. *(because in any m.s* d *is a continuous function, i.e.* $x_n \to x,\ y_n \to y \Longrightarrow d(x_n, y_n) \to d(x, y)$*)*
Hence L_n *is Cauchy in* (Y, d'), *so converges to a certain* $L \in Y$.
In $(*)$ *fix an* $m \geq N$ *and* $x \in A$ *arbitrary and let* $m \to \infty$. *Then we get:*

$\forall\varepsilon > 0\ \exists N \in \mathbb{N} :\ \forall n \geq N, \forall x \in A,\ d'(f_n(x), f(x)) < \varepsilon.$

In this last expression letting $x \to a$, $x \in A$ *we get:*

$\forall\varepsilon > 0, \exists N \in \mathbb{N} :\ \forall n \geq N, \lim_{x\to a} d'(f_n(x), f(x)) \leq \varepsilon.$

As, $\lim_{x\in A,\, x\to a} f_n(x) = L_n$ *we conclude that* $\lim_{n\to\infty} L_n = \lim_{x\in A,\, x\to a} f(x)$.

Example 10.1.11 $f_n : [0, 1[\to \mathbb{R},\ f_n(x) = x^n$
Then, $\lim_{n\to\infty} \lim_{x<1,\, x\to 1} f_n(x) \neq \lim_{x<1,\, x\to 1} \lim_{n\to\infty} f_n(x)$. *Hence,* $f_n \nrightarrow f \equiv 0$ *uniformly on* $[0, 1[$.

10.2 Continuity, Differentiation and Integration

10.2.1 Uniform Convergence and Continuity

Theorem 10.2.1 *Let* $f_n : A \to Y$ *be a sequence of functions and* $a \in A$. *Suppose that* (Y, d') *is complete and*

1. *each* f_n *is continuous at* a.
2. $(f_n)_{n\in\mathbb{N}}$ *converges uniformly on* A *to some function* $f : A \to Y$.

Then f *is continuous at* a.

Proof 10.2.2 *By the theorem 10.1.9,*

$$\begin{aligned}\lim_{x\in A,\, x\to a} f(x) &= \lim_{x\in A,\, x\to a}[\lim_{n\to\infty} f_n(x)]\\ &= \lim_{n\to\infty} \lim_{x\in A,\, x\to a} f_n(x)\\ &= \lim_{n\to\infty} f_n(a) = f(a)\end{aligned}$$

Hence f *is continuous at* a.

Remark: By contrapositive, if each f_n is continuous on A, $f_n \to f$ pointwise on A and f is not continuous at some point $a \in A$ then the convergence is not uniform on A. For instance, if $f_n = x^n$ on $[0, 1]$, then $f_n \to f = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } x \neq 1 \end{cases}$
As f is not continuous at $a = 1$, we conclude that the convergence is not uniform on $[0, 1]$.

10.2.2 Metric Nature of Uniform Convergence

Let E be any set and $\mathbb{B}(E)\{f : E \to \mathbb{R} : f$ is bounded $\}$. For $f, g \in \mathbb{B}(E)$, let $d_\infty(f, g) = \sup_{x\in E} |f(x) - g(x)|$. This is a metric on $\mathbb{B}(E)$.

For $f_n, f \in \mathbb{B}(E)$, $d_\infty(f_n, f) \to 0 \Leftrightarrow f_n \to f$ uniformly on E.

For that reason d_∞ is called the *metric of uniform convergence.*

We have already seen that $(\mathbb{B}(E), d_\infty)$ is complete.

Now, let $C_b(E) = \{f : E \to \mathbb{R} : f$ is continuous and bounded on $E\}$. On $C_b(E)$ put the supremum metric, d_∞. Obviously, $C_b(E) \subseteq \mathbb{B}(E)$

Theorem 10.2.3 *$(C_b(E), d)$ is complete.*

Proof 10.2.4 *1. $C_b(E) \subseteq \mathbb{B}(E)$.*

2. $(\mathbb{B}(E), d)$ is complete.

3. $C_b(E)$ is closed in $\mathbb{B}(E)$ by the theorem 10.2.1, indeed let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C_b(E)$ that converges (for the supremum metric) to some $f \in (B)(E)$. But then by the theorem 10.2.1 f is continuous on E, so $f \in C_b(E)$.

4. In any complete m.s (X, d), a subspace (M, d) is complete iff M is closed in X.

***Particular Case**: Let $K \subseteq X$ be a compact set and $C(K) = \{f : K \to \mathbb{R} : f$ is continuous on $K\}$, then obviously each $f \in C(K)$ is bounded on K. Then $(C(K), d)$ is a complete m.s. In particular the space $C([0, 1])$ is a complete m.s under the supremum metric. (i.e. uniform convergence metric)*

Example 10.2.5 *$C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} : \lim_{|x|\to\infty} f(x) = 0,\ f$ is continuous$\}$ is complete.*

Example 10.2.6 *Let $f_n : [0, \infty[\to \mathbb{R}$, $f_n(x) = \dfrac{1}{1 + x^n}$.*

Then $$f_n(x) \to f(x) = \begin{cases} 1, & \text{if } 0 \le x < 1 \\ \frac{1}{2}, & \text{if } x = 1 \\ 0, & \text{if } x > 1 \end{cases}$$

As f is not continuous on $[0, \infty[$ and each f_n is continuous on the same set, we conclude that convergence of f_n to f is not uniform on $[0, \infty[$.

10.2.3 Uniform Convergence and Integration

Theorem 10.2.7 *Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions. Suppose that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to some $f : [a, b] \to \mathbb{R}$ on $[a, b]$. Then,*

1. $F_n(x) = \int_a^x f_n(t)dt \to F(x) = \int_a^x f(t)dt$ uniformly on $[a, b]$.

2. $\int_a^b |f_n(t) - f(t)|dt \to 0$, *as* $n \to \infty$.

3. $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b (\lim_{n\to\infty} f_n(x))dx$.

Proof 10.2.8 *Since* $f_n \to f$ *uniformly on* $[a, b]$*, we have:*
$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N, \forall x \in [a, b], |f_n(x) - f(x)| < \varepsilon.$

Hence, for $n \geq N$ *and* $\forall x \in [a, b]$,
$|\int_a^x f_n(t)dt - \int_a^x f(t)dt| \leq \int_a^b |f_n(t) - f(t)|dt \leq \int_a^x (\varepsilon)dt \leq \int_a^b (\varepsilon)dt \leq \varepsilon(b - a).$

So, $F_n \to F$ *uniformly on* $[a, b]$*. And 2 and 3 follows from this.*

Example 10.2.9 *Let* $f_n : [0, 1] \to \mathbb{R}$, $f_n(x) = \begin{cases} 2n^2 x & \text{if } 0 \leq x \leq 1/2n \\ 2n - 2n^2 x & \text{if } 1/2n \leq x \leq 1/n \\ 0 & \text{if } 1/n \leq x \leq 1 \end{cases}$

Then, as we have seen, $f_n \to f \equiv 0$ *pointwise on* $[0, 1]$.
$\int_0^1 f_n(x)dx = \dfrac{1}{2}, \forall n \in \mathbb{N}, \lim_{n\to\infty} \int_0^1 f_n(x)dx = \dfrac{1}{2} \neq \int_0^1 f(x)dx.$

Hence, the convergence is not uniform on $[0, 1]$.

Example 10.2.10 $\lim_{n\to\infty} \int_0^a \cos nx dx = \left.\dfrac{\sin nx}{n}\right|_0^a = \dfrac{\sin na}{n} \to 0$. *But* $(\cos nx)_{n\in\mathbb{N}}$ *converges iff* $x = 2k\pi$, $k \in \mathbb{Z}$.

10.2.4 Uniform Convergence and Differentiation

Theorem 10.2.11 *Let* $f_n : [a, b] \to \mathbb{R}$*be a sequence of continuously differentiable functions. (i.e.* $f_n'(x)$ *exists and continuous on* $[a, b]$ $\forall n \in \mathbb{N}$*) Suppose that:*

1. $(f_n'(x))_{n\in\mathbb{N}}$ *converges uniformly on* $[a, b]$ *to some function* $g : [a, b] \to \mathbb{R}$.

2. *For some* $x_0 \in [a, b]$*, the numerical sequence* $(f_n'(x_0))_{n\in\mathbb{N}}$ *converges to some* $L \in \mathbb{R}$.

Then $(f_n)_{n\in\mathbb{N}}$ *converges uniformly to some* $f : [a, b] \to \mathbb{R}$*. This* f *is differentiable and* $f' = g$ *on* $[a, b]$.

Proof 10.2.12 *By "Newton - Leibniz Theorem"* $f_n(x) = f_n(x_0) + \int_{x_0}^x f_n'(t)dt$*. Then, by theorem 10.2.7,* $f_n(x) \to f(x) = L + \int_{x_0}^x g(t)dt$ *uniformly on* $[a, b]$*. Hence,* $f'(x) = g(x)$ *on* $[a, b]$.

Example 10.2.13 *Let* $f_n(x) = \dfrac{\sin nx}{n}$ *on* $[a, b]$*. Then* $f_n \to f \equiv 0$ *uniformly on* $[a, b]$.
$(f_n)_{n\in\mathbb{N}}$ *is continuously differentiable and* $f_n'(x) = \cos nx$*. But* $(f_n')_{n\in\mathbb{N}}$ *does not converge even pointwise.*

10.3 Series of Functions

Definition 10.3.1 *Let E be a set and $f_n : E \to \mathbb{R}$ be a sequence of functions. The formal sum* $\sum_{n=0}^{\infty} f_n(x)$ *is said to be a* ***"series of functions"*** *on the set E.*

10.3.1 Pointwise Convergence of Function Series

Definition 10.3.2 *If, for each $x \in E$, the "numerical series"* $\sum_{n=0}^{\infty} f_n(x)$ *converges, then we say that the function series converges* ***pointwise*** *on E.*

In this case, we obtain a function $f : E \to \mathbb{R}$, by putting $f(x) = \lim_{n\to\infty} S_n(x)$, where $S_n = f_0 + f_1 + \cdots + f_n$. By definition, f is the "pointwise sum of the series". We write this as $f(x) = \sum_{n=0}^{\infty} f_n(x)$ **pointwise on** E.

For instance, for every $x \in \mathbb{R}$, the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges. So it defines a function $f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!}$. We know that $f(x) = e^x$.

Remark Let f_n be any sequence of functions on a set E. To this sequence associate the function series $f_0 + \sum_{n=0}^{\infty}(f_{n+1} - f_n)$. Let $S_n = f_0 + \sum_{k=0}^{n-1}(f_{k+1} - f_k) = f_n$ be the partial sum of this series. Consequently, the series $f_0 + \sum_{n=0}^{\infty}(f_{n+1} - f_n)$ converges pointwise on E iff f_n converges pointwise on E.

From this we conclude that:

1. If E is a subset of some m.s. (X, d) and each term g_n of the series $\sum_{n=0}^{\infty} g_n(x)$ is continuous on E and $g(x) = \sum_{n=0}^{\infty} g_n(x)$ (pointwise on E), the function g need not to be continuous on E. e.g. let $f_n(x) = x^n$, $0 \le x \le 1$ and $g_0 = f_0$, $g_n = f_{n+1} - f_n$. Then g is the pointwise limit of f_n on $[0, 1]$, which is not continuous.

2. Similarly if each $g_n : [a, b] \to \mathbb{R}$ continuous and we have $g(x) = \sum_{n=0}^{\infty} g_n(x)$ pointwise on $[a, b]$, in general we don't have: $\int_a^b \left(\sum_{n=0}^{\infty} g_n(x)\right) dx = \sum_{n=0}^{\infty} \left(\int_a^b g_n(x)dx\right)$.

3. If each $g_n : [a,b] \to \mathbb{R}$ is continuously differentiable on $[a,b]$ and $g(x) = \sum_{n=0}^{\infty} g_n(x)$ pointwise on $[a,b]$, in general we do not have: $\left(\sum_{n=0}^{\infty} g_n(x)\right)' = \sum_{n=0}^{\infty} g_n'(x)$.

10.3.2 Uniform Convergence of the Function Series

Let E be any set and $\sum_{n=0}^{\infty} f_n(x)$ be a series of functions on E. Let $S_n = f_0 + f_1 + \cdots + f_n$ be the partial sum of this series. So we have a "sequence" of functions, S_n.

Definition 10.3.3 *If this S_n converges uniformly on E to a function f, then we say that $\sum_{n=0}^{\infty} f_n(x)$ **converges uniformly to f on** E. We write this as, $f = \sum_{n=0}^{\infty} f_n(x)$ uniformly on E. This is equivalent to say that S_n is uniformly Cauchy, i.e.*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N} :\ \forall n \geq N,\ \forall p \in \mathbb{N},\ \sup_{x\in E} \left| \sum_{k=n}^{n+p} f_k(x) \right| < \varepsilon.$$

Remark: From this definition we conclude that a "necessary" (not sufficient) condition for the uniform convergence of the series $\sum_{n=0}^{\infty} f_n(x)$ on E, is the uniform convergence of f_n to *zero* on E.

For instance, the series $\sum_{n=0}^{\infty} x^n$ converges pointwise on $[0,1[$ but not uniformly since $x^n \to 0$ pointwise but not uniformly.

Example 10.3.4 *Let $\sum_{n=0}^{\infty} f_n(x)$ be a series and $S_n = f_0 + f_1 + \cdots + f_n$ be its partial sum. If for some sequence $x_n \in E$, $(S_m(x_n))_{n\in\mathbb{N}}$ does not converge, then $f_n(x_n) \nrightarrow 0$ and so this series cannot converge uniformly.*

Example 10.3.5 *Consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. For every $x \in \mathbb{R}$, this series converges to e^x. Is this convergence uniform on $\mathbb{R}$?*
A necessary condition for uniform convergence is the uniform convergence of $f_n(x) = \frac{x^n}{n!}$ to zero on $\mathbb{R}$, i.e. $\sup_{x\in\mathbb{R}} \left|\frac{x^n}{n!}\right| \to 0$.
But $\sup_{x\in\mathbb{R}} \left|\frac{x^n}{n!}\right| \geq \frac{n^n}{n!} \to \infty$. So convergence of this series to e^x is not uniform on $\mathbb{R}$.

However, on every compact interval $[a, b]$ convergence is uniform. To see this, we use the Cauchy condition: $\sup_{a\le x\le b}\left|\sum_{k=n}^{n+p}\frac{x^k}{k!}\right| \le \sum_{k=n}^{n+p}\frac{|b|^k}{k!} \to 0$, *as* $n\to\infty$, *if* $|b|\ge a$.

Example 10.3.6 *Consider the series* $\sum_{n=0}^{\infty} x^n$.

Here $F_n(x) = 1 + x + \cdots + x^n = \begin{cases} \dfrac{1-x^{n+1}}{1-x} & \text{for } x\ne 1 \\ n+1, & \text{for } x = 1\end{cases}$

- *For* $|x|\ge 1$ *the series diverges since* $x^n \nrightarrow 0$.
- *For* $|x|<1$, $x^n\to 0$, *so* $F_n(x)\to f(x) = \dfrac{1}{1-x}$ *pointwise on* $]-1,1[$ *so that* $\sum_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$ *pointwise on* $]-1,1[$.
- *The convergence is not uniformly on* $]-1,1[$ *since for instance, for* $x = 1-\dfrac{1}{n+1}$,
$$F_n\left(1-\frac{1}{n+1}\right) = \frac{-(1-\frac{1}{n+1}+1)}{1-1+\frac{1}{n+1}} \to e^{-1}.$$

Thus $\lim_{x\to 1}\lim_{n\to\infty} F_n(x) = e^{-1}$ *but* $\lim_{n\to\infty}\lim_{x\to 1} F_n(x)$ *does not exist (actually* ∞*)*

- *Now let* $0<\alpha<1$, *then* $\sum_{n=0}^{\infty} x^n$ *converges uniformly on* $]-\alpha,\alpha[$ *to* $\dfrac{1}{1-x}$. *Indeed,*
$$\sup_{|x|\le\alpha}\left|F_n(x) - \frac{1}{1-x}\right| = \sup_{|x|\le\alpha}\left|\frac{x^{n+1}}{1-x}\right| \le \frac{\alpha^{n+1}}{1+\alpha} \to 0$$

Hence $\sum_{n=0}^{\infty} x^n$ *converges pointwise on* $]-1,1[$ *to* $\dfrac{1}{1-x}$, *and the convergence is uniform on* $[-\alpha,\alpha]$ *for any* $0<\alpha<1$.

10.3.3 Hereditary Properties

Theorem 10.3.7 ***(Continuity)*** *Let* A *be a subset of a m.s.* (X,d). $f_n : A\to\mathbb{R}$ *sequence of functions and* $x_0\in A$. *If;*

- *The series* $\sum_{n=0}^{\infty} f_n$ *converges uniformly on* A *and*
- *Each* f_n *is continuous at* x_0

Then the function $f = \sum_{n=0}^{\infty} f_n$ *is continuous at* x_0.

(i.e. $\lim_{x\in A,\, x\to x_0}\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty}\lim_{x\in A,\, x\to x_0} f_n(x)$*)*

Proof 10.3.8 *Let* $F_n = f_0 + f_1 + \cdots + f_n$*, then* $(F_n)_{n\in\mathbb{N}}$ *converges uniformly on* A *and each* F_n *is continuous at* x_0*. Then by theorem 10.2.1,* $\lim_{n\to\infty} F_n = \lim_{n\to\infty} \sum_{i=0}^{n} f_i = \sum_{n=0}^{\infty} f_n = f$ *is continuous at* x_0*.*

Theorem 10.3.9 ***(Integration)*** *Let* $f_n : [a,b] \to \mathbb{R}$ *be a sequence of functions. Suppose that the series* $\sum_{n=0}^{\infty} f_n$ *converges uniformly on* $[a,b]$*. Then* $\int_x^a (\sum_{n=0}^{\infty} f_n(t))dt = \sum_{n=0}^{\infty} \int_x^a f_n(t)dt$

Proof 10.3.10 *Let* $F_n = f_0 + f_1 + \cdots + f_n$*, then* $(F_n)_{n\in\mathbb{N}}$ *converges uniformly on* $[a,b]$ *to* F*. Then by theorem 10.2.7,* $\lim_{n\to\infty} \int_x^a F_n(t)dt = \sum_{n=0}^{\infty} \int_a^x f_n(t)dt = \int_a^x F(t)dt = \int_a^x (\sum_{n=0}^{\infty} f_n(t))dt$

Theorem 10.3.11 ***(Differentiation)*** *Let* $f_n : [a,b] \to \mathbb{R}$ *be continuously differentiable on* $[a,b]$*. Suppose that;*

- *the series* $\sum_{n=0}^{\infty} f_n'$ *converges uniformly on* $[a,b]$
- *for* $x_0 \in [a,b]$*, the numerical series* $\sum_{n=0}^{\infty} f_n(x_0)$ *converges.*

Then the series $\sum_{n=0}^{\infty} f_n$ *converges uniformly to* $f = \sum_{n=0}^{\infty} f_n$ *and*

$$f' = \left(\sum_{n=0}^{\infty} f_n\right)' = \sum_{n=0}^{\infty} f_n'$$

Proof 10.3.12 *Let* $F_n = f_0 + f_1 + \cdots + f_n$*, then each* F_n' *is continuous on* $[a,b]$ *and* F_n' *converges uniformly on* $[a,b]$*, also for* $x_0 \in [a,b]$*,* $(F_n(x_0))_{n\in\mathbb{N}}$ *converges in* $\mathbb{R}$*. Then by theorem 10.2.11,* F_n *converges uniformly on* $[a,b]$ *to* $F = \lim_{n\to\infty} F_n = \sum_{n=0}^{\infty} f_n$ *and* F *is differentiable on* $[a,b]$ *and that* $F' = \sum_{n=0}^{\infty} f_n'$

10.4 Tests for Uniform Convergence

Let $\sum_{n=0}^{\infty} f_n(x)$ be an arbitrary series of functions on a set E.

Theorem 10.4.1 ***(Weierstrass M-Test):*** *Suppose that* $sup_{x\in E} |f_n(x)| \leq \alpha_n$ *and that the numerical series* $\sum_{n=0}^{\infty} \alpha_n$ *is convergent. Then the series* $\sum_{n=0}^{\infty} f_n(x)$ *converges uniformly on* E*.*

Proof 10.4.2 *Let $S_n = f_0 + f_1 + \cdots + f_n$. Let us see that S_n is uniformly Cauchy on E. As $\sum_{n=0}^{\infty} \alpha_n$ is convergent its partial sum $A_n = \alpha_0 + \alpha_1 + \cdots + \alpha_n$ is a Cauchy sequence. We have: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N} : \forall n \geq N$, $\forall p \in \mathbb{N}$, $|A_{n+p} - A_n| = \alpha_{n+1} + \cdots + \alpha_{n+p} < \varepsilon$. Then $\forall n \geq N$, $\forall p \in \mathbb{N}$,*

$$\begin{aligned}\sup_{x\in E}|S_{n+p}(x) - S_n(x)| &= \sup_{x\in E}|f_{n+1}(x) + \cdots + f_{n+p}(x)| \\ &\leq \sup_{x\in E}|f_{n+1}(x)| + \cdots + \sup_{x\in E}|f_{n+p}(x)| \\ &\leq \alpha_{n+1} + \cdots + \alpha_{n+p} < \varepsilon\end{aligned}.$$

So, S_n is uniformly Cauchy on E. Hence the series $\sum_{n=0}^{\infty} f_n(x)$ converges uniformly.

Example 10.4.3 *Consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n}$. Determine the largest set $E \subseteq \mathbb{R}$ on which this series converges uniformly.*

This series converges pointwise on $[-1, 1[$, diverges at any point $x \notin [-1, 1[$. The convergence is uniform on any compact interval $[a, b] \subseteq]-1, 1[$.

Remark: Let A be a subset of some m.s. (X, d). Let $S : \bar{A} \to \mathbb{R}$ be a continuous and bounded function. Then $\sup_{x\in A}|S(x)| = \sup_{x\in \bar{A}}|S(x)|$. Hence, if each $f_n : \bar{A} \to \mathbb{R}$ is continuous, then if the convergence of the series $\sum_{n=0}^{\infty} f_n(x)$ is not uniform on $\bar{A}$. So it cannot be uniform on A.

$\star$ Weierstrass-M test cannot be applied e.g. to the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$. So we need another test.

10.4.1 Abel and Dirichlet Tests for Uniform Convergence

Consider a series of the form $\sum_{n=0}^{\infty} f_n(x)g_n(x)$, where $f_n, g_n : E \to \mathbb{R}$ are arbitrary functions defined on a set E. To study the convergence of this series, we need the "Cauchy form" of the Abel formula.

Consider the numerical series of the form $\sum_{n=1}^{\infty} a_n b_n$.

Put $A_0 = 0$, $A_1 = a_1, \ldots, A_n = a_1 + \cdots + a_n$.

Let $m > n$ be integers, $S_n = \sum_{k=1}^{n} a_k b_k$ and $S_m - S_{n-1} = \sum_{k=n}^{m} a_k b_k = a_n b_n + \cdots + a_m b_m$.

Then $a_n = A_n - A_{n-1}$, $a_{n+1} = A_{n+1} - A_n, \ldots, a_m = A_m - A_{m-1}$

So that;

$$\begin{aligned} S_m - S_n &= b_n(A_n - A_{n-1}) + b_{n+1}(A_{n+1} - A_n) + \cdots + b_m(A_m - A_{m-1}) \\ &= -b_n A_{n-1} + A_n(b_n - b_{n+1}) + \cdots + A_{m-1}(b_{m-1} - b_m) + b_m A_m \end{aligned}$$

Theorem 10.4.4 ***(Dirichlet Test)*** *Let E be a set, $f_n, g_n : E \to \mathbb{R}$ be arbitrary functions. Suppose that;*

1. *$g_n \geq 0$ is decreasing and $g_n \to 0$ uniformly on E.*
2. *$\exists M > 0$, $\sup_{x\in I} \left|\sum_{i=0}^{n} f_i(x)\right| < M$, $\forall n \in \mathbb{N}$.*

Then the series $\sum_{n=1}^{\infty} f_n(x)g_n(x)$ converges uniformly on E.

Proof 10.4.5 *Let $S_n(x) = \sum_{k=1}^{n} f_k(x)g_k(x)$, we want to show that $(S_n(x))_{n\in\mathbb{N}}$ is uniformly Cauchy on E.*
Since $g_n \to 0$ uniformly on E, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$, $\forall n \geq N$, $\sup_{x\in E} g_n(x) < \varepsilon$.
Now put $A_n = f_1 + \cdots + f_n$, with $A_0 = 0$. Then,

$$S_{n+p} - S_{n-1} = f_n g_n + f_{n+1}g_{n+1} + \cdots + f_{n+p}g_{n+p} = \sum_{k=0}^{p} A_{n+k}(g_{n+k} - g_{n+k-1}) + A_{n+p}g_{n+p}.$$

As, $|A_{n+p}(x)g_{n+p}(x)| \leq M\,|g_{n+p}(x)|$,

$$\sup_{x\in I}\left|\sum_{k=0}^{p} A_{n+k}(g_{n+k}(x) - g_n(x))\right| \leq M \sup_{x\in I}|g_{n+p}(x) - g_n(x)| \leq M\varepsilon,\ \forall x \in E.$$

Hence, S_n is uniformly Cauchy on I. So it converges uniformly on I.

Example 10.4.6 *Study the uniform convergence of the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$.*
Here, $g_n(x) = \frac{1}{n} \downarrow 0$, and $f_n(x) = \sin nx$. We know that $|\sin x + \cdots + \sin nx| \leq \frac{1}{\left|\sin \frac{x}{2}\right|}$ on any compact interval $[a,b] \subseteq]0, 2\pi[$, $\sup_{a\leq x\leq b} \frac{1}{\left|\sin\frac{x}{2}\right|} < \infty$.
Hence, this series converges uniformly on any compact interval $[a,b] \subseteq]0, 2\pi[$.

Theorem 10.4.7 ***(Abel Test)*** *Let f_n, g_n be two sequences of bounded functions and;*

1. *$\sum_{n=0}^{\infty} f_n(x)$ converges uniformly on E to a bounded function $f : E \to \mathbb{R}$.*
2. *g_n is monotone on E and converges uniformly on E to some function $g : E \to \mathbb{R}$.*

Then the series $\sum_{n=0}^{\infty} f_n(x)g_n(x)$ *converges uniformly on* E.

Proof 10.4.8 *Let* $g = \lim_{n\to\infty} g_n$. *Then* $(g - g_n)$ *decreases uniformly to* 0 *on* E.
As, $\sum_{n=0}^{\infty} f_n(x)$ *converges uniformly on* E *to a bounded function* $f : E \to \mathbb{R}$, $\exists M > 0$ *such that* $\sup_{x\in E} \left|\sum_{i=0}^{n} f_i(x)\right| \leq M$, $\forall n \geq 1$.

Hence, by the Dirichlet Test (theorem 10.4.4), $\sum_{n=0}^{\infty} f_n(x)(g(x) - g_n(x))$ *converges uniformly.*

So, $\sum_{n=0}^{\infty} f_n(x)g_n(x) = \sum_{n=0}^{\infty} f_n(x)(g(x) - g_n(x)) + g(x)\sum_{n=0}^{\infty} f_n(x)$ *converges uniformly.*

10.5 Power Series and Taylor Expansion

Let $a_n \in \mathbb{R}$ and $x_0 \in \mathbb{R}$ be given. A series of the form $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to be a **power series about** x_0.

Lemma 10.5.1 *Consider a power series* $\sum_{n=0}^{\infty} a_n x^n$. *Let* $z \in \mathbb{R}$ *be a number. Then,*

1. *If* $\sum_{n=0}^{\infty} a_n z^n$ *converges, then for any* $|x| < |z|$, *the series* $\sum_{n=0}^{\infty} a_n x^n$ *converges.*

2. *If* $\sum_{n=0}^{\infty} a_n z^n$ *diverges, then for any* $|x| > |z|$, *the series* $\sum_{n=0}^{\infty} a_n x^n$ *diverges.*

Proof 10.5.2

1. *Suppose* $\sum_{n=0}^{\infty} a_n z^n$ *converges. Then* $\limsup \sqrt[n]{(a_n z^n)} \leq 1$. *Then for* $|x| < |z|$ $\limsup \sqrt[n]{(a_n x^n)} = |x| \limsup \sqrt[n]{|m|} < |z| \limsup \sqrt[n]{|m|} \leq 1$.

   Hence, $\limsup \sqrt[n]{(a_n x^n)} < 1$. *Hence,* $\sum_{n=0}^{\infty} a_n x^n$ *converges absolutely, so converges.*

2. *Suppose that* $\sum_{n=0}^{\infty} a_n z^n$ *diverges. Then* $\limsup \sqrt[n]{(a_n z^n)} \geq 1$.

   Hence for $|x| > |z|$, $\limsup \sqrt[n]{(a_n x^n)} = |x| \limsup \sqrt[n]{|m|} > |z| \limsup \sqrt[n]{|m|} \geq 1 \implies \limsup \sqrt[n]{(a_n x^n)} > 1$.

Hence, $\sum_{n=0}^{\infty} a_n x^n$ *diverges.*

Conclusion: Given any power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$, there exists a symmetric interval about x_0 of the form $]R-x_0,\ R+x_0[$ such that:

1. $\forall x \in]R-x_0,\ R+x_0[$, the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely.

2. $\forall x \notin [R-x_0,\ R+x_0]$, the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ diverges.

Hence, the domain of convergence of any power series is an **interval**.

Example 10.5.3 *Consider the function series* $\sum_{n=1}^{\infty} \frac{\cos nx}{n}$.

- *For* $x = \pi$, $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ *converges.*
- *For* $x = -\pi$ *it also converges.*
- *But* $-\pi < 0 < \pi$, $\sum_{n=1}^{\infty} \frac{\cos n0}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ *diverges. So the domain of this series is not an interval.*

Example 10.5.4 *Consider the power series* $\sum_{n=0}^{\infty} n!x^n$. *This series converges for* $x = 0$ *only. So the "interval of convergence" for power series might be degenerated,* $[0,0] = \{0\}$.

Example 10.5.5 *Consider the power series* $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$. *This series converges for* $x = 1$ *but diverges for* $x = -1$. *It also converges for any* $|x| < 1$. *Hence its interval of convergence is* $]-1,1]$.

10.5.1 Radius of Convergence of a Power Series

Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series. Let $\rho = \limsup \sqrt[n]{|a_n|}$. (we take $\rho = \infty$ if the sequence $\sqrt[n]{|a_n|}$ is not bounded)
Let $R = \frac{1}{\rho}$. (we put $R = 0$, if $\rho = \infty$ and $R = \infty$, if $\rho = 0$)

This R is said to be the **"radius of convergence"** of the power series $\sum_{n=0}^{\infty} a_n x^n$.

$$R = \frac{1}{\limsup \sqrt[n]{|a_n|}}.$$

Theorem 10.5.6 *Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series and R be its radius of convergence. Then,*

1. *$\forall x \in \mathbb{R}$, $|x-x_0| < R$, this series converges absolutely.*
2. *$\forall x \in \mathbb{R}$, $|x-x_0| > R$, this series diverges.*
3. *$\forall x \in \mathbb{R}$, $|x-x_0| = R$, we cannot conclude.*
4. *$\forall r \in \mathbb{R}$, $0 \le r \le R$ on the compact interval $[r-x_0, r+x_0]$, the convergence is uniform.*

Proof 10.5.7

1. *Let $x \in \mathbb{R}$ be such that $|x-x_0| < R$. Then* $\limsup \sqrt[n]{|a_n(x-x_0)^n|} = |x-x_0| \limsup \sqrt[n]{|a_n|} = \frac{|x-x_0|}{R} < 1$.

   Hence the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges absolutely.

2. *Let $x \in \mathbb{R}$ be such that $|x-x_0| > R$. Then* $\limsup \sqrt[n]{|a_n(x-x_0)^n|} > 1$.

   Hence, the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ diverges.

3. *See below examples.*

4. *Let $0 \le r \le R$. Let $|x-x_0| \le r$. Then* $\sup_{|x-x_0|\le r} |a_n(x-x_0)^n| \le |a_n| r^n$. *Now, the positive series $\sum_{n=0}^{\infty} |a_n| r^n$ converges since* $\limsup \sqrt[n]{|a_n| r^n} = \frac{r}{R} < 1$. *Hence by "Weierstrass-M test" the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges uniformly on $[r-x_0, r+x_0]$.*

Example 10.5.8

1. *Consider the series $\sum_{n=0}^{\infty} \frac{x^n}{n}$. Here $R = \frac{1}{\lim \sqrt[n]{n}} = 1$. Hence;*

   for $x \in]-1, 1[$ this series converges,

   for $x = 1$ this series diverges,

   for $x = -1$ this series converges,

   for $0 < r < 1$ this series converges uniformly on $[-r, r]$.

2. *Consider the series* $\sum_{n=0}^{\infty} \frac{x^n}{n!}$. *Here* $a_n = \frac{1}{n!}$.

   As, $\frac{a_{n+1}}{a_n} \to 0$, $\lim \sqrt[n]{a_n} \to 0$.

   Hence, $R = \infty$. *i.e.,* $\forall x \in \mathbb{R}$, *the series* $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *converges. But convergence is not uniform on* $\mathbb{R}$. *Since* $\sup_{x>0} \left| \sum_{k=1}^{n} \frac{x^k}{k!} \right| > \sup_{x>0} \frac{x^n}{n!} \geq \frac{n^n}{n!} \downarrow 0$.

Theorem 10.5.6(4) says that for any compact interval $[-a, a]$*, the convergence is uniform. Hence, in theorem 10.5.6(3) we cannot take* $r = R$.

10.5.2 Differentiation and Integration of Power Series

Now, let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series and $]R-x_0,\ R+x_0[$ be its interval of convergence. Let $\tilde{x}$ be any point in this interval so that $|\tilde{x} - x_0| < R$. Let $|\tilde{x} - x_0| < r < R$, so that $\tilde{x} \in [r - x_0, r + x_0]$.

Now, for each $x \in]R-x_0,\ R+x_0[$, let $f(x) = \sum_{n=0}^{\infty} a_n\,(x-x_0)^n$ be the sum of this series. Since each term of the series is a continuous function and convergence is uniform on the interval $[r - x_0, r + x_0]$, f is continuous at $\tilde{x}$. Thus, f is continuous on $]R - x_0,\ R + x_0[$.

Recall: If $a_n \geq 0$, $b_n \geq 0$ and $a_n \to L$, then $\limsup(a_n b_n) = \lim a_n \limsup b_n$

Lemma 10.5.9 *Let* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ *be a power series. Then the radius of convergence of the series* $\sum_{n=0}^{\infty} n a_n(x-x_0)^{n-1}$ *and* $\sum_{n=0}^{\infty} \frac{a_n}{n+1}(x-x_0)^{n+1}$ *are the same as the radius of convergence of the series* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

Proof 10.5.10 *Let* R *be the radius of convergence of the series* $\sum_{n=0}^{\infty} a_n(x-x_0)^n$. *Then,* $R = \limsup \sqrt[n]{|a_n|}$.

Note that $\lim_{n\to\infty} |n|^{\frac{1}{n-1}} = 1$ *and* $\lim_{n\to\infty} \left|\frac{1}{n+1}\right|^{\frac{1}{n+1}} = 1$

- $$\begin{aligned} \limsup \sqrt[n-1]{|n a_n|} &= \limsup \left[|n|^{\frac{1}{n-1}}\,|a_n|^{\frac{1}{n-1}}\right] = \lim |n|^{\frac{1}{n-1}} \limsup |a_n|^{\frac{n}{n-1}\frac{1}{n}} \\ &= \limsup \left[\sqrt[n]{|a_n|}\right]^{\frac{n}{n-1}} = R \end{aligned}$$

- $\limsup \sqrt[n+1]{\frac{|a_n|}{n+1}} = \lim \sqrt[n+1]{\frac{1}{n+1}} \limsup \sqrt[n+1]{|a_n|} = \limsup \left[\sqrt[n]{|a_n|}\right]^{\frac{n}{n+1}} = R.$

Theorem 10.5.11 *Let $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ be a power series and R be its radius of convergence and for $x \in]R-x_0,\ R+x_0[,\ f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ be the sum of this series.*
Then on the interval $]R-x_0, R+x_0[,\ f$ is infinitely differentiable and

$$f'(x) = \sum_{n=0}^{\infty} na_n(x-x_0)^{n-1},\ \dots\ , f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)\dots(n-(k+1))a_n(x-x_0)^{n-k}.$$

Proof 10.5.12 *Since the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ has the same radius of convergence as the initial series and that every $\tilde{x} \in [r-x_0, r+x_0]$ belongs to a compact interval $[a,b] \subseteq]R-x_0, R+x_0[$ and that the series $\sum_{n=0}^{\infty} na_n(x-x_0)^{n-1}$ converges uniformly on $[a,b]$, by theorem 10.3.11 the function f is differentiable at $\tilde{x}$ and $f'(\tilde{x}) = \sum_{n=0}^{\infty} na_n(\tilde{x}-x_0)^{n-1}$. The rest is just an iteration of this.*
Hence, if $f(x) = \sum_{n=0}^{\infty} a_n(x-x_0)^n$ for $x \in]R-x_0,\ R+x_0[$, then $a_n = \frac{f^{(n)}(x_0)}{n!}$ from $f^{(n)}(x_0) = n!a_n$.
So, $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$. Then, if f is given as a power series, this series is just the Taylor series of f at x_0. Also, for any compact interval $[a,b] \subseteq]R-x_0, R+x_0[,\ f$ is integrable on $[a,b]$ and $\int_a^b f(x)dx = \int_a^b \left(\sum_{n=0}^{\infty} a_n(x-x_0)^n\right) dx = \sum_{n=0}^{\infty} a_n \int_a^b (x-x_0)^n dx.$

Lemma 10.5.13 *Let $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=0}^{\infty} b_n x^n$ be two power series. Suppose that for some $\alpha > 0$, these series converge on $]-\alpha, \alpha[$ and $\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n$, for each $x \in]-\alpha, \alpha[$. Then for each $n \in \mathbb{N}$, $a_n = b_n$.*

Proof 10.5.14 *Let $f(x) = sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n,\ \forall x \in]-\alpha, \alpha[$.*

Then $a_n = b_n = \frac{f^{(n)}(0)}{n!},\ \forall n.$

Conclusion:

1. If a function $f:]-\alpha,\alpha[\to\mathbb{R}$ can be represented as the sum of a power series $\sum_{n=0}^{\infty} a_n x^n$, then this is the only power series whose sum is f on $]-\alpha,\alpha[$.
2. If there is a function $f:]-\alpha,\alpha[\to\mathbb{R}$ and by any method we represent f as $f(x)=\sum_{n=0}^{\infty} a_n x^n$ then this power series is necessarily the "Taylor series" of f on $]-\alpha,\alpha[$.

Example 10.5.15 *Let* $f(x)=\ln(1+x),\ -1<x<1$, *then* $f'(x)=\dfrac{1}{1+x}$.
For $-1<x<1$, $\dfrac{1}{1+x}=\sum_{n=0}^{\infty}(-x)^n$.
Now let $0<\alpha<1$ *and* $x\in[-\alpha,\alpha]$, *then*

$$\ln(1+x)=\int_0^x \frac{dt}{1+t}=\int_0^x\left(\sum_{n=0}^{\infty}(-t)^n\right)dt=\sum_{n=0}^{\infty}\int_0^x(-t)^n dt=\sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}.$$

Hence $\ln(1+x)=\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{n+1}}{n+1}$ *and this is the Taylor series of* $f(x)=\ln(1+x)$ *on* $]-1,1[$.

10.5.3 Analytic Functions

Now let $f(x)=\sum_{n=0}^{\infty} a_n x^n$ with radius of convergence $\rho>0$, on $]-\rho,\rho[$. We have seen that such an f is infinitely differentiable on $]-\rho,\rho[$.

Question: Is an infinitely differentiable function $f:A\to\mathbb{R}$, analytic on A?

Definition 10.5.16 *Let* $A\subseteq\mathbb{R}$ *be an open set and* $f:A\to\mathbb{R}$ *be a function. Let* $x_0\in A$. *We say that* f ***is real analytic at*** x_0 *if there is* $R=R(x_0)>0$ *such that* $]R-x_0,R+x_0[\subseteq A$, *and on* $]R-x_0,R+x_0[$, f *is representable as a power series. (i.e.* $f(x)=\sum_{n=0}^{\infty} a_n(x-x_0)^n$, *for some* $a_n\in\mathbb{R}$.*)*

If f *is analytic at every* $x_0\in A$, *then we say that* f *is* ***analytic on*** A.

Example 10.5.17 *Let* $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & \text{if } x\neq 0\\ 0, & \text{if } x=0\end{cases}$ *This* f *is called as the "Cauchy function".*
Let us see that at every $x\in\mathbb{R}$, f *is infinitely differentiable and that* $f^{(n)}(x_0)=0,\ \forall n\in\mathbb{N}$.
• *For* $x\neq 0$, *the functions* $\varphi(x)=-\dfrac{1}{x^2}$ *and* $\psi(x)=e^x$ *are infinitely differentiable. So the composition* $\psi\circ\varphi(x)=e^{-\frac{1}{x^2}}$ *is infinitely differentiable on* $\mathbb{R}\setminus\{0\}$.

• *For* $x_0 = 0$, $\lim_{x\neq 0,\, x\to 0} \dfrac{f(x)-f(0)}{x-0} = \lim_{x\neq 0,\, x\to 0} \dfrac{e^{-\frac{1}{x^2}}}{x} = \lim_{x\neq 0, x\to 0} \dfrac{1}{xe^{\frac{1}{x^2}}} = 0.$

Hence, $\begin{cases} f'(0) = 0 \\ f'(x) = \dfrac{2}{x^3}e^{-\frac{1}{x^2}} \end{cases}$

$$\lim_{x\neq 0, x\to 0} \frac{f'(x)-f'(0)}{x} = \lim_{x\neq 0,\, x\to 0} \frac{\frac{2}{x^3}e^{-\frac{1}{x^2}}}{x} = \lim_{x\neq 0, x\to 0} \frac{2}{x^4 e^{\frac{1}{x^2}}} = 0.$$

Remark: If $P(x)$ is any polynomial then $\lim_{x\neq 0, x\to 0} P(x)e^{\frac{1}{x^2}} = \infty$. Hence form this we deduce that $f^{(n)}(0) = 0$, $\forall n \in \mathbb{N}$. If this f were analytic at $x_0 = 0$, we would have $R > 0$ such that $\forall x \in]-R, R[$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Since, then $a_n = \dfrac{f^{(n)}(0)}{n!}$ on $]-R, R[$, f would be identically zero, which is not the case, since $f(x) \neq 0$ for $x \neq 0$. Then f is infinitely differentiable but not analytic. In "complex analysis" every differentiable function is analytic. In real analysis this is not the case.

10.5.4 Continuity at the Boundary of Interval of Convergence

Consider a power series $\sum_{n=0}^{\infty} a_n x^n$. Let $]-\rho, \rho[$ be its interval of convergence and for $x \in]-\rho, \rho[$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$. We know that f is continuous on $]-\rho, \rho[$.

Question: What can we say about the continuity of f at $-\rho$ or ρ?

Example 10.5.18 *For* $|x| < 1$, *the geometric series* $\sum_{n=0}^{\infty} x^n$ *converges and* $f(x) = \dfrac{1}{1+x} = \sum_{n=0}^{\infty} x^n$ *on* $]-1, 1[$.

Now $\lim_{x\to -1, x>-1} f(x) = \dfrac{1}{2}$.

But $\sum_{n=0}^{\infty} \lim_{x\to -1,\, x>-1} x^n = \sum_{n=0}^{\infty} (-1)^n$ *does not exist.*

So that $\lim_{x\to -1,\, x>-1} (\sum_{n=0}^{\infty} x^n)$ *exists but it is not* $\sum_{n=0}^{\infty} (\lim_{x\to -1,\, x>-1} x^n)$.

Theorem 10.5.19 ***(Abel's Theorem)*** *Let* $\sum_{n=0}^{\infty} a_n x^n$ *be a power series and* 1 *be its radius of convergence and* $f :]-1, 1[\to \mathbb{R}$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

1. *If the numerical series* $\sum_{n=0}^{\infty} a_n$ *converges then the series* $\sum_{n=0}^{\infty} a_n x^n$ *converges uniformly on* $[0,1]$*, so that* f *is continuous at* $x=1$ *and*

$$\lim_{x\to 1,\, x<1} f(x) = \lim_{x\to 1,\, x<1}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty}\lim_{x\to 1,\, x<1} a_n x^n = \sum_{n=0}^{\infty} a_n.$$

2. *If the numerical series* $\sum_{n=0}^{\infty}(-1)^n a_n$ *converges then the series* $\sum_{n=0}^{\infty} a_n x^n$ *converges uniformly on* $[-1,0]$ *and* f *is continuous at* $x=-1$ *and*

$$\sum_{n=0}^{\infty}\lim_{x\to -1,\, x>-1} a_n x^n = \sum_{n=0}^{\infty} a_n(-1)^n = \lim_{x\to -1,\, x>-1} f(x).$$

Proof 10.5.20 1. *Let* $S_n(x) = \sum_{k=0}^{n} a_k x^k$ *and suppose that* $\sum_{n=0}^{\infty} a_n$ *converges. We want to prove that* $(S_n)_{n\in\mathbb{N}}$ *is uniformly Cauchy on* $[0,1]$*.*

Let $T_n = a_0 + a_1 + \cdots + a_n$*, as* T_n *is convergent by hypothesis,* T_n *is Cauchy so;* $\forall\varepsilon > 0, \exists N\in\mathbb{N}, \forall N, \forall p\in\mathbb{N}, |a_n + a_{n+1} + \cdots + a_{n+p}| < \varepsilon$

Fix $n \ge N$*. Let* $A_p = a_n + a_{n+1} + \cdots + a_{n+p}$*, then*

$$\begin{aligned} S_{n+p}(x) - S_{n-1}(x) &= a_n x^n + a_{n+1}x^{n+1} + \cdots + a_{n+p}x^{n+p} \\ &= A_0 x^n + (A_1 - A_0)x^{n+1} + \ldots + (A_p - A_{p-1})x^{n+p} \\ &= A_0(x^n - x^{n+1}) + A_1(x^{n+1} - x^{n+2}) + \cdots + A_{p-1}(x^{n+p-1} - x^{n+p}) + A_p x^{n+p} \\ &= A_0 x^n(1-x) + A_1 x^{n+1}(1-x) + \cdots + A_{p-1}x^{n+p-1}(1-x) + A_p x^{n+p} \end{aligned}$$

As by Cauchy condition for T_n $|A_i| < \varepsilon$ *for* $0 \le i \le p$*; for* $0 \le x \le 1$*,*

$$\begin{aligned} |S_{n+p}(x) - S_{n-1}(x)| &\le (1-x)x^n[\varepsilon + \varepsilon x + \ldots + \varepsilon x^{p-1}] + \varepsilon x^{n+p} \\ &= \varepsilon(1-x)x^n\frac{1-x^p}{1-x} + \varepsilon x^{n+p} \\ &= \varepsilon[x^n - x^{n+p} + x^{n+p}] = \varepsilon x^n \end{aligned}$$

So for $n \ge N, \forall p\in\mathbb{N}$, $\sup_{0\le x\le 1}|S_{n+p}(x) - S_{n-1}(x)| \le \sup_{0\le x\le 1}\varepsilon x^n \le \varepsilon$*.*

Hence S_n *is uniformly Cauchy on* $[0,1]$*.*

Hence f *is continuous at* $x=1$ *and* $\lim_{x\to 1,\, x<1} f(x) = \sum_{n=0}^{\infty} a_n$

2. *Use similar steps to prove.*

Example 10.5.21 $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^n}{n}$ *for* $-1 < x < 1$*.*

We have seen that $\sum_{n=0}^{\infty}\frac{(-1)^n}{n}$ *converges.*

Hence by above theorem $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n} = \lim_{x \to 1,\, x<1} \ln(1+x) = \ln(2)$.

10.6 Continuous but Nowhere Differentiable Functions

Start with the function $f(x) = |x|$, for $-1 \leq x \leq 1$. Then extend this function periodically with period 2.

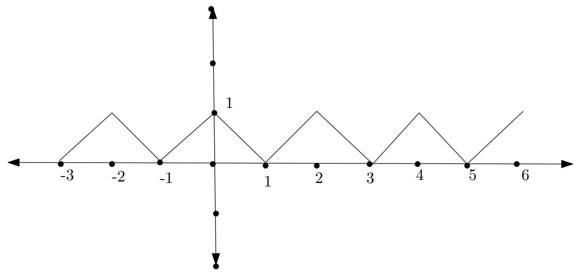

Figure 10.1: Graph of the function

Let g be this function. $g(x) = \begin{cases} |x| \text{ for } -1 \leq x \leq 1 \\ g(x+2) = g(x)\ \forall x \in \mathbb{R} \end{cases}$

In particular $|g(x)| \leq 1$. For any $x, y \in \mathbb{R}$, $|g(x) - g(y)| \leq ||x| - |y|| \leq |x-y|$. Let $\delta_n = \dfrac{\pm 1}{2 \cdot 4^n}$. Observe that for any $x \in \mathbb{R}$ there is no integer in between $]x, x \pm \delta_n[$. This implies that the points x and $x \pm \delta$ are on the same line that composes the graph of g. Hence $|g(x+\delta_n) - g(x)| \leq \delta_n |g'(c)| \leq \delta_n$ for some $x < c < x + \delta_n$.

Now, let $f(x) = \displaystyle\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k g(4^k x)$. By Weierstrass-M test, f is continuous on $\mathbb{R}$. Let us see that f has no tangent.

Theorem 10.6.1 *This f is nowhere differentiable on $\mathbb{R}$.*

Proof 10.6.2 *Fix a point $x \in \mathbb{R}$. Let us prove that* $\dfrac{|f(x+\delta_n) - f(x)|}{\delta_n} \to \infty$. *Since $\delta_n \to 0$, as $n \to \infty$, this proves that f is not differentiable at x_0.*

We have, $$\frac{|f(x+\delta_n) - f(x)|}{\delta_n} = \frac{\displaystyle\sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k g(4^k(x+\delta_n)) - g(4^k x)}{\delta_n}$$

For $k > n$, $g(4^k(x+\delta_n)) - g(4^k x) = g\left(4^k x + \dfrac{4^k}{2 \cdot 4^n}\right) - g(4^k x) = 0$, *since g is periodical with period 2.*

For $k=n$, $\left|g(4^k(x+\delta_n)) - g(4^k x)\right| = \left|g\left(4^k x + \frac{1}{2}\right) - g(4^k x)\right| \leq \frac{1}{2}$

Next, we use the following:

$$|a_1 + \cdots + a_n| \geq |a_1| - |a_2 + \cdots + a_n| \geq |a_1| - \sum_{k=2}^{n} |a_k|.$$

Hence, $\dfrac{f(x+\delta_n) - f(x)}{\delta_n} = \dfrac{\left(\frac{3}{4}\right)^n g(4^n(x+\delta_n)) - g(4^n x)}{\delta_n} + \displaystyle\sum_{k=1}^{n-1} \dfrac{\left(\frac{3}{4}\right)^k g(4^k(x+\delta_n)) - g(4^k x)}{\delta_n}$.

$$\begin{aligned} \text{So, } \left|\frac{f(x+\delta_n) - f(x)}{\delta_n}\right| &\geq 3^n - \sum_{k=1}^{n-1} \frac{\left(\frac{3}{4}\right)^k \left|g(4^k(x+\delta_n)) - g(4^k x)\right|}{\delta_n} \\ &\geq 3^n - \sum_{k=1}^{n} \left(\frac{3}{4}\right)^k \frac{4^k |\delta_n|}{|\delta_n|} \\ &= 3^n - \sum_{k=1}^{n-1} 3^k = 3^n - \frac{3^n - 1}{2} = \frac{3^n + 1}{2} \to \infty \end{aligned}$$

Hence, f *is not differentiable at* x.

Theorem 10.6.3 ***(Weierstrass Approximation Theorem)*** *Every continuous function on a compact interval* $f : [0,1] \to \mathbb{R}$ *is a uniform limit of a sequence of polynomial functions* $P_n(x)$ *on* $[0,1]$.

Proof 10.6.4 *Observe that* $1 = 1^n = (x + (1-x))^n = \displaystyle\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k}$. *Hence,*

$$f(x) = \sum_{k=0}^{n} f(x) \binom{n}{k} x^k (1-x)^{n-k}.$$

Because of this let $P_n(x) = \displaystyle\sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$. *Hence,* P_n *is a polynomial of degree* n. *These polynomials are "Bernstein polynomials associated to* f*".*

Let us see that $\sup_{0 \leq x \leq 1} |f(x) - P_n(x)| \to 0$, *as* $n \to \infty$.

As f *is continuous on* $[0,1]$ *and* $[0,1]$ *is compact,* f *is uniformly continuous. So, we have:*

$$\forall \varepsilon > 0\ \exists \delta > 0 : \forall x, y \in [0,1],\ (|x-y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon).$$

Hence, if $\left|x - \dfrac{k}{n}\right| \leq \delta$, *then* $\left|f(x) - f\left(\frac{k}{n}\right)\right| < \varepsilon$.

For this reason, $f(x) - P_n(x) = \displaystyle\sum_{k=0}^{n} \left(f(x) - f\left(\frac{k}{n}\right)\right) \varphi_n(x)$, *where* $\varphi_n(x) = \dbinom{n}{k} x^k (1-x)^{n-k}$.

Then, $f(x) - P_n(x) = \displaystyle\sum_{\left|x - \frac{k}{n}\right| < \delta} \left(f(x) - f\left(\frac{k}{n}\right)\right) \varphi_k(x) + \sum_{\left|x - \frac{k}{n}\right| \geq \delta} \left(f(x) - f\left(\frac{k}{n}\right)\right) \varphi_k(x)$,

where the sums are over k.

Then for $M = \sup_{0\le x\le 1}|f(x)|$,

$$\begin{aligned}|f(x)-P_n(x)| &\le \sum_{\left|x-\frac{k}{n}\right|<\delta}\left(f(x)-f\left(\frac{k}{n}\right)\right)\varphi_n(x)+\sum_{\left|x-\frac{k}{n}\right|\ge\delta}\left(f(x)-f\left(\frac{k}{n}\right)\right)\varphi_n(x)\\ &\le \varepsilon\sum_{k=0}^{n}\varphi_k(x)+\sum_{\left|x-\frac{k}{n}\right|\ge\delta}\left(f(x)-f\left(\frac{k}{n}\right)\right)\varphi_k(x)\\ &\le \varepsilon+2M\sum_{\left|x-\frac{k}{n}\right|\ge\delta}\varphi_k(x)\end{aligned}$$

Let us see that $\sum_{\left|x-\frac{k}{n}\right|\ge\delta}\varphi_k(x)\to 0$, *as* $n\to\infty$.

$$\left|x-\frac{k}{n}\right|\ge\delta \Longrightarrow \left(x-\frac{k}{n}\right)^2\ge\delta^2\Longrightarrow \frac{1}{\delta^2}\left(x-\frac{k}{n}\right)^2\ge 1.$$

Hence, $\sum_{\left|x-\frac{k}{n}\right|\ge\delta}\varphi_k(x)\le\frac{1}{\delta^2}\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2\varphi_k(x)$.

$$\begin{aligned}\textit{Now, }\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2\varphi_k(x) &= \sum_{k=0}^{n}\left(x^2-2x\frac{k}{n}+\left(\frac{k}{n}\right)^2\right)\varphi_k(x)\\ &= x^2-\frac{2x}{n}\sum_{k=0}^{n}k\varphi_k(x)+\frac{1}{n^2}\sum_{k=0}^{n}k^2\varphi_k(x)\end{aligned}$$

To estimate last quantities, form $(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}x^ky^{n-k}$, *differentiate with respect to* x, *and multiply by* x *to get;*

$$nx(x+y)^{n-1}=\sum_{k=0}^{n}k\binom{n}{k}x^ky^{n-k}.$$

Then differentiating twice and multiplying by x^2 *we get:*

$$x^2n(n-1)(x+y)^{n-2}=\sum_{k=0}^{n}k(k-1)\binom{n}{k}x^ky^{n-k}.$$

Then replace y *by* $1-x$ *to get* $nx=\sum_{k=0}^{n}k\varphi_k(x)$ *and* $n(n-1)x^2=\sum_{k=0}^{n}k(k-1)\varphi_k(x)$. *Then*

$$\begin{aligned}x^2-2\frac{x}{n}\sum_{k=0}^{n}k\varphi_k(x)+\frac{1}{n^2}\sum_{k=0}^{n}k^2\varphi_k(x) &= x^2-2x^2+\frac{1}{n^2}\left[n(n-1)x^2+nx\right]\\ &= \frac{x-x^2}{n}\le\frac{1}{n}\to 0\end{aligned}$$

Hence, $\sup_{0\le x\le 1}|f(x)-P_n(x)|\to 0$, *as* $n\to\infty$.

10.7 Exercises

1. For each of the following sequences of functions, study the convergence (pointwise or uniform) on the given sets.

   (a) $f_n(x) = x^n(1-x)^n$, $x \in [0,1]$.

   (b) $f_n(x) = \dfrac{1}{1+nx^2}$, $x \in \mathbb{R}$.

   (c) $f_n(x) = \dfrac{(1+x)^n - 1}{(1+x)^n + 1}$, $x \in \mathbb{R} \setminus \{-2\}$.

   (d) For $x \in \left]0,1\right]$, $f_n(x) = \begin{cases} n \text{ if } 0 \leq x \leq \frac{1}{n} \\ 0 \text{ if } \frac{1}{n} < x \leq 1 \end{cases}$

   (e) $f_n(x) = \dfrac{\sin nx}{1+n^2x}$, $x \in \mathbb{R}$.

   (f) Let $\{\alpha_n : n \in \mathbb{N}\} = \mathbb{Q} \cap [0,1]$ and $f_n : [0,1] \to \mathbb{R}$ be $f_n(x) = \begin{cases} 1 \text{ if } x \in \{\alpha_0, ..., \alpha_n\} \\ 0 \text{ otherwise} \end{cases}$

2. Let $f_n : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$ be given by $f_n(x) = \begin{cases} \dfrac{\sin^2 nx}{n \sin x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

   (a) Show that f_n continuous on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and $f_n \to f \equiv 0$ pointwise on this interval.

   (b) Let $0 < \alpha < \frac{\pi}{2}$ be any number. Show that $f_n \to f \equiv 0$ uniformly on $[\alpha, \frac{\pi}{2}]$.

   (c) Show that $(f_n(\frac{\pi}{2n}))$ converges. Find this limit and conclude that $(f_n)_{n\in\mathbb{N}}$ does not converge uniformly on $[-\frac{\pi}{2}, \frac{\pi}{2}]$ to $f \equiv 0$.

3. Let for $0 \leq x \leq \frac{\pi}{2}$, $f_n(x) = n(\cos x)^n \sin x$

   (a) Study the convergence (pointwise or uniform) of the sequence $(f_n)_{n\in\mathbb{N}}$ on $[0, \frac{\pi}{2}]$. Let f be its limit function.

   (b) Evaluate the integrals $\int_0^{\frac{\pi}{2}} f_n(x)dx$ and $\int_0^{\frac{\pi}{2}} f(x)dx$.

   (c) Find $\lim_{n\to\infty} \int_0^{\frac{\pi}{2}} f_n(x)dx$ and compare it with $\int_0^{\frac{\pi}{2}} f(x)dx$.

4. Let $f_n(x) = n^2 \sin \dfrac{x}{n^2}$, $x \in [0, 2\pi]$.

(a) Show that the sequence $(f_n'(x))_{n\in\mathbb{N}}$ converges uniformly on $[0, 2\pi]$ to some function g.

(b) Deduce that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $[0, 2\pi]$ to some function f and $f' = g$.

5. Answer the same questions as in the exercise 2 for the sequence $f_n(x) = n \sin \frac{x}{n^2}$ for $x \in [0, 2\pi]$.

6. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of functions such that f_n' exists and continuous on $[a, b]$ for each $n \in \mathbb{N}$. If for each $x \in [a, b]$ and $n \in \mathbb{N}$, $|f_n'(x)| \leq 1$ and $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $[a, b]$ to some function f, show that then $f_n \to f$ uniformly on $[a, b]$.

7. Let $a_n \in \mathbb{R}$, $a_n \to 1$. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = a_n x^2$. Study the convergence (pointwise and uniform) of the sequence $(f_n)_{n\in\mathbb{N}}$ on $\mathbb{R}$.

8. Let $f_n : [0, +\infty[\; \to \mathbb{R}$, $f_n(x) = \dfrac{nx}{2 + n^2x^2}$

(a) Study the convergence (pointwise and uniform) of the sequence $(f_n)_{n\in\mathbb{N}}$ on $[0, +\infty[$.

(b) Show that, for each $\alpha > 0$, the convergence is uniform on $[0, +\infty[$.

9. Let $f_n : [0, +\infty[\to \mathbb{R}, f_n(x) = nx^r e^{-nx}$, where $0 < r \leq 1$ is a fixed number.

(a) Study the convergence of $(f_n)_{n\in\mathbb{N}}$ on $[0, +\infty[$. Determine the limit function

(b) Let $a > 0$. Is the convergence uniform on $[a, +\infty[$?

10. Let $C_0(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}$: f is continuous on $\mathbb{R}$, and $\lim_{|x|\to\infty} f(x) = 0\}$. Show that $C_0(\mathbb{R})$ under the supremum metric, is a complete metric space.

11. Let $f_n \in C_0(\mathbb{R})$ and suppose that there exists a constant $M > 0$ such that, for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$, $|f_n(x)| \leq M$.

12. Let for $x \in \mathbb{R}$, $f_n(x) = e^{\frac{x^2}{n}}$. Show that $f_n(x) \to f(x) \equiv 1$ for each $x \in \mathbb{R}$. Is there an $M > 0$ such that $|f_n(x)| \leq M$ for all $x \in \mathbb{R}$. Show that given any $x \in \mathbb{R}$ there is an $M_x > 0$ such that $e^{\frac{x^2}{n}} \leq M_x$ for all $n \in \mathbb{N}$.

13. Let C_0 be the space of the sequences $x = (x_n)_{n\in\mathbb{N}}$ that converge to zero. Equip C_0 with the supremum metric $d(x, y) = \sup_{n\in\mathbb{N}} |x_n - x_m|$. Show that the metric space (C_0, d) is complete.

14. Let C_{00} be the space of the almost null sequences $(x_n)_{n\in\mathbb{N}}$, i.e. $x_n = 0$ for all but finitely many $n \in \mathbb{N}$. Show that C_{00} is dense in C_0 for the above metric.

15. Let $a_n = \dfrac{1}{2} + \dfrac{1}{4} + \cdots + \dfrac{1}{2^n}$ and for $x \in \mathbb{R}$, $f_n(x) = x^{a_n}$. Find the pointwise limit of $f(n)_{n\in\mathbb{N}}$ on $\mathbb{R}$. Show that for $x \in [0, 1]$, $|f_n(x) - f(x)| \leq 1 - a_n$, where f is the limit function of $(f_n)_{n\in\mathbb{N}}$. Is the convergence uniform on $[0, 1]$?

16. For $x \in [0, +\infty[$, let $f_n(x) = nxe^{-nx}$

    (a) Show that $(f_n)_{n\in\mathbb{N}}$ converges pointwise on $[0, +\infty[$ to some function f.

    (b) Show that the convergence is not uniform on $[0, +\infty[$ but for any $\alpha > 0$, it is uniform on $[\alpha, +\infty[$.

17. Let $b > 0$. Compute $\int_0^b f_n(x)dx$. Is $\lim_{n\to\infty} \int_0^b f_n(x)dx = \int_0^b f(x)dx$?

    Is $\lim_{b\to\infty} \lim_{n\to\infty} \int_0^b f_n(x)dx = \lim_{n\to\infty} \lim_{b\to\infty} \int_0^b f_n(x)dx$?

18. Let $f_n : [0, +\infty[\to \mathbb{R}$, $f_n(x) = \dfrac{xe^{-\frac{x}{n}}}{n}$.

    (a) Find the pointwise limit of f of the sequence $(f_n)_{n\in\mathbb{N}}$ on $[0, +\infty[$.

    (b) Show that the convergence $f_n \to f$ is not uniform on $[0, +\infty[$ but it is on $[0, b]$ for any $b > 0$.

    (c) Compare the iterated limits $\lim_{n\to\infty} \lim_{b\to\infty} \int_0^b f_n(x)dx$ and $\lim_{b\to\infty} \lim_{n\to\infty} \int_0^b f_n(x)dx$.

19. Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \dfrac{e^{-n^2x^2}}{n}$

    (a) Show that $(f_n)_{n\in\mathbb{N}}$ converges uniformly on $\mathbb{R}$ be some function $f : \mathbb{R} \to \mathbb{R}$

    (b) Show that $f_n' \to f'$ pointwise on $\mathbb{R}$, but the convergence is not uniform on any interval $[-a, a]$ $(a > 0)$.

Chapter 11

Riemann Integral

1. **Definition and Existence**
2. **Properties of the Riemann Integration**
3. **Fundamental Theorem of Calculus**
4. **Improper Integrals**

11.1 Definition and Existence

Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $m = \inf_{a \le x \le b} f(x),\ M = \sup_{a \le x \le b} f(x)$.

Definition 11.1.1 *A **partition** P of $[a, b]$ is a finite set of points in $[a, b]$ such that, if $P = \{x_0, x_1, \cdots x_n\}$, then $a = x_0 < x_1 < \cdots x_n = b$. By $\mathbb{P}([a, b])$ we denote the set of all the possible partitions of $[a, b]$. The set $\mathbb{P}([a, b])$ is ordered by inclusion: $P_1 \le P_2 \iff P_1 \subseteq P_2$. In this case P_2 is said to be a "refinement" of P_1.*

In any partition $P = \{x_0, x_1, \cdots x_n\}$ of $[a, b]$ for $i = 1, 2, \ldots, n$ put $m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$ and $M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$

To any partition $P = \{x_0, x_1, \cdots x_n\}$ of $[a, b]$ we associate three sums:

$$L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$$

$$U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

$$R(f, P) = \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}), \text{ where } \xi_i \in [x_{i-1}, x_i] \text{ is any point. ("Riemann Sum")}$$

It is clear that $m(b - a) \le L(f, P) \le R(f, P) \le U(f, P) \le M(b - a)$.

Definition 11.1.2 *A function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable on** $[a, b]$, if $\sup_{P\in\mathbb{P}([a,b])} L(f, P) = \inf_{P\in\mathbb{P}([a,b])} U(f, P)$.*
If this is the case and $I = \sup_{P\in\mathbb{P}([a,b])} L(f, P) = \inf_{P\in\mathbb{P}([a,b])} U(f, P)$. Then we say that, I is the Riemann integral of f on $[a, b]$ and we denote this I by $\int_a^b f(x)dx$.
If the function is Riemann integrable on $[a, b]$, $\int_a^b f(x)dx = \sup_{P\in\mathbb{P}([a,b])} L(f, P) = \inf_{P\in\mathbb{P}([a,b])} U(f, P)$.

$\star$ $R[a, b]$ is the set of all R-integrable functions on $[a, b]$.

Example 11.1.3 *Let $f : [0, 1] \to \mathbb{R}$, $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$ ("Dirichlet function").*
Let $P = \{x_0, x_1, \cdots, x_n\}$ be any partition of $[0, 1]$. Then for each $i \in \{1, \ldots, n\}$, we have
$m_i = \inf_{x_{i-1}\le x\le x_i} f(x) = 0$
$M_i = \sup_{x_{i-1}\le x\le x_i} f(x) = 1$
Hence, $L(f, P) = 0$, $U(f, P) = 1$. So, $\sup_{P\in\mathbb{P}([a,b])} L(f, P) = 0 \neq 1 = \inf_{P\in\mathbb{P}([a,b])} U(f, P)$. Hence f is not Riemann integrable on $[0, 1]$.

Example 11.1.4 *Let $f : [a, b] \to \mathbb{R}$, $f(x) = c$. Then for any $P \in \mathbb{P}([a, b])$,*
$L(f, P) = c(a - b),\ U(f, P) = c(a - b)$.
So f is R-integrable and $\int_a^b f(x)dx = c(b - a)$.

Example 11.1.5 *Let $y_1, \ldots, y_n$ be finitely many arbitrary points in $[0, 1]$ and $f : [0, 1] \to \mathbb{R}$ be the function defined by $f(x) = \begin{cases} c_i & \text{if } x = y_i \\ 0 & \text{otherwise.} \end{cases}$ where $c_i \in \mathbb{R}^+$ are given numbers.*
Let $P = \{x_0, x_1, \cdots x_n\}$ be any partition of $[0, 1]$. Then form the sums $L(f, P)$ and $U(f, P)$. Then $L(f, P) = 0$ and $U(f, P) \neq 0$ but $\inf_{p\in\mathbb{P}([a,b])} U(f, P) = 0$.

Lemma 11.1.6 *Let $R([a, b]) = \{f : [a, b] \to \mathbb{R} : f \text{ is R-integrable}\}$. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and P, P' be two arbitrary partitions of $[a, b]$. Then,*

1. *If $P \subseteq P'$, then $L(f, P) \le L(f, P')$, $U(f, P) \ge U(f, P')$*
2. *$L(f, P) \le U(f, P')$, $\forall P, P' \in \mathbb{P}([a, b])$.*

Proof 11.1.7 *1. Suppose $P \subseteq P'$. First suppose that $P' = P \cup \{x^*\}$.*
Then, let $k_i = \inf_{x_{i-1}\le x\le x^*} f(x)$, $\tilde{k}_i = \inf_{x_{i-1}\le x\le x_i} f(x)$. Then

$$\begin{aligned} L(f, P') &= m_1(x_1 - x_0) + \cdots + m_{i-1}(x_{i-1} - x_{i-2}) + m_i(x_i - x_{i-1})k_i(x^* - x_{i-1}) \\ &\quad + \tilde{k}_i(x_i - x^*) + m_{i+1}(x_{i+1} - x_i) + \cdots + m_n(x_n - x_{n-1}) \\ &\ge L(f, P) \end{aligned}$$

Similarly, $U(f, P') \le U(f, P)$.
Now, if $P' \setminus P = \{x_1^*, \ldots, x_k^*\}$ then applying the above reasoning to $P \cup \{x_1^*\}$, then to $P \cup \{x_1^*\} \cup \{x_2^*\}, \ldots,$ we get $L(f, P') \ge L(f, P)$ and $U(f, P') \le U(f, P)$.

2. *Let* $\tilde{P} = P \cup P'$. *Then* $\tilde{P} \in \mathbb{P}([a,b])$ *so that* $\tilde{P} \supseteq P$ *and* $\tilde{P} \supseteq P'$. *By 1 above,* $L(f,P) \le L(f,\tilde{P}) \le U(f,\tilde{P}) \le U(f,P')$.

   Conclusion*: Let* $A = \{L(f,P) : P \in \mathbb{P}([a,b])\}$, $A = \{L(f,P) : P \in \mathbb{P}([a,b])\}$. *Then,* A, B *are two bounded subsets of* $\mathbb{R}$. *Moreover,* $\forall x \in A$, $\forall y \in B$, $x \le y$.

Theorem 11.1.8 *Let* $f : [a,b] \to \mathbb{R}$ *be a bounded function. Then* f *is Riemann integrable on* $[a,b] \Leftrightarrow \forall \varepsilon > 0$, *there is a partition* $P \in \mathbb{P}([a,b])$ *such that* $U(f,P) - L(f,P) < \varepsilon$.

Proof 11.1.9 $(\Rightarrow)$ *Suppose* f *is Riemann integrable on* $[a,b]$. *Then,* $I = \sup_{P\in\mathbb{P}([a,b])} L(f,P) = \inf_{P\in\mathbb{P}([a,b])} U(f,P)$, *where* $I = \int_a^b f(x)dx$.

Hence, $\forall \varepsilon > 0$, $\exists P_1 \in \mathbb{P}([a,b]) : L(f,P_1) > I - \dfrac{\varepsilon}{2}$, $\exists P_2 \in \mathbb{P}([a,b]) : U(f,P_2) < I + \dfrac{\varepsilon}{2}$.

Let $P = P_1 \cup P_2$, *then* $L(f,P) > I - \dfrac{\varepsilon}{2}$, $U(f,P) < I + \dfrac{\varepsilon}{2}$.

Hence $U(f,P) - L(f,P) < \varepsilon$.

$(\Leftarrow)$ *Conversely suppose that given any* $\varepsilon > 0$ *there is a* $P \in \mathbb{P}([a,b])$ *such that* $U(f,P) - L(f,P) < \varepsilon$.

This implies that $\sup A = \inf B$. *Hence* f *is R-integrable.*

Theorem 11.1.10 *Let* $f : [a,b] \to \mathbb{R}$ *be a bounded function. Then* $f \in R([a,b])$ *iff there is a number* $I \in \mathbb{R}$ *such that* $\forall \varepsilon > 0$, $\exists P_0 \in \mathbb{P}([a,b]) : \forall P \in \mathbb{P}([a,b])$, $P \supseteq P_0 \Rightarrow |R(f,P) - I| < \varepsilon$, *for any choices of* ξ_i*'s in the definition of* $R(f,P)$.

Proof 11.1.11 $(\Rightarrow)$ *Suppose first that* $f \in R([a,b])$ *on* $[a,b]$. *Let* $I = \int_a^b f(x)dx$ *and* $\varepsilon > 0$ *be arbitrary. Then as above,* $\exists P_0 \in \mathbb{P}([a,b])$, $L(f,P) > I - \dfrac{\varepsilon}{2}$, $U(f,P) < I + \dfrac{\varepsilon}{2}$. *Hence, for any* $P \supseteq P_0$, $L(f,P) > I - \dfrac{\varepsilon}{2}$, $U(f,P) < I + \dfrac{\varepsilon}{2}$.

Then since for any choices of ξ_i*'s,* $L(f,P) \le R(f,P) \le U(f,P)$, *we conclude that* $-\dfrac{\varepsilon}{2} < R(f,P) - I < \dfrac{\varepsilon}{2}$, *i.e.* $|R(f,P) - I| < \dfrac{\varepsilon}{2}$.

$(\Leftarrow)$ *Conversely, assume that:* $\exists I \in \mathbb{R}$, $\forall \varepsilon > 0$, $\exists P_0 \in \mathbb{P}([a,b]) : \forall P \in \mathbb{P}([a,b])$, $P \supseteq P_0 \Rightarrow |R(f,P) - I| < \varepsilon$, *for any choices of* ξ_i*'s in the definition of* $R(f,P)$.

Take such a $P \supseteq P_0$, *so that* $|R(f,P) - I| < \varepsilon$, *for every choices of* ξ_i*'s in the definition of* $R(f,P)$.

As $R(f,P) = \displaystyle\sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1})$, *where* $\xi_i \in [x_{i-1}, x_i]$, *choose* ξ_i*'s such that* $f(\xi_i) - m_i < \dfrac{\varepsilon}{b-a}$. *Then* $|L(f,P) - R(f,P)| < \varepsilon$. *So,* $|L(f,P) - I| < 2\varepsilon$.

If we choose ξ_i*'s such that* $M_i - f(\xi_i) < \dfrac{\varepsilon}{b-a}$, *then* $|U(f,P) - I| < 2\varepsilon$.

Hence, $U(f,P) - L(f,P) < 4\varepsilon$. *So by theorem 11.1.8* f *is R-integrable on* $[a,b]$.

Lemma 11.1.12 *Let $f : [\alpha, \beta] \to \mathbb{R}$ be a bounded function. Then,*

$$\begin{aligned}\sup_{\alpha \le x \le \beta} f(x) - \inf_{\alpha \le x \le \beta} f(x) &= \sup\{f(x) - f(y) : \alpha \le x, y \le \beta\} \\ &= \sup\{|f(x) - f(y)| : \alpha \le x, y \le \beta\}\end{aligned}$$

Theorem 11.1.13 *1. If $f \in R[a,b]$, then $|f| \in R[a,b]$.*

2. If $f, g \in R[a,b]$, then $f + g \in R[a,b]$.

3. If $f \in R([a,b])$ and $c \in \mathbb{R}$ is a constant, then $cf \in R([a,b])$.

Proof 11.1.14 *1. Suppose $f \in R[a,b]$. Then, $\forall \varepsilon > 0$, $\exists P_0 \in \mathbb{P}([a,b]) : U(f,P) - L(f,P) < \varepsilon$. (i.e. $\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) < \varepsilon$)*

Hence, $M_i - m_i = \sup_{x_{i-1} \le x \le x_i} f(x) - \inf_{x_{i-1} \le x \le x_i} f(x)$

$$\begin{aligned}M_i - m_i &= \sup\{f(x) - f(y) : x, y \in [x_{i-1}, x_i]\} \\ &\ge \sup\{|f(x)| - |f(y)| : x, y \in [x_{i-1}, x_i]\} \\ &= \sup_{x_{i-1} \le x \le x_i} |f(x)| - \inf_{x_{i-1} \le x \le x_i} |f(x)|\end{aligned}.$$

Hence, $U(|f|, P) - L(|f|, P) \le U(f,P) - L(f,P) < \varepsilon$.

Hence by theorem 11.1.8, $|f| \in R[a,b]$.

2. Suppose $f, g \in R[a,b]$. Then, $\forall \varepsilon > 0$, $\exists P_1 \in \mathbb{P}([a,b]) : U(f,P_1) - L(f,P_1) < \frac{\varepsilon}{2}$

$\forall \varepsilon > 0$, $\exists P_2 \in \mathbb{P}([a,b]) : U(g,P_2) - L(g,P_2) < \frac{\varepsilon}{2}$

Let $P = P_1 \cup P_2$. Then $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$ and $U(g,P) - L(g,P) < \frac{\varepsilon}{2}$.

Let $P = \{x_0, \ldots, x_n\}$. Then;

$\sup_{x_{i-1} \le x \le x_i}(f(x) + g(x)) \le \sup_{x_{i-1} \le x \le x_i} f(x) + \sup_{x_{i-1} \le x \le x_i} g(x)$ and

$\inf_{x_{i-1} \le x \le x_i}(f(x) + g(x)) \ge \inf_{x_{i-1} \le x \le x_i} f(x) + \inf_{x_{i-1} \le x \le x_i} g(x)$.

Hence, $U(f+g,P) - L(f+g,P) \le U(f,P) - L(f,P) + U(g,P) - L(g,P) < \varepsilon$.

So, $f + g \in R[a,b]$.

3. If $c = 0$, this is trivial.

If $c > 0$, then for any $P \in \mathbb{P}([a,b]) : U(cf,P) - L(cf,P) = c[U(f,P) - L(f,P)]$. From this we see that $cf \in R[a,b]$. If $c < 0$, then $cf \in R([a,b])$, too.

Conclusion:

1. $R([a,b])$ is a vector space over $\mathbb{R}$.
2. $\forall f \in R([a,b])$, $|f| \in R([a,b])$.

   Hence, $\forall f \in R([a,b])$, $f^+ = \frac{|f| + f}{2}$, $f^- = \frac{f - |f|}{2}$ are in $R([a,b])$.

3. For $f, g \in R([a,b])$, $\sup\{f,g\} = \dfrac{|f-g|+f+g}{2}$, $\inf\{f,g\} = \dfrac{f+g-|f-g|}{2}$ are in $R([a,b])$.
   So, $R([a,b])$ is a "lattice".

Proposition 11.1.15 *If $f \in R([a,b])$ then $f^2 \in R([a,b])$.*

Proof 11.1.16 *Suppose $f \in R([a,b])$. Then, $\forall \varepsilon > 0$, $\exists P \in \mathbb{P}([a,b]) : U(f,P) - L(f,P) < \varepsilon$. Let $P = \{x_0, \ldots, x_n\}$. $\tilde{M} = \sup_{a \le x \le b} |f(x)|$. Then,*

$$\sup_{x_{i-1} \le x \le x_i} f^2(x) - \inf_{x_{i-1} \le x \le x_i} f^2(x) \le 2\tilde{M}(\sup_{x_{i-1} \le x \le x_i} f(x) - \inf_{x_{i-1} \le x \le x_i} f(x)).$$

Hence, $U(f^2,P) - L(f^2,P) < 2\varepsilon\tilde{M}$. So, $f^2 \in R([a,b])$.
***Conclusion**: Let $f, g \in R([a,b])$. Then $f \cdot g = \dfrac{(f+g)^2 - f^2 - g^2}{2}$, so that $f \cdot g \in R([a,b])$. Hence, $R([a,b])$ is a ring.*

Theorem 11.1.17 *Every continuous function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof 11.1.18 *Let $f \in C([a,b])$ be a continuous function. Then f is uniformly continuous on $[a,b]$. So, $\forall \varepsilon > 0$, $\exists \eta > 0$, $\forall x, y \in [a,b]$, $|x-y| < \eta \Rightarrow |f(x) - f(y)| < \dfrac{\varepsilon}{b-a}$.*
Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ such that $\sup_{1 \le i \le n} |x_i - x_{i-1}| < \eta$. Then $\sup_{x_{i-1} \le x \le x_i} f(x) - \inf_{x_{i-1} \le x \le x_i} f(x) = \sup\{|f(x)| - |f(y)| : x, y \in [x_{i-1}, x_i]\} < \dfrac{\varepsilon}{b-a}$, i.e. $M_i - m_i \le \dfrac{\varepsilon}{b-a}$.
Then, $U(f,P) - L(f,P) = \displaystyle\sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \le \varepsilon$. Hence, $f \in R([a,b])$.

Theorem 11.1.19 *Every monotone function $f : [a,b] \to \mathbb{R}$ is Riemann integrable.*

Proof 11.1.20 *Suppose f is increasing. If $f(a) = f(b)$ then f is constant. So, we can assume that $f(a) < f(b)$.*
Let $\varepsilon > 0$ be any number.
Let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a,b]$ such that $x_i - x_{i-1} < \dfrac{\varepsilon}{f(b) - f(a)}$ for $i = 1, \ldots, n$.
Then, $\sup_{x_{i-1} \le x \le x_i} f(x) - \inf_{x_{i-1} \le x \le x_i} f(x) = f(x_i) - f(x_{i-1})$. Hence,

$$\begin{aligned} U(f,P) - L(f,P) &= \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))(x_i - x_{i-1}) \\ &\le \frac{\varepsilon}{f(b) - f(a)} \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \varepsilon \end{aligned}$$

Definition 11.1.21 *A subset $E \subseteq \mathbb{R}$ is said to be* ***negligible*** *if $\forall \varepsilon > 0$ we can find countably many intervals (a_i, b_i) such that, $E \subseteq \cup_{i \in \mathbb{N}}(a_i, b_i)$ and $\sum_{i=1}^{\infty}(b_i - a_i) < \varepsilon$.*

Example 11.1.22 *Any countable set $E = \{x_n : n \in \mathbb{N}\}$ is negligible.*
Indeed, let $\varepsilon > 0$ be any number. Let $I_n = \left]x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}}\right[$. Then, $E \subseteq \cup_{n \in \mathbb{N}} I_n$ and $\sum_{n=0}^{\infty} l(I_n) = \sum_{n=0}^{\infty} \frac{\varepsilon}{2^{n+1}} < \varepsilon$.

Example 11.1.23 *There are also uncountable negligible sets. E.g. the usual Cantor set $E \subseteq [a, b]$ is uncountable and negligible.*

Now, let $f : [a, b] \to \mathbb{R}$ be a function and $D_f = \{x \in [a, b] : f \text{ is discontinuous at } x\}$. $D_f \subseteq [a, b]$.

E.g. • if f is monotone, D_f is countable, so negligible.

• if f is $f(x) = \begin{cases} 1 \text{ if } x \in [a,b] \cap \mathbb{Q} \\ 0 \text{ if } x \notin [a,b] \cap \mathbb{Q} \end{cases}$, then $D_f = [a, b]$.

• if f is $f(x) = \begin{cases} x \sin \frac{1}{x} \text{ if } x \neq 0 \\ 0 \ \text{ if } x = 0 \end{cases}$, then $D_f = \emptyset$.

Theorem 11.1.24 *Let $f : [a, b] \to \mathbb{R}$ be any function. Then, $f \in R([a, b])$ iff*
1) f is bounded on $[a, b]$
2) D_f is negligible, ie f is continuous "almost everywhere".

11.2 Properties of the Riemann Integral

Theorem 11.2.1 *For $f \in R([a, b])$, let $T(f) = \int_a^b f(x)dx$. Then $T : R([a, b]) \to \mathbb{R}$ is a linear mapping, i.e. $T(f + g) = T(f) + T(g)$ and $T(cf) = cT(f)$.*

Proof 11.2.2 • *Let $f, g \in R([a, b])$. So we have:*
$\forall \varepsilon > 0, \exists P_0' \in \mathbb{P}([a, b])\ \forall P \in \mathbb{P}([a, b]) : P \supseteq P_0' \Rightarrow \left|R(f, P) - \int_a^b f(x)dx\right| < \frac{\varepsilon}{2}$. (1)
$\forall \varepsilon > 0, \exists P_0'' \in \mathbb{P}([a, b])\ \forall P \in \mathbb{P}([a, b]) : P \supseteq P_0'' \Rightarrow \left|R(g, P) - \int_a^b g(x)dx\right| < \frac{\varepsilon}{2}$. (2)
Let $P_0 = P_0' \cup P_0''$. Then for any $P \supseteq P_0$, (1) and (2) are satisfied for every choices of ξ_i's.
As, $R(f+g, P) = R(f, P)+R(g, P)$, for $P \supseteq P_0$ $\left|R(f + g, P) - \int_a^b f(x)dx - \int_a^b g(x)dx\right| < \varepsilon$.
As, $f + g \in R([a, b])$ and the integral of a function is unique,
$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx$.
• *For $c \in \mathbb{R}$, observe that $R(cf, P) = cR(f, P)$ for any $P \in \mathbb{P}([a, b])$. From this we deduce that $\int_a^b cf(x)dx = c \int_a^b f(x)dx$.*

Theorem 11.2.3 *For* $f \in R([a,b])$, $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

Proof 11.2.4 *We know that if* $f \in R([a,b])$ *then* $|f| \in R([a,b])$*, too.*
Moreover, $f^+, f^- \in R([a,b])$ *and* $f = f^+ + f^-$. *Hence, by theorem 11.2.1,*
$\int_a^b f(x)dx = \int_a^b f^+(x)dx - \int_a^b f^-(x)dx$.
Hence, $\left|\int_a^b f(x)dx\right| = \left|\int_a^b f^+(x)dx - \int_a^b f^-(x)dx\right| \leq \int_a^b f^+(x)dx + \int_a^b f^-(x)dx = \int_a^b |f(x)|\,dx$.

Warning: $|f| \in R([a,b]) \not\Rightarrow f \in R([a,b])$. Let $f : [0,1] \to \mathbb{R}$ $f(x) = \begin{cases} -1 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is irrational} \end{cases}$

Then, $|f| = 1$. So $|f| \in R([a,b])$.
But f is discontinuous at every $x \in [0,1]$, so $f \notin R([a,b])$.

Example 11.2.5 *Let* $[c,d] \subseteq [a,b]$ *be any subinterval and* $\varphi = \chi_{[c,d]} : [a,b] \to \mathbb{R}$. *Then,* $\varphi \in R([a,b])$. *So, for any* $f \in R([a,b])$, $f \circ \varphi$ *is integrable. Hence,* f *is integrable on* $[c,d]$.
Now, let $f \in R([a,b])$. *Then for every choices of* ξ_i*'s,*
$\forall \varepsilon > 0,\ \exists P_0 \in \mathbb{P}([a,b])\ \forall P \in \mathbb{P}([a,b]) : P \supseteq P_0 \Rightarrow \left|R(f,P) - \int_a^b f(x)dx\right| < \varepsilon$.
We can always assume that the points $c, d \in P$.
Let $P_1 = P \cap [a,c],\ P_2 = P \cap [c,d],\ P_3 = P \cap [d,b]$
Then $R(f,P) = R(f,P_1) + R(f,P_2) + R(f,P_3)$. *From this we conclude that,*
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^d f(x)dx + \int_d^b f(x)dx$.

Example 11.2.6 ***Perturbation of Riemann Integrable Functions:***
- *Let* $y_0 \in [a,b]$ *any point. Then* $\varphi = \chi_{\{y_0\}}$. *Then* $\int_a^b \varphi(x)dx = 0$.
- *Now, let* $y_0, y_1, \ldots, y_n \in [a,b]$ *and* $c_0, \ldots, c_n \in \mathbb{R}$. *Let* $\varphi = \sum_{i=1}^{n} c_i \chi_{\{y_i\}}$. *Then,*

$$\int_a^b \varphi(x)dx = \sum_{i=1}^{n} c_i \int_a^b \chi_{\{y_i\}}(x)dx = 0.$$

Hence, $\forall f \in R([a,b])$, $f + \varphi \in R([a,b])$ *and* $\int_a^b (f(x) + \varphi(x))dx = \int_a^b f(x)dx$.
- *Now, let* $\mathbb{Q} \cap [0,1] = \{y_n : n \in \mathbb{N}\}$. *Let* $\varphi_n = \chi_{\{y_0,\ldots,y_n\}} = \chi_{\{y_0\}} + \cdots + \chi_{\{y_n\}}$. *Then,* $\varphi_n \in R([a,b])$ *and* $\int_0^1 \varphi_n(x)dx = 0$.

However, $\varphi_n \to \varphi$, $\varphi(x) = \begin{cases} 1 \ \textit{if } x \in \mathbb{Q} \cap [0,1] \\ 0 \ \textit{if } x \notin \mathbb{Q} \cap [0,1] \end{cases}$ *and* $\varphi \notin R([a,b])$.
Hence, $R([a,b])$ *is not closed under pointwise limit.*

Theorem 11.2.7 ***(Uniform Convergence and Riemann Integration)*** *Let* $f_n \in R([a,b])$ *be a sequence of Riemann integrable functions. Suppose that* $(f_n)_{n\in\mathbb{N}}$ *converges uniformly on* $[a,b]$ *to some function* $f : [a,b] \to \mathbb{R}$. *Then* $f \in R([a,b])$ *and* $\int_a^b |f_n(x) - f(x)|\,dx \to 0$, *so* $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

Proof 11.2.8 *Since $f_n \to f$ uniformly on $[a,b]$, we have:*

$\forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N \ \sup_{x\in[a,b]} |f_n(x) - f(x)| < \dfrac{\varepsilon}{b-a}.$ $(\star)$

Now, fix an $n \geq N$. Since $f_n \in R([a,b])$, $\exists P \in \mathbb{P}([a,b])$ such that

$$U(f,P) - L(f,P) = \sum_{i=1}^{n}(M_{i,n} - m_{i,n})(x_i - x_{i-1}) < \frac{\varepsilon}{b-a}.$$

Then, by $(\star)$, $f_n(x) - \varepsilon \leq f(x) \leq f_n(x) + \varepsilon$, $\forall x \in [a,b]$.

If $M_i = \sup_{x_{i-1}\leq x\leq x_i} f(x)$, $m_i = \inf_{x_{i-1}\leq x\leq x_i} f(x)$, $|M_{i,n} - M_i| < \varepsilon$ and $|m_{i,n} - m_i| < \varepsilon$ for all $i \geq 1$, then

$$U(f,P) - L(f,P) = U(f,P) - U(f_n,P) + U(f_n,P) - L(f_n,P) + L(f_n,P) - L(f,P) \leq \varepsilon.$$

Hence, $f \in R([a,b])$.

Now, from $(\star)$: $\forall n \geq N \ \int_a^b |f_n(x) - f(x)|\,dx \leq \int_a^b \dfrac{\varepsilon}{b-a}dx \leq \varepsilon$.

Hence, $\int_a^b |f_n(x) - f(x)|\,dx \to 0 \Longrightarrow \int_a^b f_n(x)dx \to \int_a^b f(x)dx$.

$\star$ Now, for $f,g \in R([a,b])$, put $d(f,g) = \int_a^b |f(x) - g(x)|\,dx$.

Then, $d(f,g) = d(g,f)$, as well as, $d(f,g) \leq d(f,h) + d(h,g)$, $\forall h \in R([a,b])$.

However, $d(f,g) = 0 \not\Rightarrow f = g$.

Theorem 11.2.9 *Let $f : [a,b] \to \mathbb{R}$ be a continuous, positive function, i.e. $f(x) \geq 0$, $\forall x \in [a,b]$. Then, $\int_a^b f(x)dx = 0$ iff $f \equiv 0$ on $[a,b]$.*

Proof 11.2.10 $(\Rightarrow)$ *Suppose that, there is an $x_0 \in [a,b]$, $f(x_0) > 0$. Then taking $\varepsilon = \dfrac{f(x_0)}{2}$ and writing the continuity of f at x_0, we get:*

$\exists \eta > 0, \forall x \in]x_0 - \eta, x_0 + \eta[\cap [a,b]$, *we have* $f(x) > \dfrac{f(x_0)}{2}$.

Let $[c,d] \subseteq]x_0 - \eta, x_0 + \eta[\cap [a,b]$. $(c < d)$. Then, since $f \geq 0$ on $[c,d]$,

$\int_a^b f(x)dx \geq \int_c^d f(x)dx \geq \int_c^d \dfrac{f(x_0)}{2}dx = \dfrac{f(x_0)}{2}(d-c) > 0$, *which is not possible.*

Hence, $\int_a^b f(x)dx = 0 \Rightarrow f \equiv 0$ on $[a,b]$.

$(\Leftarrow)$ *The converse is trivial.*

Corollary 11.2.11 *For $f,g \in C([a,b])$, $d(f,g) = \int_a^b |f(x) - g(x)|\,dx$, then d is a metric on $C([a,b])$.*

11.3 Fundamental Theorem of Calculus

Theorem 11.3.1 ***(Fundamental Theorem of Calculus)*** *Let $f \in R([a,b])$ be a Riemann integrable function. For $x \in]a,b[$, let $F(x) = \int_a^x f(t)dt$. If at some point $x_0 \in]a,b[$, f is continuous then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.*

Proof 11.3.2 *Suppose f is continuous at x_0. Then we have:*

$$\forall\varepsilon > 0,\ \exists\eta > 0 : \begin{cases} t \in [a,b] \\ |t - x_0| < \eta \end{cases} \Longrightarrow |f(t) - f(x_0)| < \varepsilon$$

Since, $x_0 \in]a,b[$, we can assume that $[x_0 - \eta, x_0 + \eta] \subseteq [a,b]$.

Now, let $h \in \mathbb{R}$ be a number such that $|h| < \eta$, so that $x_0 + h \in [a,b]$.

Then, $\dfrac{F(x_0+h) - F(x_0)}{h} = \dfrac{1}{h}\left[\int_a^{x_0+h} f(t)dt - \int_a^{x_0} f(t)dt\right] = \dfrac{1}{h}\int_{x_0}^{x_0+h} f(t)dt.$

Hence, $\dfrac{F(x_0+h) - F(x_0)}{h} - f(x_0) = \dfrac{1}{h}\int_{x_0}^{x_0+h} [f(t) - f(x_0)]\, dt.$

Hence, since $|h| < \eta$, $\left|\dfrac{F(x_0+h) - F(x_0)}{h} - f(x_0)\right| \le \dfrac{1}{|h|}\int_{x_0}^{x_0+h} |f(t) - f(x_0)| dt \le \varepsilon.$

So, $\lim_{h\to 0,\ h\ne 0} \dfrac{F(x_0+h) - F(x_0)}{h} = f(x_0)$. *Hence, if $f \in C([a,b])$, then $F(x) = \int_a^x f(t)dt$ is differentiable on $]a,b[$ and $F'(x) = f(x)$ on $]a,b[$.*

Remark: For any $f \in R([a,b])$, the function $F(x) = \int_a^x f(t)dt$ is continuous on $[a,b]$. (absolutely continuous) Hence, if $f \in C([a,b])$ and $F(x) = \int_a^x f(t)dt$, from $F' = f$ on $]a,b[$, we conclude that $F' = f$ on $[a,b]$. So, we can write the above theorem as follows:

$\int_a^x F'(t)dt = F(x) - F(a)$ or $\int_a^b F'(t)dt = F(b) - F(a)$. ("Newton-Leibniz Theorem")

Definition 11.3.3 *Let $f : [a,b] \to \mathbb{R}$ be a function. If there exists a function $F : [a,b] \to \mathbb{R}$ such that F is differentiable on $]a,b[$ and $F'(x) = f(x)$ for $x \in]a,b[$, then we say that F is a* ***primitive*** *(or* ***antiderivative****) of f.*

An antiderivative, if exists is unique up to an additive constant. Indeed, if $\begin{matrix} F' = f \\ G' = f \end{matrix}$ on $]a,b[$, then $(F - G)' = 0$ on $]a,b[$. Hence, $F - G$ =constant, so they are the same up to a constant. We have just proved that every continuous function $f : [a,b] \to \mathbb{R}$ has an antiderivative: $F(x) = \int_a^x f(t)dt$.

Theorem 11.3.4 *For any $f \in R([a,b])$, $F(x) = \int_a^x f(t)dt$ is absolutely continuous, so of the bounded variation on $[a,b]$.*

Proof 11.3.5 *We have to show that: $\forall\varepsilon > 0, \exists\eta > 0 : \forall(a_1,b_1),\dots,(a_n,b_n)$, non-overlapping subintervals of $[a,b]$ satisfying $\sum_{i=1}^n (b_i - a_i) < \eta$, we have $\sum_{i=1}^n |F(b_i) - F(a_i)| < \varepsilon$*

Indeed, let $(a_1,b_1),\dots,(a_n,b_n)$ be arbitrary non-overlapping subintervals of $[a,b]$. Then, as $|F(b_i) - F(a_i)| = \left|\int_{a_i}^{b_i} f(t)dt\right| \le \int_{a_i}^{b_i} |f(t)|\, dt$, $\sum_{i=1}^n |F(b_i) - F(a_i)| \le \sum_{i=1}^n \int_{a_i}^{b_i} |f(t)|\, dt$. If $M = \sup_{a\le t\le b} |f(t)|$, then $\sum_{i=1}^n \int_{a_i}^{b_i} |f(t)|\, dt \le M\sum_{i=1}^n (b_i - a_i)$. Thus, given $\varepsilon > 0$, if we choose

$0 < \eta < \dfrac{\varepsilon}{M}$, *then we see that the definition of the absolute continuity of* F *is satisfied. Hence,* F *is absolutely continuous on* $[a, b]$.

Theorem 11.3.6 ***(Mean Value Theorem for Integration)*** *Let* $f : [a, b] \to \mathbb{R}$ *be a continuous function,* $m = \inf_{a \leq x \leq b} f(x)$, $M = \sup_{a \leq x \leq b} f(x)$. *Then* $\exists c \in [a, b]$ *such that* $\dfrac{1}{b-a} \int_a^b f(x)dx = f(c)$.

Proof 11.3.7 *We know that* $m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a), \forall P \in \mathbb{P}([a, b])$. *As,* $\int_a^b f(x)dx = \sup_{P \in \mathbb{P}([a,b])} L(f, P)$, *we see that* $m \leq \dfrac{1}{b-a} \int_a^b f(x)dx \leq M$.

As, $f : [a, b] \to \mathbb{R}$ *is continuous,* $f([a, b]) = [m, M]$. *Hence,* $\dfrac{1}{b-a} \int_a^b f(x)dx \in [m, M]$. *So,* $\dfrac{1}{b-a} \int_a^b f(x)dx = f(c)$, *for some* $c \in [a, b]$.

Theorem 11.3.8 ***(Integration by Parts)*** *Let* $F, G : [a, b] \to \mathbb{R}$ *two continuously differentiable functions. Then* $(FG)' = F'G + FG'$, *so that* $F'G = (FG)' - FG'$. *Hence,* $\int_a^b F'(x)G(x)dx = \int_a^b (F(x)G(x))'dx - \int_a^b F(x)G'(x)dx = F(x)G(x)|_a^b - \int_a^b F(x)G'(x)dx$.

Theorem 11.3.9 ***(Change of Variable Formula)*** *Let* $u : [c, d] \to \mathbb{R}$ *be a continuously differentiable function with* $u(c) = a$, $u(d) = b$. *Let* $F : [a, b] \to \mathbb{R}$ *be another continuously differentiable function. Then,* $\int_a^b F'(x)dx = \int_c^d F'(u(x))u'(x)dx$.

Proof 11.3.10 *Let* $G : [c, d] \to \mathbb{R}$, $G(x) = F(u(x))$. *Then,* $G'(x) = F'(u(x))u'(x)$, *so that*

$$\begin{aligned} \int_c^d F'(u(x))u'(x)dx &= \int_c^d G'(x)dx = G(x)|_c^d \\ &= F(u(x))|_c^d = F(u(d)) - F(u(c)) = F(b) - F(a) \\ &= \int_a^b F'(x)dx \end{aligned}$$

11.4 Improper Integrals

Riemann integral is defined for a bounded function on a compact interval $[a, b]$. If either the interval on which we work is not compact, or the function with which we work is not bounded, then we cannot define Riemann integral. Instead of it we define the "improper Riemann integral". e.g., $\int_0^1 \dfrac{dx}{\sqrt{x}}$, $\int_1^\infty \dfrac{dx}{x^2}$.

Definition 11.4.1 *Let* $f : [a, \infty[\to \mathbb{R}$ *be a function such that for each* $b \geq a$, f *is Riemann integrable on* $[a, b]$, *so that* $\int_a^b f(x)dx$ *exists. If* $\lim_{b \to \infty} \int_a^b f(x)dx$ *exists, then we say that the improper integral* $\int_a^\infty f(x)dx$ *exists or converges. So, in this case by definition,*

$\int_a^\infty f(x)dx = \lim_{b \to \infty} \int_a^b f(x)dx$.

Similarly, if $f :]a, b] \to \mathbb{R}$ *is such that, for each* $a < c < b$, f *is R-integrable on* $[c, b]$ *and* $\lim_{c\to a,\ c>a} \int_c^b f(x)dx$ *exists, then we say that the improper integral,* $\int_a^b f(x)dx$ *exists or converges. In this case,* $\int_a^b f(x)dx = \lim_{c\to a,\ c>a} \int_c^b f(x)dx$.

If $f :]-\infty, \infty[\to \mathbb{R}$ *is such that, for all* $a \le b$, $f \in R([a, b])$ *and* $\lim_{b\to\infty,\ a\to\infty} \int_a^b f(x)dx$ *exists, then* $\int_{-\infty}^{\infty} f(x)dx$ *exists or converges.*

In this case, $\int_{-\infty}^{\infty} f(x)dx = \lim_{b\to\infty,\ a\to\infty} \int_a^b f(x)dx$.

Warning Consider the improper integral $\int_{-\infty}^{\infty} \frac{2x}{1+x^2}dx$.

As, $\lim_{b\to\infty,\ a\to\infty} \int_a^b \frac{2x}{1+x^2}dx = \lim_{b\to\infty,\ a\to\infty} \ln(1+x^2)|_a^b$, the improper integral $\int_{-\infty}^{\infty} \frac{2x}{1+x^2}dx$ does not exists. But, $\lim_{a\to\infty} \int_{-a}^{a} \frac{2x}{1+x^2}dx = \lim_{a\to\infty} \ln(1+x^2)|_{-a}^{a} = 0$.

If $f :]-\infty, \infty[\to \mathbb{R}$ and $\lim_{a\to\infty} \int_{-a}^{a} f(x)dx$ exists, then this is denoted by (CV)

$\int_{-\infty}^{\infty} f(x)dx = \lim_{a\to\infty} \int_{-a}^{a} f(x)dx$. (Cauchy Mean Value of an improper integral)

Example 11.4.2 *For* $p < 1$, $\int_0^1 \frac{dx}{x^p}$ *converges. For* $p \ge 1$, *it diverges.* $\int_0^1 \frac{dx}{x^p} = \lim_{c\to 0,\ c>0} \int_c^1 \frac{dx}{x^p}$, *for* $x \ne 1$, $\int_c^1 \frac{dx}{x^p} = \left.\frac{x^{-p+1}}{1-p}\right|_c^1 = \frac{1}{1-p} - \frac{c^{-p+1}}{1-p}$.

If $p < 1$, $\lim_{c\to 0, c>0} \frac{c^{-p+1}}{1-p} = 0$, *so that* $\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}$.

If $p \ge 1$, $\int_0^1 \frac{dx}{x^p}$ *diverges.*

Proposition 11.4.3 *Let* $f : [a, \infty[\to \mathbb{R}$ *be a function such that* $\forall b \ge a$, $f \in R([a, b])$. *If there is* $M > 0$, *such that* $\forall b \ge a$, $\int_a^b |f(x)|dx \le M$, *then the improper integral converges.*

Proof 11.4.4 *Let* $F(b) = \int_a^b |f(x)|dx$. *Then* F *is increasing and* $F(b) \le M$. *So,* $\lim_{b\to\infty} F(b)$ *exists, i.e.* $\int_a^{\infty} |f(x)|dx$ *exists.*

Now, let $b' \ge b \ge a$. *Then,* $\left|\int_b^{b'} f(x)dx\right| \le \int_b^{b'} |f(x)|\, dx \to 0$, *as* $b \to \infty$, $b' \to \infty$. *Hence, the Cauchy condition is satisfied as* $b \to \infty$, *for* $G(b) = \int_a^b f(x)dx$. *Hence,* $\lim_{b\to\infty} G(b)$ *exists, i.e.* $\int_a^{\infty} f(x)dx$ *converges.*

Example 11.4.5 *Show that the improper integral* $\int_1^{\infty} \frac{\sin x}{x^2}dx$ *converges. Indeed,* $\int_1^b \left|\frac{\sin x}{x^2}\right| dx \le \int_1^b \frac{1}{x^2}dx \le M$, *for all* $b \ge 1$. *Hence,* $\int_1^{\infty} \frac{\sin x}{x^2}dx$ *converges. But, it may happen that* $\int_a^{\infty} |f(x)|\, dx$ *diverges but* $\int_a^{\infty} f(x)dx$ *converges.*

11.5 Exercises

1. Let $f \in \mathbb{R}[a,b]$. For $n \in \mathbb{N}, n \geq 1$, put

$$\sigma_n = \frac{b-a}{n}\left[f(a) + f\left(a + \frac{b-a}{n}\right) + \cdots + f\left(a + k\frac{b-a}{n}\right) + \cdots + f\left(a + (n-1)\frac{b-a}{n}\right)\right]$$

$$\sum_n = \frac{b-a}{n}\left[f\left(a + \frac{b-a}{n}\right) + \cdots + f\left(a + k\frac{b-a}{n}\right) + \cdots + f\left(a + (n-1)\frac{b-a}{n}\right) + f(b)\right]$$

Show that the sequences $(\sigma_n)_{n\geq 1}$ and $(\sum_n)_{n\geq 1}$ converge and

$$\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \sum_n = \int_a^b f(x)dx$$

2. Find $\alpha = \lim_{n\to\infty} \frac{\pi}{n} \sum_{k=0}^{n-1} \frac{1}{2 + \cos\frac{k\pi}{n}}$. Hint $(\alpha = \int_0^\pi \frac{d_x}{2 + \cos x})$.

3. Find $\beta = \lim_{n\to\infty} \ln \frac{1}{n^n} \sqrt[n]{(n+1)(n+2)...(n+n)}$. Hint $(\beta = \int_0^1 \ln(1+x)dx)$.

4. Find $\gamma = \lim_{n\to\infty} \sum_{k=1}^{pn} \frac{1}{n+k}$ $(p \geq 1$ fixed). Hint $\left(\gamma = \int_0^1 \frac{d_x}{x + \frac{1}{p}}\right)$

5. Evaluate $\int_0^1 x dx$ using
   (a) the definition of the integrals
   (b) the above question

6. Evaluate $\alpha_n = \int_0^2 x^n dx$ and show that $\lim_{n\to\infty} \alpha_n^{\frac{1}{n}} = 2$.

7. Can you prove that $\lim_{n\to\infty} \left[\int_0^{\frac{\pi}{2}} (\sin x)^n dx\right]^{\frac{1}{n}} = 1$?

8. Let $f : [a,b] \to \mathbb{R}$ be a continuous positive function and $M = \sup_{a\leq x\leq b} f(x)$. Prove that $\lim_{n\to\infty} \left[\int_a^b (f(x))^n dx\right]^{\frac{1}{n}} = M$

9. Prove that the series $\sum_{k=1}^{\infty} \frac{\sin kx}{k^p}$ and $\sum_{k=1}^{\infty} \frac{\cos kx}{k^p}$ $(p > 0)$ converge uniformly on any closed interval that does not contain an integer multiple of 2π

10. Let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \begin{cases} \frac{1}{n} & \text{if } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n} \\ 0 & \text{elsewhere} \end{cases}$. Prove that the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[0,1]$, but that the Weierstrass M-test fails.

11. Show that each of the following series converges uniformly on the indicated interval.

    (a) $\sum_{k=1}^{\infty} \frac{1}{x^2 + k^2}$, $0 \leq x < \infty$

    (b) $\sum_{k=0}^{\infty} e^{-kx} x^k$, $0 \leq x \leq \infty$

    (c) $\sum_{k=1}^{\infty} k^2 e^{-kx}$, $1 \leq x < \infty$

    (d) $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{x + k}$, $0 \leq x < \infty$

12. Find the radius of convergence of the following series

    (a) $\sum_{k=1}^{\infty} \frac{3^k}{k^3} x^k$

    (b) $\sum_{k=1}^{\infty} (1 - \frac{1}{k})^k x^k$

    (c) $\sum_{k=1}^{\infty} (1 - \frac{1}{k})^{k^2} x^k$

    (d) $\sum_{k=1}^{\infty} \frac{1}{4^k} (x + 1)^{2k}$

13. Determine the sum of each of the series

    (a) $\sum_{k=1}^{\infty} k x^k$, $|x| < 1$

    (b) $\sum_{k=1}^{\infty} k^2 x^k$, $|x| < 1$

(c) $\sum_{k=1}^{\infty} \frac{k}{2^k}$

(d) Let $I_n = \int_0^1 (1-x^2)^n d_x$. Show that $\lim_{n\to\infty} I_n = 0$.

(e) Let $0 < \alpha < 1$, and $J_n = \int_\alpha^1 (1-x^2)^n dx$. Prove that $\lim_{n\to\infty} \frac{J_n}{I_n} = 0$.

14. Let $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ be a continuous function with $\frac{\delta f}{\delta y}$ continuous on $[a,b] \times \mathbb{R}$. Put $F(y) = \int_a^b f(x,y)dx$. Show that F is differentiable on $\mathbb{R}$ and $F'(y) = \int_a^b \frac{\delta}{\delta y} f(x,y)dx$.

15. Let $f : [a,b] \to \mathbb{R}$ be continuous $u, v : [c,d] \to [a,b]$ be continuously differentiable, i.e. u', v' exist and continuous on $[c,d]$. Put $g(x) = \int_{u(x)}^{v(x)} f(t)dt$. Show that g is differentiable on $]c,d[$ and find g'.

16. Let $f \in \mathbb{R}[a,b]$. Let $F(x) = \int_a^z f(t)dt$. Show that F is the difference of two monotone increasing functions.

17. Let $f : [0, 2\pi] \to \mathbb{R}$ be a continuous function. Assume that for each $n \in \mathbb{N}$, $\int_0^{2\pi} f(x)\cos nx dx = \int_0^{2\pi} f(x)\sin nx dx = 0$. Show that then $f \equiv 0$ on $[0, 2\pi]$.

18. Let $f : [a,b] \to \mathbb{R}$ be a function which is zero at all $x \in [a,b]$ except at finitely many points $x_1, x_2, ..., x_N$ in $[a,b]$. Show that $\int_a^b f(x)d_x = 0$.

19. Let $f, g : [a,b] \to \mathbb{R}$ be two Riemann Integrable functions such that $f(x) = g(x)$ for all $x \in [a,b]$ except at finitely many points. Show that $\int_a^b f(x)d_x = \int_a^b g(x)d_x$.

20. Let $f \in \mathbb{R}[-a, 0]$. Show that

(a) if f is even (i.e. $f(x) = f(-x)$), then $\int_{-a}^a f(x)d_x = 2\int_0^a f(x,y)dx$.

(b) if f is odd (i.e. $f(-x) = -f(x)$), then $\int_{-a}^a f(x)dx = 0$.

21. Find $F'(x)$ where F is defined on $[0,1]$ as follows:

(a) $F(x) = \int_0^x \frac{dt}{1+t^2}$

(b) $F(x) = \int_0^x \cos t^2 dt$

(c) $F(x) = \int_0^{x^2} \cos t^2 dt$

(d) $F(x) = \int_x^1 \sqrt{1+t^2} dt$

22. Let $f : [a,b] \to \mathbb{R}$ be continuous and $g \in \mathbb{R}[a,b]$, $g(x) \geq 0$ for all $x \in [a,b]$. Show that there exists $x \in [a,b]$ such that $\int_a^b f(x)g(x)dx = f(c)\int_a^b g(x)dx$.

23. Suppose that $f : [0,1] \to \mathbb{R}$ is continuous. Prove that $\lim_{n\to\infty} \int_0^1 f(x^n)dx = f(0)$.

Chapter 12

The Space $C(K)$

12.1 Generalities About $C\left(K\right)$

Let (X, d) be any m.s , $K \subseteq X$ a compact subset and $C(K) = \{f : K \to \mathbb{R} : f \text{ is continuous on } K\}$. On $C(K)$ we shall always put the "supremum metric": $d_\infty(f, g) = \sup_{x \in K} |f(x) - g(x)|$. We know that $(C(K), d_\infty)$ is a complete metric space.

In this chapter we want to study some properties of this metric space. In particular we want:
a) to characterize the **compact** subsets of $C(K)$
b) to characterize the **dense** subsets of $C(K)$.

12.1.1 Cantor's Diagonal Method

Theorem 12.1.1 *Let $E = \{x_0, x_1, \ldots, x_n, \ldots\}$ be a countable set, (X, d) a compact metric space and $f_n : E \to X$ any sequence of functions. Then, $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that for each $x_i \in E$, the sequence $(f_{n_k}(x_i))_{k\in\mathbb{N}}$ converges in X.*

Proof 12.1.2 *Start with x_0 and consider the sequence $(f_n(x_0))_{n\in\mathbb{N}}$. Since this sequence lies in the compact set X, it has a convergent subsequence, denote it $(f_{n_{k,0}}(x_0))_{k\in\mathbb{N}}$. Here, $n_{0,0} < n_{1,0} < n_{2,0} < \cdots < n_{k,0} < \cdots$*

Now, start with the sequence $(f_{n_{k,0}}(x_1))_{k\in\mathbb{N}}$. As this sequence lies in the compact m.s X, it has a convergent subsequence, denote it $(f_{n_{k,1}}(x_1))_{k\in\mathbb{N}}$. Here, $n_{0,1} < n_{1,1} < \cdots < n_{k,1} < \cdots$

Moreover, $\{n_{k,1} : k \in \mathbb{N}\} \subseteq \{n_{k,0} : k \in \mathbb{N}\}$

Next, start with the sequence $(f_{n_{k,0}}(x_2))_{k\in\mathbb{N}}$. For the same reason this sequence has a convergent subsequence, denote it $(f_{n_{k,2}}(x_2))_{k\in\mathbb{N}}$. Here, $n_{0,2} < n_{1,2} < n_{2,2} < \cdots < n_{k,2} < \cdots$ and $\{n_{k,2} : k \in \mathbb{N}\} \subseteq \{n_{k,1} : k \in \mathbb{N}\} \subseteq \{n_{k,0} : k \in \mathbb{N}\} \subseteq \cdots$

In this way we get subsequences $f_{n_{k,p}}$ of f_n such that $(f_{n_{k,p}}(x_p))_{k\in\mathbb{N}}$ converges and

$$\cdots \subseteq \left\{f_{n_{k,p}} : k \in \mathbb{N}\right\} \subseteq \left\{f_{n_{k,p-1}} : k \in \mathbb{N}\right\} \subseteq \cdots \subseteq \left\{f_{n_{k,o}} : k \in \mathbb{N}\right\}.$$

It follows that, for $i \le p$, $(f_{n_{k,p}}(x_i))_{k\in\mathbb{N}}$ converges since this is a subsequence of the convergent subsequence $(f_{n_{k,i}}(x_i))_{k\in\mathbb{N}}$.

Now, let $(f_{n_{k,k}})_{k\in\mathbb{N}}$ be the diagonal of this infinite matrix:

$$\begin{pmatrix} f_{n_{0,0}} & f_{n_{1,0}} & \cdots & f_{n_{k,0}} & \cdots \\ f_{n_{0,1}} & f_{n_{1,1}} & & f_{n_{k,1}} & \cdots \\ \vdots & & \ddots & \vdots & \cdots \\ f_{n_{0,p}} & & \cdots & f_{n_{k,p}} & \cdots \\ \vdots & & \vdots & \vdots & \vdots \end{pmatrix}$$

Then for each $x_i \in E$, except for first i terms, $(f_{n_{k,k}}(x_i))_{k\in\mathbb{N}}$ is a subsequence of $(f_{n_{k,i}}(x_i))_{k\in\mathbb{N}}$. So, $(f_{n_{k,k}}(x_i))_{k\in\mathbb{N}}$ converges.

12.1.2 Pointwise and Uniformly Bounded Sets of Functions

Definition 12.1.3 *Let $A \subseteq X$ be any set and H be a set of functions $f : A \to \mathbb{R}$. If, for each $x_0 \in A$ given, the set $\{f(x_0) : f \in H\}$ is bounded then we say that H is **pointwise bounded** on A. If, $\{f(x) : x \in A,\ f \in H\}$ is bounded, then we say that H is **uniformly bounded** on A.*

Thus, H is pointwise bounded on $A \Leftrightarrow \forall x \in A,\ \exists M_x > 0 : \forall f \in H,\ |f(x)| < M_x$.

H is uniformly bounded on $A \Leftrightarrow \exists M > 0,\ \forall x \in A,\ \forall f \in H,\ |f(x)| < M$.

Example 12.1.4 *Let $H = \{f_n : n \in \mathbb{N}\}$, $f_n(x) = \dfrac{nx^2}{1+nx}$, $0 \le x \le \infty$. For any $x_0 \in [0,\infty[$ fixed, $|f_n(x)| \le x_0 \Longrightarrow H$ is pointwise bounded.*

But, $\sup_{n\in\mathbb{N},\, x\in[0,\infty[} |f_n(x)| = \infty \Longrightarrow H$ is not uniformly bounded.

Lemma 12.1.5 *Let E be any set and $\mathbb{B}(E)$ be the space of bounded functions $f : E \to \mathbb{R}$, with the supremum metric $d_\infty(f,g) = \sup_{x\in E} |f(x) - g(x)|$. Let $f_n, f \in \mathbb{B}(E)$ be such that $d_\infty(f_n, f) \to 0$ (i.e. $f_n \to f$ uniformly on E). Then the set $H = \{f_n : n \in \mathbb{N}\}$ is uniformly bounded on E.*

⋆ So "uniform boundedness" is a necessary condition for uniform convergence, but it is far away from being sufficient. E.g. $f_n(x) = \sin(nx)$ is uniformly bounded on $\mathbb{R}$, but it does not even converge pointwise.

12.1.3 Equi-continuity of a set of Functions

Let $A \subseteq X$, $x \in A$ a point and $f : A \to \mathbb{R}$ be a function. In the definition of continuity of f at x we have: $\forall \varepsilon > 0$, $\exists \eta > 0 : \begin{cases} \forall y \in A \\ d(x,y) < \eta \end{cases} \Longrightarrow |f(x) - f(y)| < \varepsilon.$

In this definition, η depends not only on x and ε, but also on f.

• Now, suppose that we have finitely many functions. $f_1, f_2, \ldots, f_n : A \to \mathbb{R}$ all of them continuous at x_i. So, $\forall \varepsilon > 0$, $\exists \eta_i > 0 : \begin{cases} \forall y \in A \\ d(x,y) < \eta_i \end{cases} \Longrightarrow |f_i(x) - f_i(y)| < \varepsilon.$

If $\eta = \inf\{\eta_1, \ldots, \eta_n\}$, then $\eta > 0$ and $\begin{cases} \forall y \in A \\ d(x,y) < \eta \end{cases} \Longrightarrow |f_i(x) - f_i(y)| < \varepsilon$, $\forall i = 1, 2, \ldots, n$.

• Now, suppose that we have infinitely many $f_n : A \to \mathbb{R}$, each of them are continuous at x_i. Hence, $\forall \varepsilon > 0$, $\exists \eta_i > 0 : \begin{cases} \forall y \in A \\ d(x,y) < \eta_n \end{cases} \Longrightarrow |f_n(x) - f_n(y)| < \varepsilon.$

Here $\inf\{\eta_n : n \in \mathbb{N}\}$ might be zero!

Question: Is there an $\eta > 0$ that works for all f_n?

Example 12.1.6 *Let $f_n : \mathbb{R} \to \mathbb{R}$, $f_n(x) = \cos nx$, $x = 0$. Is there an $\eta > 0$ such that $|y - 0| < \eta \Longrightarrow |f_n(x) - f_n(y)| < \varepsilon$, $\forall n \in \mathbb{N}$?*
Let us fix $\varepsilon = \frac{1}{2}$. Is there an $\eta > 0$ such that $|y| < \eta \Longrightarrow |\cos ny - 1| < \frac{1}{2}$ for all $n \in \mathbb{N}$?
Let n be large enough to have $\frac{\pi}{n} < \eta$. Then, with $y = \frac{\pi}{n}$, $|\cos ny - 1| = |-2| = 2 > \frac{1}{2}$.
Hence, there is no $\eta > 0$ satisfying $|y| < \eta \Longrightarrow |f_n(x) - f_n(y)| < \varepsilon$.

Definition 12.1.7 *Let H be a set of functions $f : A \to \mathbb{R}$ and $x_0 \in A$ a point. We say that H is **equi-continuous** at x_0 if we have:*

$$\forall \varepsilon > 0,\ \exists \eta > 0 : \begin{cases} \forall y \in A \\ d(x_0,y) < \eta \end{cases} \Longrightarrow |f(x_0) - f(y)| < \varepsilon,\ \forall f \in H.$$

or equivalently; $\forall \varepsilon > 0$, $\exists \eta > 0 : \begin{cases} \forall y \in A \\ d(x_0,y) < \eta \end{cases} \Longrightarrow \sup_{f \in H} |f(x_0) - f(y)| < \varepsilon$. (Here η depends on H but not on any particular $f \in H$)
If H is equi-continuous at every $x_0 \in A$, then we say that H is equi-continuous on A.

Example 12.1.8 *Let $H = \{f : [a,b] \to \mathbb{R} : |f(x) - f(y)| \leq k|x - y|, \forall x, y \in [a,b]\}$, $k > 0$ a fixed number (does not depend on f). Then H is equi-continuous on $[a,b]$.*
Indeed, $\forall x \in [a,b]$, $\forall \varepsilon > 0$, let $\eta = \frac{\varepsilon}{k+1}$, $\forall y \in [a,b]$
Then, $|x - y| < \eta \Longrightarrow |f(x) - f(y)| < \varepsilon$, $\forall f \in H$.

Lemma 12.1.9 *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuous functions. Suppose that f_n converges uniformly on $[a,b]$ to some $f : [a,b] \to \mathbb{R}$. Let $H = \{f_n : n \in \mathbb{N}\}$. Then H is equi-continuous on $[a,b]$.*

Proof 12.1.10 *Since $f_n \to f$ uniformly on $[a,b]$, we have: $\forall \varepsilon > 0$, $\exists N \in \mathbb{N} : \forall n \geq N$* $\sup_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon$*. Now, let $x_0 \in [a,b]$ be any point. Then*

$$\exists \eta > 0 : \begin{cases} \forall y \in [a,b] \\ |y - x_0| < \eta \end{cases} \Longrightarrow |f_i(x_0) - f_i(y)| < \varepsilon,\ 0 \leq i \leq N.$$

Then, for this η,

$$|x_0 - y| < \eta \Longrightarrow |f_n(x_0) - f_n(y)| \leq |f_n(x_0) - f(x_0)| + |f(x_0) - f(y)| + |f(y) - f_n(y)| < 3\varepsilon,$$ $\forall n \geq N$.

Hence, $|x_0 - y| < \eta \Longrightarrow \sup_{n\in\mathbb{N}} |f_n(x_0) - f_n(y)| < 3\varepsilon$. So, H is equi-continuous on $[a,b]$.

⋆ So equi-continuous is a necessary condition for uniform convergence.

Lemma 12.1.11 *Fix a point $x \in K$, let $\delta_x : C(K) \to \mathbb{R}$ be the "Dirac function", i.e. $\delta_x(f) = f(x)$. Then δ_x is continuous.*

12.2 Ascoli - Arzela Theorem

Let $H \subseteq C(K)$ be a subset.

Theorem 12.2.1 ***(Main Lemma)*** *Let $D \subseteq K$ be a dense subset of K, f_n be sequence of functions in H. Suppose that:*

1. *H is equi-continuous on K.*
2. *For each $x \in D$, $(f_n)_{n\in\mathbb{N}}$ converges pointwise on D.*

Then $(f_n)_{n\in\mathbb{N}}$ converges uniformly on K.

Proof 12.2.2 *Let us see that $(f_n)_{n\in\mathbb{N}}$ is uniformly Cauchy on K. Let us first write what we have:*

$$\forall x \in K,\ \forall \varepsilon > 0,\ \exists \eta_x > 0 \text{ such that } \begin{cases} \forall y \in K \\ d(x,y) < \eta_x \end{cases} \Longrightarrow \sup_{n\in\mathbb{N}} |f_n(x) - f_n(y)| < \varepsilon. \quad (1)$$

$$\forall x_0 \in D,\ \forall \varepsilon > 0,\ \exists N_{x_0} \in N \text{ such that } \forall n,m \geq N_{x_0},\ |f_n(x_0) - f_m(x_0)| < \varepsilon. \quad (2)$$

As, $K \subseteq \cup_{x\in K} B_{\eta_x}(x)$ and K is compact, there exists $x_1, x_2, \ldots, x_p \in K$ such that

$$K \subseteq B_{\eta_{x_1}}(x_1) \cup B_{\eta_{x_2}}(x_2) \cup \cdots \cup B_{\eta_{x_p}}(x_p), \text{ for some } p \in \mathbb{Z}^+.$$

As, $\overline{D} = K$, $B_{\eta_{x_i}}(x_i) \cap D \neq \emptyset$. Let $\tilde{x}_i \in B_{\eta_{x_i}}(x_i) \cap D$. Let $N = \sup\left\{N_{x_1}, \ldots, N_{x_p}\right\}$, so that

$$\forall n,m \geq N,\ |f_n(\tilde{x}_i) - f_m(\tilde{x}_i)| < \varepsilon. \quad (3)$$

Let $n, m \geq N$ and $x \in K$ be arbitrary. (So N does not depend on x).

Since, $K \subseteq B_{\eta_{x_1}}(x_1) \cup B_{\eta_{x_2}}(x_2) \cup \cdots \cup B_{\eta_{x_p}}(x_p)$, $x \in B_{\eta_{x_i}}(x_i)$ *for some* $i \in \{1, \ldots, p\}$.
Then, $|f_n(x) - f_m(x)| \leq |f_n(x) - f_n(x_i)| + |f_n(x_i) - f_m(x_i)| + |f_m(x_i) - f_m(x)| < 3\varepsilon$, *by* $(1), (2), (3)$ *above.*

Hence, $(f_n)_{n\in\mathbb{N}}$ *is uniformly cauchy on* K*. So, converges uniformly on* K *to some continuous function* $f : K \to \mathbb{R}$*.*

Lemma 12.2.3 *A pointwise bounded, equi-continuous set* $H \subseteq C(K)$ *is uniformly bounded.*

Proof 12.2.4 *Let* $H \subseteq C(K)$ *be equi-continuous on* K *and be pointwise bounded. So,*

$\forall x \in K$, $\forall \varepsilon > 0$, $\exists \eta_x > 0$ *such that* $\begin{cases} \forall y \in K \\ d(x,y) < \eta \end{cases} \Longrightarrow \sup_{n\in\mathbb{N}} |f_n(x) - f_n(y)| < \varepsilon.$ (4)

$\forall x \in K$, $\exists M_x > 0$ *such that* $\sup_{f\in H} |f(x)| \leq M_x$. (5)

From (4) *we get that* $K \subseteq \cup_{x\in K} B_{\eta_x}(x)$*. As* K *is compact,*

$K \subseteq B_{\eta_{x_1}}(x_1) \cup B_{\eta_{x_2}}(x_2) \cup \cdots \cup B_{\eta_{x_p}}(x_p)$ *for some* $i \in \{1, \ldots, p\}$*.*

Then for any $x \in K$*, (so* $x \in B_{\eta_{x_i}}(x_i)$*), for* $M = \sup_{1\leq i\leq p} M_{x_i}$ *we have:*

$|f(x)| = |f(x) - f(x_i) + f(x_i)| \leq |f(x_i) - f(x)| + |f(x_i)| \leq M_{x_i} + \varepsilon \leq M + \varepsilon.$

Hence, $\sup_{f\in H} \sup_{x\in K} |f(x)| \leq \tilde{M}$*. (*$\tilde{M} \geq M + \varepsilon$*). So,* H *is uniformly bounded on* K*.*

Recall: Let (Y, d) be a complete metric space and $H \subseteq Y$ a set. Then;

H is compact $\Longleftrightarrow \begin{cases} H \text{ is totally bounded} \\ H \text{ is closed} \end{cases}$

Theorem 12.2.5 ***(Ascoli - Arzela)*** *Let* H *be a subset of* $C(K)$*. Then,*

H *is compact* $\Longleftrightarrow \begin{cases} H \text{ is equi-continuous on } K \\ H \text{ is pointwise bounded on } K \\ H \text{ is closed in } C(K) \end{cases}$

Proof 12.2.6 $(\Rightarrow)$ *Assume that* H *is compact. So, it is closed and totally bounded.*

Hence, $\forall \varepsilon > 0$, $\exists f_1, \ldots, f_p \in H : H \subseteq B_\varepsilon(f_1) \cup \cdots \cup B_\varepsilon(f_p)$*. Since any finite set of continuous functions is equi-continuous,*

$\forall x \in K$, $\exists \eta_x > 0$ *such that* $\begin{cases} \forall y \in K \\ d(x,y) < \eta_x \end{cases} \Longrightarrow \sup_{1\leq i\leq p} |f_i(x) - f_i(y)| < \varepsilon.$ (6)

Now, let f *be arbitrary, say* $f \in B_\varepsilon(f_i)$*. So,* $\sup_{y\in K} |f(y) - f_i(y)| < \varepsilon$. (7)

So, $\begin{cases} \forall y \in K \\ d(x,y) < \eta_x \end{cases} \Longrightarrow |f(x) - f(y)| \leq |f(x) - f_i(x)| + |f_i(x) - f_i(y)| + |f_i(y) - f(y)| < 3\varepsilon.$

Hence, H *is equi-continuous on* K*.*

Next, for $x \in K$ *fixed, let* $\delta_x : C(K) \to \mathbb{R}$ *be the mapping defined by* $\delta_x(f) = f(x)$*. This* δ_x *is a continuous mapping. Hence,* $\delta_x(H)$ *is compact in* $\mathbb{R}$*, so bounded. But,*

$\delta_x(H) = \{f(x) : f \in H\}$*. So* H *is pointwise bounded on* K*.*

($\Leftarrow$) *Conversely, suppose that the 3 conditions of the theorem are satisfied. As, K is a compact metric space and every compact metric space is separable, K has a countable dense subset $\epsilon = \{x_0, \ldots x_n, \ldots\}$ with $\bar{\epsilon} = K$. Also, H is equi-continuous and pointwise bounded on K. By lemma 12.2.3 H is uniformly bounded, i.e. $\exists M > 0$ such that $\forall x \in K$, $\forall f \in H$, $f(x) \in [-M, M]$.*

Let now $(f_n)_{n\in\mathbb{N}}$ be a sequence in H. $(f_n)_{n\in\mathbb{N}}$ has a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ that converges pointwise on ϵ. As, $(f_{n_k})_{k\in\mathbb{N}} \in H$ by lemma 12.2.1 $(f_{n_k})_{k\in\mathbb{N}}$ converges uniformly to some $f \in H$ since H is closed. Thus, every sequence f_n in H has a convergent subsequence.

Hence, H is compact.

Theorem 12.2.7 *Let $f_n : [a, b] \to \mathbb{R}$ be sequence of functions such that f_n' exists and $|f_n'(x)| \leq M$, $\forall x[a, b]$. Also assume that $(f_n(a))_{n\in\mathbb{N}}$ is bounded. Then for $x, y \in [a, b]$, $|f_n(x) - f_n(y)| \leq f'(c)|x - y| \leq M|x - y|$, for some $c \in]x, y[$. Hence, $(f_n)_{n\in\mathbb{N}}$ is equi-continuous on $[a, b]$.*

Also, $|f_n(x)| = |f_n(x) - f_n(a) + f_n(a)| \leq |f_n(x) - f_n(a)| + |f_n(a)| \leq M(b-a) + |f_n(a)| \leq \tilde{M}$. Hence, $(f_n)_{n\in\mathbb{N}}$ is pointwise bounded. So $(f_n)_{n\in\mathbb{N}}$ has a uniformly convergent subsequence.

Example 12.2.8 *Let $f_n : [0, 1] \to \mathbb{R}$ be a pointwise convergent sequence and $|f_n'(x)| \leq 1$. Then, $(f_n)_{n\in\mathbb{N}}$ converges uniformly.*

Indeed, let $f_n \to f$ pointwise on $[0, 1]$ So in particular $(f_n(0))_{n\in\mathbb{N}}$ is bounded. Hence, by the preceding example f_n has a subsequence that converges uniformly on $[0, 1]$ to f. This shows that f is the only cluster point of f_n. Since f_n belongs to a compact set $H = \{f_n : n \in \mathbb{N}\}$, $f_n \to f$ uniformly on $[0, 1]$.

12.3 Stone - Weierstrass Theorem

In this section we are looking for dense subsets of $C(K)$. First consider the space $(C([0, 1]), d_\infty)$

• Let A be the set of all the polynomial functions $P : [0, 1] \to \mathbb{R}$. Since every polynomial is continuous, $A \subseteq C([0, 1])$. Properties of A:

1. A is a vector space over $\mathbb{R}$
2. A is a ring
3. A has a unit element 1
4. $\forall x_0 \neq y_0 \quad x_0, y_0 \in [0, 1]$, $\exists P \in A : P(x_0) \neq P(y_0)$.

Weierstrass Approximation Theorem says that $\bar{A} = C([0, 1])$.

• Now, let $B = \left\{ \sum_{k=0}^{n} c_k e^{n_k x} : n_k \in \mathbb{N},\, c_k \in \mathbb{R},\, 0 \leq x \leq 1 \right\}$. Then,

1. B is a vector space over $\mathbb{R}$
2. B is a ring
3. B has a unit element 1
4. $\forall x_0 \neq y_0,\ x_0, y_0 \in [0,1]\ \exists f \in B : f(x_0) \neq f(y_0)$

• Now, let $C = \left\{ (a_0 + \sum_{k=0}^{n} a_k \cos kx + b_k \sin kx) : n, k \in \mathbb{N}, a_k, b_k \in R, 0 \leq x \leq 2\pi \right\}$. Then,

1. C is a vector space over $\mathbb{R}$
2. C is a ring
3. C has a unit element 1
4. $\forall x_0 \neq y_0,\ x_0, y_0 \in [0,1],\ \exists f \in C : f(x_0) \neq f(y_0)$.

• Now, let $\varphi : [0,1] \to \mathbb{R}$ be a continuous, strictly increasing function. $D = \left\{ \sum_{k=0}^{n} a_k \varphi^k(x) : a_k \in \mathbb{R},\ k \in \mathbb{N}\right.$
Then,

1. D is a vector space over $\mathbb{R}$
2. D is a ring
3. D has a unit element 1
4. $\forall x_0 \neq y_0,\ x_0, y_0 \in [0,1],\ \exists f \in D : f(x_0) \neq f(y_0)$

Definition 12.3.1 *A subset A of $C(K)$ is said to be an **algebra** if it is both a vector space and a ring. In other words,*

1. $\forall f, g \in A, \forall d, u \in \mathbb{R} : d \cdot f + u \cdot g \in A.$

2. $\forall f, g \in A,\ f \cdot g \in A.$

*If also $1 \in A$ then we say that A is **unital**.*
*If $\forall x \neq y,\ \exists f \in A : f(x) \neq f(y)$, then we say that A **separates the points of** K.*

In the above examples A, B, C, D are unital subalgebras that separate the points of $[0,1]$.

Lemma 12.3.2 *Let A be a unital subalgebra of $C(K)$. Then for $f \in A$, $|f| \in \bar{A}$, too.*

Proof 12.3.3 *Let $f \in A$. Let $M = \sup_{x\in K} |f(x)|$. Then $\frac{f(x)}{M} \in [-1,1]$. We know that the set of polynomials is dense in $C([-1,1])$.*
Let $h(x) = |x| \in C([-1,1])$. Then, $\forall \varepsilon > 0$, $\exists p_n(x) = a_n x^n + \cdots + a_0$ a polynomial such that $\sup_{|x|<1} |p(x) - |x|| < \varepsilon$.
Then the function $p\left(\frac{f(x)}{M}\right) = a_n \left(\frac{f(x)}{M}\right)^n + \cdots + a_1 \frac{f(x)}{M} + a_0 \in A$ since A is a unital algebra. Then, $\sup_{|x|<1} \left| p\left(\frac{f(x)}{M}\right) - \frac{|f(x)|}{M} \right| < \varepsilon$. Hence, $\frac{|f(x)|}{M} \in \bar{A} \Longrightarrow |f| \in \bar{A}$.

Corollary 12.3.4 *Let A be a unital subalgebra of $C(K)$. Then, for any $f, g \in A : \sup\{f,g\}$ and $\inf\{f,g\}$ are in $\bar{A}$.*

Proof 12.3.5 $\sup\{f,g\} = \frac{|f-g| + f + g}{2}$, $\inf\{f,g\} = \frac{f+g-|f-g|}{2}$. *Hence by lemma 12.3.2* $\sup\{f,g\}$ *and* $\inf\{f,g\}$ *are in* $\bar{A}$.

Lemma 12.3.6 *Let A be a unital subalgebra of $C(K)$ separating the points of K. Then for each $x \neq y$ ($x, y \in K$), for each $\alpha, \beta \in \mathbb{R}$, there is $\varphi_{x,y} \in A$ such that* $\begin{matrix} \varphi_{x,y}(x) = \alpha \\ \varphi_{x,y}(y) = \beta \end{matrix}$

Proof 12.3.7 *Let $x \neq y$ and $\alpha, \beta \in \mathbb{R}$ be given. Since A separates the points of K, there is an $f \in A$ such that $f(x) \neq f(y)$. Let $\varphi_{x,y} = \alpha + (\beta - \alpha)\frac{f - f(x)}{f(y) - f(x)}$. Then, $\varphi_{x,y} \in A$ and*

$$\begin{matrix} \varphi_{x,y}(x) = \alpha \\ \varphi_{x,y}(y) = \beta \end{matrix}$$

Theorem 12.3.8 ***(Stone - Weierstrass)*** *Any unital subalgebra A of $C(K)$ separating the points of K is dense in $C(K)$.*

Proof 12.3.9 *Let A be a unital subalgebra of $C(K)$ separating the points of K.*
Let $f \in C(K)$ and $\varepsilon > 0$ be given. We have to show that there is $g \in A$ such that $\sup_{x\in K} |f(x) - g(x)| < \varepsilon$. *Let $x \in K$ be given. Then for each $y \in K$, there is a $\varphi_{x,y} \in A$ such that $\varphi_{x,y}(x) = f(x)$, $\varphi_{x,y}(y) = f(y)$*
As $\varphi_{x,y}(y) = f(y), \varphi_{x,y}(y) < f(y) + \varepsilon$. Hence, since both $\varphi_{x,y}$ and f are continuous at y, there is $\eta_y > 0$ such that $\forall z \in K \cap B_{\eta_y}(y) : \varphi_{x,y}(z) < f(z) + \varepsilon$. As $K \subseteq \cup_{y\in K} B_{\eta_y}(y)$, $K \subseteq B_{\eta_{y_1}}(y_1) \cup \cdots \cup B_{\eta_{y_k}}(y_k)$, for some $y_1, y_2, \ldots y_k \in K$.
Hence, $\forall z \in B_{\eta_{y_i}}(y_i) \cap K : \varphi_{x,y}(z) < f(z) + \varepsilon$ and $\varphi_{x,y_i}(x) = f(x)$.
Let $\psi_x = \sup\{\varphi_{x,y_1}, \ldots, \varphi_{x,y_k}\}$. Then by corollary 12.3.4, $\psi_x \in A$ and $\forall z \in K$,

$\psi_x(z) < f(z) + \varepsilon$, $\psi_x(x) = f(x)$.

From $\psi_x(x) = f(x) > f(x) - \varepsilon$, by continuity of ψ_x and f, there is an $\eta_x > 0$ such that $\forall z \in B_{\eta_n}(x) \cap K$, $\psi_x(z) > f(z) - \varepsilon$. Hence, from $K \subseteq \cup_{x\in K} B_{\eta_n}(x)$, we get;

$K \subseteq B_{\eta_{x_1}}(x_1) \cup \cdots \cup B_{\eta_{x_p}}(x_p)$.
So, there are $\psi_{x_1}, \ldots, \psi_{x_p} \in \bar{A}$ *such that* $\psi_{x_i}(z) > f(z) - \varepsilon$, $\psi_{x_i}(z) < f(z) + \varepsilon$, $\forall z \in K$.
Let $g = \inf\{\psi_{x_1}, \ldots, \psi_{x_p}\}$. *Then* $g \in \bar{A}$ *and* $\forall z \in K$, $f(z) - \varepsilon < g(z) < f(z) + \varepsilon$, *i.e.* $\sup_{x \in K} |f(x) - g(x)| < \varepsilon$. *This proves that* $\bar{A}$ *is dense in* $C(K)$. *So,* $\bar{A} = C(K)$.

Example 12.3.10 *Let* $A_0 = \left\{ a_0 + \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) : a_k, b_k \in \mathbb{R},\ k \in \mathbb{N} \right\}$. *Consider* A_0 *as a subset of* $C([0, 2\pi])$. *Then* A_0 *is a unital subalgebra of* $C([0, 2\pi])$. A_0 *separates all pair of points in* $[0, 2\pi]$ *except* 0 *and* 2π.
Let $C_*([0, 2\pi]) = \{f \in C([0, 2\pi]) : f(0) = f(2\pi)\} = \{f \in C([0, 2\pi]) : f \text{ is periodic with } 2\pi \text{ period}\}$.
Now, let $K = \{z \in \mathbb{C} : |z| = 1\}$ *and consider* $C(K)$. *Let* $\varphi : [0, 2\pi] \to K$, $\varphi(t) = (\cos t, \sin t)$.
Let $T : C(K) \to C_*([0, 2\pi])$, $T(f) = f \circ \varphi$. *Then* T *is continuous and onto.*
Let $A = \{p(z) : p \text{ is a polynomial}\}$. $A \subseteq C(K)$, A *is a unital subalgebra and separates the points of* K. *Hence,* $\bar{A} = C(K)$.
Now, if we identify $z \in K$, *with* $z = (\cos x, \sin x)$, *then*

$$T(A) = \left\{ a_0 + \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) : a_k, b_k \in \mathbb{R},\ k \in \mathbb{N} \right\} = A_0.$$

Hence, since A *is dense in* $C(K)$ *and* T *is onto,* A_0 *is dense in* $C_*([0, 2\pi])$.
Thus, if $f : [0, 2\pi] \to \mathbb{R}$ *is continuous and* 2π *periodical then there exists a sequence of trigonometric polynomials,* $P_n(x) = a_0 + \sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx)$ *that* ***converges uniformly*** *to* f.

Theorem 12.3.11 *The metric space* $(C(K), d_\infty)$ *is separable.*

Proof 12.3.12 *We know that:*

1. *Every compact m.s is separable.*
2. *Every separable m.s is second countable, i.e. there exists a sequence* $(\beta_n)_{n \in \mathbb{N}}$ *of open sets such that any other open set is a union of some of these* β_n*'s.*

So, let $(\beta_n)_{n \in \mathbb{N}}$ *be such a sequence for the m.s* (K, d). *For each* $n \in \mathbb{N}$ *let* $f_n(x) = d(x, \beta_n^c)$. $(x \in K$, $\beta_n \subseteq K$, $\beta_n^c = K \backslash \beta_n)$ *Then,* f_n *is continuous and for any* $x \neq y$ $(x, y \in K)$ *for some* $n \in \mathbb{N}$, $(f_n(x) \neq f_n y)$. *Let* $E = \{1, f_0, f_1, \ldots, f_n, \ldots\}$ *and* A *be the subalgebra generated by* E.
A typical element of A *is of the form:* $g = \sum a_{n,p,q,\ldots,r} f_1^n, f_2^p, \ldots, f_k^r$. *Then by Stone - Weierstrass Theorem* A *is dense in* $C(K)$. *Let,* B *be the set of the same type of elements as in* A *but with rational coefficients. Then,* B *is dense in* A. *So,* B *is dense in* $C(K)$. *As* B *is countable,* $C(K)$ *is separable.*

Warning: If K is topologically compact but not metric, then $C(K)$ is in general is not separable.

12.4 Exercises

1. Let (K, d) be a compact metric space and $C(K) = \{f : K \to \mathbb{R}\text{: } f \text{ is continuous }\}$

   (a) Let $(f_n)_{n\in\mathbb{N}}$ be a uniformly convergent sequence in $C(K)$. Show that the set $H = \{f_n : n \in \mathbb{N}\}$ is uniformly bounded and equi-continuous at each $x \in K$.

   (b) Let H be an equi-continuous subset of $C(K)$. Show that $\overline{H}$ is also equi-continuous on K.

   (c) Let $x_0 \in K$, $H \subseteq C(K)$ be equi-continuous on K and $H(x_0) = \{f(x_0) : f \in H\}$ be bounded. If K is connected, show that then H is uniformly bounded.

   (d) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C(K)$, $x_0 \in K$ and $x_n \to x_0$. If $\{f_n : n \in \mathbb{N}\}$ is a equi-continuous at x_0 and $f_n(x_0) \to f(x_0)$. Show then $f_n(x_n) \to f(x_0)$.

2. Let $H = \{f \in C[0, 1] : f' \text{ exists and } |f(x)| + |f'(x)| \leq 1\ \forall\ x \in [0, 1]\}$. Show that every sequence $(f)_{n\in\mathbb{N}}$ in H has a uniformly convergent subsequence.

3. Let $H \subseteq C(K)$, $E \subseteq K$ be dense in K and H be equi-continuous at each $x \in E$. Show that

   (a) $\forall\, \eta > 0\ K \subseteq \cup_{x\in E} B_\eta(x)$

   (b) H is also equi-continuous on K.

4. Let (K, d) be a compact metric space. For a subset F of K, let $I(F) = \{f \in C(K) : f(F) = \{0\}\}$. Show that;

   (a) $I(F)$ is a closed ideal of $C(K)$

   (b) $I(F) = I(\overline{F})$

   (c) $F_1 \subseteq F_2 \to I(F_2) \subseteq I(F_1)$.

5. Let I be a closed ideal of $C(K)$ and $F = \{x \in K : \forall\, f \in I,\ f(x) = 0\}$, i.e. $F = \cap_{f\in I} f^{-1}(\{0\})$. Show that F is closed and $I \subseteq C(K)$ iff $\cap_{f\in I} f^{-1}(\{0\}) = \emptyset$

6. Let I be a closed ideal of $C(K)$. Show that $I = C(K)$ iff $\cap_{f\in I} f^{-1}(\{0\}) = \emptyset$.

7. Let A be a subalgebra of $C(K)$. Show that if $\overset{\circ}{A} \neq \emptyset$ then $A = C(K)$

8. Let $f \in C[0, 1]$. If, for each $n \in \mathbb{N}$, $\int_0^1 x^n f(x)dx = 0$ then $f \equiv 0$ on $[0, 1]$.

9. Let $C_*[0, 2\pi] = \{f \in C(K) : f(0) = f(2\pi)\}$ and $A = \{\sum_{k=0}^{n} (a_k \cos kx + b_k \sin kx) : x \in [0, 2\pi],\ n \in \mathbb{N},\ a_k, b_k \in \mathbb{R}\}$ Show that A is a subalgebra of $C_*[0, 2\pi]$ and A is dense in it.

10. Let $f \in C_*[0, 2\pi]$. If $\int_0^1 f(x) \sin nx dx = \int_0^{2\pi} f(x) \cos nx dx = 0$. Show that then $f \equiv 0$ on $[0, 2\pi]$.

11. Let $A = \{\sum_{k=1}^{n} f_k(x) g_k(y) : f_k, g_k \in C[0,1],\, n \in \mathbb{N}\}$. Show that A is dense in $C([0,1] \times [0,1])$.

Chapter 13

Baire Category Theorem

1. **Generalities**
2. **Basic Notations**
3. **Various Forms of the Baire Category Theorem**
4. **Baire's Great Theorem Without Proof**
5. **Applications**

13.1 Generalities

Let (X,d) be a metric space, $f_n : X \to \mathbb{R}$ be sequence of *continuous* functions.
Suppose f_n converges pointwise on K to some function f. We know that f need not be continuous.
What does "f is not continuous" mean? Does it say that f is discontinuous everywhere?
Questions:

1. How discontinuous is this f?
2. Given $g : X \to \mathbb{R}$ how to recognize that g is the pointwise limit of a sequence of continuous functions $(g_n)_{n\in\mathbb{N}}$?

Example 13.1.1 *Let* $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$
***Question**: Is there a sequence* $f_n : \mathbb{R} \to \mathbb{R}$ *of continuous functions such that* $\forall x \in \mathbb{R}$, $f(x) = \lim_{n\to\infty} f_n(x)$*? (not possible)*

Example 13.1.2 *Let* $f : \mathbb{R} \to \mathbb{R}$ *be the derivative of some* $g : \mathbb{R} \to \mathbb{R}$*. Then,* $\forall x \in \mathbb{R}$, $f(x) = \lim_{n\to\infty} \dfrac{g\left(x+\frac{1}{n}\right) - g(x)}{\frac{1}{n}} = f_n(x)$*, i.e.* $\lim_{n\to\infty} f_n(x) = f(x)$*. (each* f_n *is continuous)*

Definition 13.1.3 *Let $B_1(x)$ be the set of the functions $f : X \to \mathbb{R}$, such that $f(x) = \lim_{n\to\infty} f_n(x)$, $\forall x \in X$ for some sequence of continuous functions $f_n : X \to \mathbb{R}$. The functions that belong to $B_1(x)$ are said to be* ***Baire-1 functions****.*
Similarly we define $B_2(x) = \{g : X \to \mathbb{R} : g(x) = \lim_{n\to\infty} f_n(x)$ for some $f_n \in B_1(x)\}$.
$B_3(x) = \{h : X \to \mathbb{R} : h(x) = \lim_{n\to\infty} f_n(x)$ for some $g_n \in B_2(x)\}$.
$\vdots$

So, the above questions become:

1. How discontinuous a Baire-1 function may be?
2. Given a function $f : X \to \mathbb{R}$, how to recognize it is a Baire-1 function or not?

Thus, our task in this chapter is to characterize Baire-1 functions.

13.2 Basic Notations

13.2.1 G_δ-sets, F_σ-sets

Let (X, d) be a metric space.

Definition 13.2.1 *A set $A \subseteq X$ is said to be a G_δ**-set**, if it is possible to present A as an intersection countably many open sets O_n, i.e. $A = \cap_{n\in\mathbb{N}} O_n$.*
*A set $B \subseteq X$ is said to be a F_σ**-set** if it is possible to present B as a union of countably many closed sets F_n. i.e., $B = \cup_{n\in\mathbb{N}} F_n$.*

Example 13.2.2
1. *In any m.s (X, d), every open set is a G_δ-set and every closed set is a F_σ-set.*
2. *A is a G_δ-set $\iff$ A^c is a F_σ-set.*
3. *Let $X = \mathbb{R}$. Then,* $\begin{array}{l} [a,b] = \cap_{n\geq 1} \left]a - \frac{1}{n}, a + \frac{1}{n}\right[\\ \{a\} = \cap_{n\geq 1} \left]a - \frac{1}{n}, a + \frac{1}{n}\right[\\ ]a,b[= \cup_{n\geq 1} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] \end{array}$.

Question $\mathbb{Q}$ is a F_σ-set. Is $\mathbb{Q}$ a G_δ-set?

13.2.2 Nowhere Dense Sets

Let (X, d) be a metric space.

Definition 13.2.3 *A set $A \subseteq X$ is said to be* ***nowhere dense*** *if $\overset{\circ}{(\bar{A})} = \emptyset$.*

Example 13.2.4 *In $\mathbb{R}$, $A = \mathbb{N}$, $A = \mathbb{Z}$ are nowhere dense. But $\mathbb{Q}$ is not nowhere dense.*

Example 13.2.5 *Let $X = \mathbb{R}^2$, $A = \mathbb{R} \times \{0\}$. Then A is closed in $\mathbb{R}^2$ and $\overset{\circ}{A} = \emptyset$. So, $\mathbb{R} \times \{0\}$ is nowhere dense in $\mathbb{R}^2$.*

Example 13.2.6 *A closed set $F \subseteq X$ is nowhere dense iff $\overline{X \setminus F} = X$. Recall for any $A \subseteq X$, $\left(\overset{\circ}{A}\right)^c = \overline{(A^c)}$ and $\left(\overline{A}\right)^c = (\overset{\circ}{A^c})$*

Example 13.2.7 *For any closed set $F \subseteq X$, $A = \partial F = F \setminus \overset{\circ}{F}$ is nowhere dense.*
For any open set $B \subseteq X$, $A = \partial B = \overline{B} \setminus B$ is nowhere dense.

Example 13.2.8 *The union of finitely many nowhere dense sets is nowhere dense. A subset of a nowhere dense set is nowhere dense.*

13.2.3 First and Second Category Sets, Residual Sets

Let (X, d) be a metric space and $M \subseteq X$ a set.

Definition 13.2.9 *We say that "M **is of the first category in** X" if it is possible to present M as a <u>countable</u> union of nowhere dense sets.*

Example 13.2.10 *In $\mathbb{R}$, $\mathbb{Q}$ is of the first category. $\mathbb{Q} = \cup_{k=1}^{\infty} A_k$, $A_k = \frac{1}{k}\mathbb{Z}$. Then A_k is closed and $\overset{\circ}{A_k} = \emptyset$.*
In $\mathbb{R}$, any countable set is of the first category.

Warning To be of the "first category" is a relative notion. e.g. $\mathbb{N}$ as a subset of $\mathbb{R}$ is of the first category, but if we consider $\mathbb{N}$ as a metric space of its own with the metric $d(n, m) = |n - m|$, then $\mathbb{N}$ is discrete and $\mathbb{N}$ is not of the first category in itself.

Example 13.2.11 *If $M_0, M_1, \dots M_n, \dots \subseteq X$ are of the first category in X, then the union $M = \cup_{n \in \mathbb{N}} M_n$ is also of the first category in X. Indeed, if $M_n = \cup_{k \in \mathbb{N}} A_{n,k}$ with $\overset{\circ}{\overline{A_{n,k}}} = \emptyset$, then $M = \cup_{(n,k) \in \mathbb{N} \times \mathbb{N}} A_{n,k}$ and $\overset{\circ}{\overline{A_{n,k}}} = \emptyset$, $\forall (n, k) \in \mathbb{N} \times \mathbb{N}$.*

Definition 13.2.12 *A subset M of a metric space (X, d) is said to be of the **second category in** X, if M is not of the first category in X. Thus, a subset M of X is either of the first category or of the second category in X.*

Definition 13.2.13 *A subset M of a metric space (X, d) is said to be a **residual set** iff $X \setminus M$ is of the first category in X.*

13.3 Various Forms of the Baire Category Theorem

Theorem 13.3.1 *A complete metric space (X, d) is of the second category in X, i.e. if we write X as $\cup_{n\in\mathbb{N}} A_n$ for some sets A_n, then for at least one $n \in \mathbb{N}$, $\overset{\circ}{\overline{A_n}} \neq \emptyset$.*

Proof 13.3.2 *Suppose that for some sets $A_n \subseteq X$, we have $\cup_{n\in\mathbb{N}} A_n = X$ and $\forall n \in \mathbb{N}$ $\overset{\circ}{\overline{A_n}} = \emptyset$. Since $\cup_{n\in\mathbb{N}} A_n = X$, $\cup_{n\in\mathbb{N}} \overline{A_n} = X$, too. Then, since $\overset{\circ}{\overline{A_n}} = \emptyset$, the open set $O_n = X \setminus \overline{A_n}$ is dense in X. So, it is enough to prove the following lemma.*

Lemma 13.3.3 *Let (X, d) be a complete m.s and O_n's are open, dense sets. Then the set $D = \cap_{n\in\mathbb{N}} O_n$ is a dense G_δ-set.*

Proof 13.3.4 *We have to prove that $\forall x \in X$, $\forall \varepsilon > 0 : B_\varepsilon(x) \cap D \neq \emptyset$.*
Fix an $x \in X$ and an $\varepsilon > 0$. As $\overline{O_0} = X$, $B_\varepsilon(x) \cap O_0 \neq \emptyset$. As, $B_\varepsilon(x) \cap O_0$ is open, for any $x_0 \in B_\varepsilon(x) \cap O_0$ there is $\varepsilon_0 > 0$ such that $B'_{\varepsilon_0}(x_0) \subseteq B_\varepsilon(x) \cap O_0$. We can and do assume that $\varepsilon_0 < \dfrac{\varepsilon}{2}$.
Since $\overline{O_1} = X$, $B_{\varepsilon_0}(x_0) \cap O_1 \neq \emptyset$. Hence for any $x_1 \in B_{\varepsilon_0}(x_0) \cap O_1$, there is $\varepsilon_1 < \dfrac{\varepsilon_0}{2}$ such that $B'_{\varepsilon_1}(x_1) \subseteq B_{\varepsilon_0}(x_0) \cap O_1$.
As $\overline{O_2} = X$, $B_{\varepsilon_0}(x_0) \cap O_2 \neq \emptyset$. Hence for any $x_2 \in B_{\varepsilon_0}(x_0) \cap O_2$, there is $\varepsilon_2 < \dfrac{\varepsilon_1}{2}$ such that $B'_{\varepsilon_2}(x_2) \subseteq B_{\varepsilon_1}(x_1) \cap O_2, \ldots$ etc.
In this way we get sets $B_{\varepsilon_n}(x_n)$ such that
i) $B_\varepsilon(x) \supseteq B'_{\varepsilon_0}(x_0) \supseteq \cdots \supseteq B'_{\varepsilon_n}(x_n) \supseteq \cdots$ and $\varepsilon_n < \dfrac{\varepsilon}{2^n} \to 0$.
ii) $B'_{\varepsilon_n}(x_n) \subseteq B'_{\varepsilon_{n-1}}(x_{n-1}) \cap O_n$.
Hence, (X, d) being complete, by "Cantor's Nested Interval" theorem:
$\cap_{k\in\mathbb{N}} B'_{\varepsilon_k}(x_k) \neq \emptyset$.
Let $y \in \cap_{k\in\mathbb{N}} B'_{\varepsilon_k}(x_k)$ be any point. Then by (ii), $y \in O_n$, $\forall n \in \mathbb{N}$ and by (i) $y \in B_\varepsilon(x)$. Then, $y \in B_\varepsilon(x) \cap D$. Hence, $B_\varepsilon(x) \cap D \neq \emptyset$ and D is dense in X.

Example 13.3.5 *The set $M = \mathbb{R} \setminus \mathbb{Q}$ is of the second category in $\mathbb{R}$.*
Indeed, we know that:
i) $\mathbb{Q}$ is of the first category in $\mathbb{R}$.
ii) The union of two first category sets is of the first category.
So, if $\mathbb{R} \setminus \mathbb{Q}$ was of the first category in $\mathbb{R}$, then since $\mathbb{R} = \mathbb{Q} \cup (\mathbb{R} \setminus \mathbb{Q})$, $\mathbb{Q}$ would also be of the first category in $\mathbb{R}$.

Example 13.3.6 *The space $(C(K), d_\infty)$ is of the second category in itself.*

Another form of theorem 13.3.1 is this:

Theorem 13.3.7 *Let (X,d) be a complete m.s. If for some closed sets F_n's, we have $X = \cup_{n\in\mathbb{N}} F_n$, then the set $\cup_{n\in\mathbb{N}} (\overset{\circ}{F_n})$ is dense in X.*

Proof 13.3.8 *Let $A_n = \partial F_n = F_n \setminus \overset{\circ}{F_n}$. We know that A_n is closed and $\overset{\circ}{A_n} = \emptyset$. Hence, $O_n = X \setminus A_n$ is a dense open set. So, by lemma 13.3.3 $\cap_{n\in\mathbb{N}} O_n$ is dense in X.*

Let us see that $\cap_{n\in\mathbb{N}} O_n \subseteq \cup_{n\in\mathbb{N}} (\overset{\circ}{F_n})$. Let $x \in \cap_{n\in\mathbb{N}} O_n$ be any point. Hence, $x \in O_n$, $\forall n \in \mathbb{N}$. Now, since $X = \cup_{n\in\mathbb{N}} F_n$, $x \in F_p$ for some $p \geq 0$. As, $\left\{\begin{array}{l} x \in O_p \\ x \in F_p \end{array}\right. \Longrightarrow \left\{\begin{array}{l} x \notin A_p \\ x \in F_p \end{array}\right. \Longrightarrow x \in \overset{\circ}{F_p}$.

Hence, $\cap_{n\in\mathbb{N}} O_n \subseteq \cup_{n\in\mathbb{N}} (\overset{\circ}{F_n})$.

These three results (13.3.1,13.3.3,13.3.7) are known as "Baire Category Theorems".

Theorem 13.3.9 *In any complete metric space (X,d) :*

1. *Every residual set is of the second category in X.*
2. *Every dense $G_\delta -$ set is of the second category in X.*

Proof 13.3.10

1. *Let $M \subseteq X$ be a residual set, i.e. $X \setminus M$ is of the first category in X. If M was of the first category in X, $M \cup X \setminus M = X$ would be of the first category in X, too, which is not possible since (X,d) is complete.*
2. *Let M be a dense G_δ-set. Let us see that M is residual, i.e. $X \setminus M$ is of the first category. Now, since M is a G_δ-set, $M = \cap_{n\in\mathbb{N}} O_n$ for some open sets O_n's. Since, $M \subseteq O_n$, each O_n is dense in X.*

   Hence, $F_n = X \setminus O_n$ is a closed nowhere dense set. So, $\cup_{n\in\mathbb{N}} F_n = X \setminus M$ is of the first category in X.

Example 13.3.11

1. *In $\mathbb{R}$, $M = \mathbb{R} \setminus \mathbb{Q}$ is a residual set.*
2. *In $\mathbb{R}$, $\mathbb{Q}$ is NOT a G_δ-set.*
3. *In any complete m.s (X,d), any set $M \subseteq X$ such that $\overset{\circ}{M} \neq \emptyset$ cannot be of the first category in X.*

Proposition 13.3.12 *In a complete metric space (X,d), if M is of the first category, then $\overline{X \setminus M} = X$.*

Proof 13.3.13 *As M is of the first category in X, $M = \cup_{n\in\mathbb{N}} A_n$ with $\overset{\circ}{\overline{A_n}} = \emptyset$, $\forall n \in \mathbb{N}$. Hence, by lemma 13.3.3, with $O_n = X \setminus \overline{A_n}$, $D = \cap_{n\in\mathbb{N}} O_n$ is dense in X. But, $M \cap D = \emptyset$. So, we have $\overset{\circ}{\overline{M}} = \emptyset$.*

Example 13.3.14

1. *In a complete m.s. (X,d), a set $M \subseteq X$ and its complement M^c both may be of the second category in X.*
2. *In a complete m.s. (X,d), any set M containing a second category set is of the second category.*

13.4 A Study of Discontinuous Functions (Baire's Great Theorem)

Let $f : \mathbb{R} \to \mathbb{R}$ be the "Riemann function", i.e. $f(x) = \begin{cases} 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 1 \text{ if } x = 0 \\ \dfrac{1}{n} \text{ if } x \in \mathbb{Q} \end{cases}$

Then f is continuous at every $x \in \mathbb{R} \setminus \mathbb{Q}$ and discontinuous at every $x \in \mathbb{Q}$. Hence, $C_f = \{x \in \mathbb{R} : f \text{ is continuous at } x\} = \mathbb{R} \setminus \mathbb{Q}$, a G_δ-set.

Question: Let (X, d) be a m.s $f : X \to \mathbb{R}$ be an arbitrary function and $C_f = \{x \in X : f \text{ is continuous at } x\}$. What is the structure of C_f ?

Theorem 13.4.1 *For any metric space (X, d), for any $f : X \to \mathbb{R}$, C_f is a $G_\delta -$ set. (may be $\emptyset$)*

Proof 13.4.2 *Let $\tilde{f} : X \to \mathbb{R}$, $\tilde{f}(x) = \arctan(f(x))$. Since, $\arctan : \mathbb{R} \to \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ is a homeomorphism, $\tilde{f}$ is continuous at a point $x \in X \iff f$ is continuous at x. The advantage of $\tilde{f}$ over f is $\tilde{f}$ is bounded. So, if necessary replacing f by $\tilde{f}$, we can assume that f is bounded.*

Definition 13.4.3 *For any set $A \subseteq X$, $A \neq \emptyset$, let $O(f, A) = \sup_{x\in A}(f(x)) - \inf_{x\in A}(f(x))$. This quantity is said to be the* ***oscillation of*** *f* ***on*** *A.*

Clearly, $B \subseteq A \Longrightarrow O(f, B) \leq O(f, A)$.

Fix a point $x_0 \in X$. Let $O(f, x_0) = \lim_{n\to\infty} O(f, B_{\frac{1}{n}}(x_0))$. This limit exists. The number is said to be the ***oscillation of*** *f* ***at*** *x_0.*

Proposition 13.4.4 *$\forall \alpha > 0$, the set $A_\alpha = \{x \in X : O(f, x) < \alpha\}$ is an open set.*

Proof 13.4.5 *Let $x_0 \in A$ be any point. So $O(f, x_0) < \alpha$. Hence, there is $n > 1$ such that $O(f, B_{\frac{1}{n}}(x_0)) < \alpha$. Then, $B_{\frac{1}{n}}(x_0) \subseteq A_\alpha$. Hence, A_α is open.*

Theorem 13.4.6 *The function f is continuous at a point $x_0 \in X$ iff $O(f, x_0) = 0$.*

Conclusion: C_f is a G_δ-set.

$$\begin{aligned} C_f &= \{x \in X : f \text{ is continuous at } x\} \\ &= \{x \in X : O(f, x) = 0\} \\ &= \cap_{n\geq 1} \left\{x \in X : O(f, x) < \tfrac{1}{n}\right\} \end{aligned}$$

Example 13.4.7 • *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$. Then, $C_f = \emptyset$.*

- *If $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$, then $C_f = \mathbb{R}$.*
- *But we know that $\mathbb{Q}$ is not a G_δ-set. So, there is no $f : \mathbb{R} \to \mathbb{R}$ such that $C_f = \mathbb{Q}$.*

Question Given a G_δ-set, $A \subseteq \mathbb{R}$, is there a function such that $C_f = A$?

Theorem 13.4.8 *Let (X, d) be a metric space. Suppose that there exists a set $M \subseteq X$ such that $\overline{M} = \overline{M^c} = X$ (i.e. $X = \mathbb{R}$, $M = \mathbb{Q}$) Let A be any G_δ-set. Then there exists a function $f : X \to \mathbb{R}$ such that $C_f = A$.*

Proof 13.4.9 *Since A is a G_δ-set, Ais of the form: $A = \cap_{n\in\mathbb{N}} O_n$, where O_n's are open. Let $O_0 = X$ and replacing O_n by $O_1 \cap \cdots \cap O_n$ we can assume that $O_0 \supseteq O_1 \supseteq \cdots \supseteq O_n \supseteq \cdots$ so that $\cap_{n\in\mathbb{N}} O_n = A$ and $A \subseteq O_n$, $\forall n \in \mathbb{N}$.*
Then, $X = A \cup (O_0 \setminus O_1) \cup (O_1 \setminus O_2) \cup \cdots \cup (O_n \setminus O_{n+1}) \cup \cdots$ Any two of these sets are disjoint. Then each $x \in X$ belongs to only one of these sets. Now, define a function as follows: Let $x \in X$ be any point.

- *If $x \in A$, let $f(x) = 0$.*
- *If $x \notin A$, then $x \in O_n \setminus O_{n+1}$, for some $n \in \mathbb{N}$. There are cases:*

(i) $x \in M \cap (O_n \setminus O_{n+1})$. In this case put $f(x) = \dfrac{1}{n+1}$.

(ii) $x \in M^c \cap (O_n \setminus O_{n+1})$. In this case put $f(x) = -\dfrac{1}{n+1}$.

Let us see that $C_f = A$. To see this, let first $a \in A$ be a point and $x_n \in X$ be any sequence converging to a.
We have to show that $f(x_n) \to f(a) = 0$. If $x_n \in A$ for all but finitely many $n \in \mathbb{N}$, then $f(x_n) = 0 \to f(a)$.
Otherwise, since $a \in A = \cap_{n\in\mathbb{N}} O_n$, $a \in O_p$, $\forall p \in \mathbb{N}$. As, O_p is open, $x_n \in O_p$, for all but finitely many $n \in \mathbb{N}$.
Hence, each $O_p \setminus O_{p+1}$ contains x_n, for only finitely many $n \in \mathbb{N}$.
Hence, $f(x) = \pm\dfrac{1}{n+1} \to 0 = f(a)$, as $n \to \infty$. Then, $A \subseteq C_f$. Now, suppose $a \notin A$. Then, a belongs to one and only one $O_p \setminus O_{p+1}$. So, $f(a) = \pm\dfrac{1}{p+1}$.
If $a \in (O_p \setminus O_{p+1}) \cap M$, then let $x_n \in M^c$ be such that $x_n \to a$. Then, $f(x_n) \leq 0$. So, $f(x_n) \nrightarrow \dfrac{1}{p+1}$.
If $a \in (O_p \setminus O_{p+1}) \cap M^c$, then $f(a) = -\dfrac{1}{p+1}$.
Then, let $x_n \in M : x_n \to a$. Then $f(x_n) \geq 0$. So, $f(x_n) \nrightarrow -\dfrac{1}{p+1}$. Hence, f is discontinuous at every $a \notin A$. So, $C_f = A$.

13.4.1 Continuity of Baire-1 Functions

Let (X, d) be a metric space. $f_n : X \to \mathbb{R}$ be a sequence of continuous functions. Suppose that, for each $x \in X$, $(f_n(x))_{n\in\mathbb{N}}$ converges to some $f : X \to \mathbb{R}$, i.e. $f_n \to f$ pointwise on X.

Question: What is C_f ?

Let, for $n \in \mathbb{N}$ and $k = 1, 2, \ldots$, $A_n(k) = \sup_{p\in\mathbb{N}} \left\{ x \in X : |f_{n+p}(x) - f(x)| \leq \frac{1}{k} \right\}$.

Theorem 13.4.10 *1.* $C_f = \cap_{k=1}^{\infty} \cup_{n\in\mathbb{N}} \overset{\circ}{A}_n (k)$

2. If (X, d) is complete, then C_f is residual. (i.e., a dense G_δ-set)

Proof 13.4.11 *1. Let* $B = \cap_{k=1}^{\infty} \cup_{n\in\mathbb{N}} \overset{\circ}{A}_n (k)$. *Let $x_0 \in B$ be a point. We want to show that f is continuous at x_0. Let $\varepsilon > 0$ be arbitrary. Then choose $k \geq 1$ such that $\frac{1}{k} < \frac{\varepsilon}{3}$.*

Since, for this k, $x_0 \in \cup_{n\in\mathbb{N}} \overset{\circ}{A}_n (k)$, $x_0 \in \overset{\circ}{A}_{n_0} (k)$, for some $n_0 \in \mathbb{N}$.

Hence, there is $\eta_k > 0$ such that $B_{\eta_k}(x) \subseteq \overset{\circ}{A}_{n_0} (k)$.

So, $\forall x \in B_{\eta_k}(x)$, $|f_{n_0+p}(x) - f(x)| \leq \frac{1}{k}$. In particular, $|f_{n_0+p}(x_0) - f(x_0)| < \frac{1}{k}$.

Now, since $f_n(x_0) \to f(x_0)$, there is $p \in \mathbb{N}$ such that $|f_{n_0+p}(x_0) - f(x_0)| \leq \frac{1}{k}$. (1)

As f_{n_0+p} is continuous at x_0, there is $\eta < \eta_k$ such that $\forall x \in B_\eta(x_0)$:

$|f_{n_0+p}(x) - f_{n_0+p}(x_0)| \leq \frac{1}{k}$. (2)

Then, for $x \in B_\eta(x_0)$,

$|f(x) - f(x_0)| \leq |f(x) - f_{n_0+p}(x)| + |f_{n_0+p}(x) - f_{n_0+p}(x_0)| + |f_{n_0+p}(x_0) - f(x_0)| \leq \frac{3}{k}$.

Hence, $B \subseteq C_f$.

• *Conversely, let $x_0 \in C_f$ and let $k \geq 1$ be any number. Since $f_n(x_0) \to f(x_0)$, for some $n \in \mathbb{N}$, $|f_n(x_0) - f(x_0)| \leq \frac{1}{2k}$. Now the function $g_n = f_n - f$ is continuous at x_0 and $|g_n(x_0)| \leq \frac{1}{2k}$. So, there is $\eta > 0$ such that $\forall x \in B_\eta(x_0)$, $|g_n(x)| \leq \frac{1}{k}$. This says that, $B_\eta(x_0) \subseteq A_n(k)$. So, $x_0 \in \overset{\circ}{A}_n (k) \Longrightarrow x_0 \in \cup_{n\in\mathbb{N}} \overset{\circ}{A}_n (k)$. This being true for any $k \geq 1$, $x_0 \in \cap_{k=1}^{\infty} \cup_{n\in\mathbb{N}} \overset{\circ}{A}_n (k)$. Thus, $C_f \subseteq B$.*

Hence, $B = C_f$.

2. Now, let for all $k \geq 1$, $n \in \mathbb{N}$ $B_n(k) = \left\{ x \in X : \sup_{p\in\mathbb{N}} |f_{n+p}(x) - f(x)| \leq \frac{1}{k} \right\}$

Clearly $B_n(k) \subseteq A_n(k)$.

Moreover, $B_n(k) = \cap_{p\in\mathbb{N}} \left\{ x \in X : |f_{n+p}(x) - f(x)| \leq \frac{1}{k} \right\}$. So, $B_n(k)$ is closed since f_{n+p} and f_n are continuous.

Moreover, since for each $x \in X$, $f_n(x)$ converges, so it is Cauchy, so $x \in B_n(k)$ for some $n \in \mathbb{N}$.

Hence, $\cup_{n\in\mathbb{N}} B_n(k) = X$. By lemma 13.3.3 $\bigcup_{n\in\mathbb{N}} \left(\overset{\circ}{B_n}(k)\right)$ is dense in X.

So, since $\bigcup_{n\in\mathbb{N}} \left(\overset{\circ}{B_n}(k)\right) \subseteq \bigcup_{n\in\mathbb{N}} \overset{\circ}{A_n}(k)$, $O_k = \bigcup_{n\in\mathbb{N}} \overset{\circ}{A_n}(k)$ is open and dense in X.

Hence, $C_f = \cap_{k\geq 1} O_k$ is a dense G_δ-set.

Example 13.4.12 *Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} 1 \text{ if } x \in \mathbb{Q} \\ 0 \text{ if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$*
Is there a sequence of continuous functions $f_n : \mathbb{R} \to \mathbb{R}$ that converges pointwise to f on $\mathbb{R}$? i.e., is f a Baire-1 function?
As, $C_f = \emptyset$, f is not a Baire-1 function. But, $\forall x \in \mathbb{R}$, $f(x) = \lim_{n\to\infty} \lim_{m\to\infty} [\cos(m!x\pi)]^{2n}$. So, f is a Baire-2 function.

Question: Given a function $f : X \to \mathbb{R}$ how to recognize that f is a Baire-1 function?

Theorem 13.4.13 ***(Baire's Great Theorem)*** *Let (X, d) be a complete metric space and $f : X \to \mathbb{R}$ be a given function. Then, f is Baire-1 $\iff \forall F \subseteq X$, closed, the restriction function $f_{|F} : F \to \mathbb{R}$ is continuous at least at one point $x_0 \in F$. (i.e. $\exists f_n : X \to \mathbb{R}$ continuous, $f_n \to f$ pointwise $\iff \forall F \subseteq X$, $\forall x_n \in F$, $x_n \to x_0 \implies f(x_n) \to f(x_0)$)*

13.5 Exercises

Let (X, d) be a metric space.

1. A function $f : X \to \mathbb{R}$ is said to be **lsc(= lower semi-continuous)** at a point $x_0 \in X$ if

   $\forall \varepsilon > 0\ \exists \eta > 0\ \forall x \in B_\eta(x_0),\ f(x) > f(x) - \varepsilon$

   usc(= upper semi-continuous) at x_0 if

   $\forall \varepsilon > 0\ \exists \eta > 0\ \forall x \in B_\eta(x_0),\ f(x) < f(x) + \varepsilon$

   (a) Show that f is usc at x_0 iff $-f$ is lsc at x_0

   (b) f is continuous at x_0 iff both usc and lsc at x_0.

   (c) f is usc at every $x \in X$ iff $\forall\ a \in \mathbb{R}$, the set $f^{-1}(]-\infty, a[)$ is open in X.

   (d) f is lsc at every $x \in X$ iff $\forall\ a \in \mathbb{R}$, the set $f^{-1}(]a, \infty[)$ is open in X.

   (e) Let $A \subseteq X$ and $f = \chi_A$. Then f is lsc iff A is open; f is usc iff A is closed.

2. Let $(f_\alpha)_{\alpha \in I}$, $f_\alpha : X \to \mathbb{R}$ be a family of continuous functions such that, for each $x \in X$, the set $\{f_\alpha(x) : \alpha \in I\}$ is bounded. Let $f(x) = \sup\{f_\alpha(x) : \alpha \in I\}$ and $g(x) = \inf\{f_\alpha(x) : \alpha \in I\}$. Show that f is lsc on X and g is usc on X.

3. Let $f{:}X \to \mathbb{R}$ be a function, $x_0 \in X$ and $g(x) = Arc\tan f(x)$. Show that f is continuous at x_0 iff f is continuous at x_0. Show also that $|g(x)| \leq \frac{\pi}{2}$ for all $x \in X$.

4. Let $f : X \to \mathbb{R}$ be a bounded function. for $x \in X$, $O(f, x) = \inf_{\eta > 0} \sup_{y,z \in B_\eta(x)} |f(y) - f(z)|$ be the oscillation of f at x_0. Show that

   (a) f is continuous at x_0 iff $O(f, x_0) = 0$.

   (b) The function $g(x) = O(f, x_0)$ is usc on X.

   (c) The set $C_f = \{x \in X : O(f, x_0) > 0\}$ is a G_δ-set.

5. Let $K \subseteq X$ be compact and $f : X \to \mathbb{R}$ be a function. Show that

   (a) If f is lsc on K, $f(K)$ is bounded from below and for some $x_0 \in K$, $f(x_0) = \inf_{x \in K} f(x)$.

   (b) If f is usc on K, $f(K)$ is bounded from above and for some $x_0 \in K$, $f(x_0) = \sup_{x \in K} f(x)$.

6. Now let (X, d) is a complete ms.

   (a) Let G_1, G_2 2 dense G_δ-sets. Show that $G_1 \cap G_2$ is also a dense G_δ-subset of X.

(b) Deduce from 6a that if $G \subseteq X$ is a dense G_δ-set then G is of the second category in X.

(c) Show that any set $G \subseteq X$ that contains a second category set M is also of the second category.

(d) If $M \subseteq X$ is of the first category in X, then show that $X \setminus M$ is dense in X.

7. Show that every subset $G \subseteq \mathbb{R}$ which is of the second category in $\mathbb{R}$ is uncountable. Deduce that every dense G_δ-subset is uncountable.

8. Let $f_n : X \to \mathbb{R}$ be a sequence of continuous functions that converges pointwise to a function f on X. Show that on some nonempty open set $B \subseteq X$ f is bounded.

9. Let $X = C([0,1])$, the space of the continuous function $f : [0,1] \to \mathbb{R}$ equipped with the metric $d(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$. Show that the set $M \subseteq X$ of the polynomial functions is of the first category in X. Deduce that $X \setminus M$ is dense in X. What does this mean?

10. Let (X,d) be a complete m.s. $A \subseteq X$ be a dense set and $f : A \to \mathbb{R}$ be a continuous function.

    (a) Show that the set $M = \{a \in X : \lim_{x\to a,\ x\in A} \text{ does not exist}\}$ is of the first category in X.

    (b) Let $G = X \setminus M$. Show that there exists a function $f^* : G \to \mathbb{R}$ extending f and continuous at each $x \in G$.