Analysis Notes
(only a draft, and the first one!)

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June 22, 2004

Contents

Foreword

This is a revolving and continuously changing textbook. The present version is far from complete. It may contain several errors, omissions, oversights etc. Do not circulate and consult it only with great suspicion. The new version can be found at www.alinesin.org.

Chapters 1 through 19 form the first semester of a four semester course.

Chapter 1

Preliminaries

1.1 Binary Operation

Let X be a set. A **binary operation** on X is just a function from $X \times X$ into X. The binary operations are often denoted by such symbols as $+$, $\times$, $\cdot$, $*$, $\circ$ etc. The result of applying the binary relation to the elements x and y of X is denoted as $x + y$, $x \times y$, $x \cdot y$, $x * y$, $x \circ y$ etc.

Examples.

i. Let X be a set and let $c \in X$ be any fixed element. The rule $x * y = c$ defines a binary operation on X. This binary operation satisfies $x*y = y*x$ for all $x, y \in X$ (commutativity) and $x*(y*z) = (x*y)*z$ (associativity).

ii. Let X be a set. The rule $x * y = x$ defines a binary operation on X. Unless $|X| \leq 1$, this binary operation is not commutative. But it is always associative.

iii. Let U be a set. Let $X := \wp(U)$ be the set of subsets of U. The rule $A * B = A \cap B$ defines a binary operation on X. This binary operation is commutative and associative. Note that $A*U = U*A = A$ for all $A \in X$. Such an element is called the **identity element** of the binary operation. The rule $A * B = A \cup B$ defines another binary operation on X, which also commutative and associative. $\emptyset$ is the identity element of this binary operation. Examples i and ii do not have identity elements unless $|X| = 1$.

iv. Let U be a set. Let $X := \wp(U)$ be the set of subsets of U. The rule $A*B = A \setminus B$ defines a binary operation on X, which is neither associative nor commutative in general. It does not have an identity element either, although it has a right identity element, namely $\emptyset$.

v. Let U be a set. Let $X := \wp(U)$ be the set of subsets of U. The rule $A * B = (A \setminus B) \cup (B \setminus A)$ defines a binary operation on X, which is commutative and associative (harder to check) and which has an identity element. Every element in this example has an **inverse element** in the sense that, if e denotes the identity element of X for this operation, then for every $x \in X$ there is a $y \in X$ (namely $y = x$) such that $x*y = y*x = e$.

Exercises.

i. Let A be a set. Let X be the set of functions from A into A. For f, $g \in X$, define the function $f \circ g \in X$ by the rule

$$(f \circ g)(a) = f(g(a))$$

for all $a \in A$. Show that this is a binary operation on X which is associative, noncommutative if $|A| > 1$ and which has an identity element. The identity element (the **identity function**) is denoted by Id_A and it is defined by the rule $\mathrm{Id}_A(a) = a$ for all $a \in A$. Show that if $|A| > 1$ then not all elements of X have inverses.

ii. Let A be a set. Let $\mathrm{Sym}(A)$ be the set of bijections from A into A. For f, $g \in \mathrm{Sym}(A)$, define the function $f \circ g \in \mathrm{Sym}(A)$ by the rule

$$(f \circ g)(a) = f(g(a))$$

for all $a \in A$ (as above). Show that this is a binary operation on X which is associative, noncommutative if $|A| > 2$ and which has an identity element. Show that every element of $\mathrm{Sym}(A)$ has an inverse.

1.2 Binary Relations

Let X be a set. A **binary relation** on X is just a subset of $X \times X$.

Let R be a binary relation on X. Thus $R \subseteq X \times X$. If $(x, y) \in R$, we will write xRy. If $(x, y) \notin R$, we will write $x \not R y$.

Binary relations are often denoted by such symbols as R, S, T, $<$, $>$, $\leq$, $\geq$, $\prec$, $\preceq$, $\ll$, $\sqsubseteq$, $\perp$, $\sim$, $\equiv$, $\simeq$, $\approx$ etc.

Examples.

i. $R = X \times X$ is a binary relation on the set X. We have xRy for all x, $y \in X$.

ii. $R = \emptyset$ is a binary relation on X. For this relation, $x \not R y$ for all x, $y \in X$.

iii. Let $R := \delta(X \times X) := \{(x, x) : x \in X\}$. Then R is a binary relation on X. We have xRy if and only if $x = y$.

iv. The set $R := \{(x, y) \in X \times X : x \in y\}$ is a binary relation on X. Thus, for all x, $y \in X$, $(x, y) \in R$ if and only if $x \in y$.

v. Let A and Y be two set and $B \subseteq A$. Let X be the set of functions from A into Y. For f, $g \in X$, set $f \equiv g$ if and only if $f(b) = g(b)$ for all $b \in B$. This is a binary relation on X. It has the following properties:

   Reflexivity. For all $f \in X$, $f \equiv f$.

   Symmetry. For all f, $g \in X$, if $f \equiv g$ then $f \equiv f$.

   Transitivity. For all f, g, $h \in X$, if $f \equiv g$ and $g \equiv h$, then $g \equiv h$.

   A relation satisfying the three properties above is called an **equivalence relation**. The relation in Example iii is also an equivalence relation.

Exercises.

i. Let A and Y be two set. Let $\wp$ be a nonempty set of subsets of A satisfying the following condition: For all B_1, $B_2 \in \wp$, there is a $B_3 \in \wp$ such that $B_3 \subseteq B_1 \cap B_2$. Let X be the set of functions from A into Y. For f, $g \in X$, set $f \equiv g$ if and only if there is a $B \in \wp$ such that $f(b) = g(b)$ for all $b \in B$. Show that this is an equivalence relation on X.

Chapter 2

Real Numbers

We will define the set of real numbers axiomatically. We will supply some number of axioms (which are by definition statements that we accept without proofs) and we will state that the set of real numbers is a set that satisfies these axioms. We will neither ask nor answer the (important) question of the existence of the set of real numbers. This question is part of Math 111. However we will prove that the set of real numbers is unique in a sense to be made precise (Theorem 3.3.1).

We will be interested just in two (binary) operations on $\mathbb{R}$, called addition and multiplication. Apart from these two operations, we will also be interested in a (binary) relation $<$.

Our definition will take some time, till page 23.

Definition 2.0.1 *A set* $\mathbb{R}$ *together with two binary operations* $+$ *and* $\times$*, two distinct constants* $0 \in \mathbb{R}$ *and* $1 \in \mathbb{R}$ *and a binary relation* $<$ *is called a* **set of real numbers** *if the axioms A1, A2, A3, A4, M1, M2, M3, M4, D, O1, O2, O3, AO, MO and C (that will be stated in this chapter) hold.*

The binary operation $+$ is called **addition**, the binary operation $\times$ is called **multiplication**, the element 0 is called **zero**, the element 1 is called **one**, the binary relation $<$ is called the **order relation**. For two elements x, $y \in \mathbb{R}$, $x+y$ will be called the **sum** of x and y; instead of $x \times y$ we will often prefer to write xy. xy will be called the **product** of x and y. If $x < y$, we will say that "x **is less than** y". We will use the expressions "greater than", "not less than" etc. freely.

We also define the binary relations $\leq$, $>$ and $\geq$ as follows:

$$\begin{array}{lcl} x \leq y & \Leftrightarrow & x < y \text{ or } x = y \\ x > y & \Leftrightarrow & y < x \\ x \geq y & \Leftrightarrow & x > y \text{ or } x = y \end{array}$$

The fact that $0 \neq 1$ (which is explicitly stated in the definition) is important and will be needed later. One cannot prove this fact from the rest of the axioms.

Indeed, the set $\{0\}$ satisfies all the axioms (take 1 to be 0) that we will state. In fact, if $0 = 1$, then there can be only one element in $\mathbb{R}$, namely 0, because, for any $x \in \mathbb{R}$,

$$x \stackrel{M2}{=} 1x = 0x \stackrel{A2}{=} 0.$$

2.1 Axioms for Addition

We start with the axioms that involve only the addition.

A1. Additive Associativity. *For any x, y, $z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.*

The first axiom tells us that when adding, the parentheses are unnecessary. For example, instead of $(x + y) + z$, we can just write $x + y + z$. Similarly, instead of $(x+y)+(z+t)$ or of $((x+y)+z)+t$, we can just write $x+y+z+t$. Although this fact (that the parentheses are useless) needs to be proven, we will not prove it. The interested reader may look at Bourbaki.

A2. Additive Identity Element. *For any $x \in \mathbb{R}$, $x + 0 = 0 + x = x$.*

A3. Additive Inverse Element. *For any $x \in \mathbb{R}$ there is a $y \in \mathbb{R}$ such that $x + y = y + x = 0$.*

A set together with a binary operation, say $+$, and an element denoted 0 that satisfies the above axioms is called a **group**. Thus $(\mathbb{R}, +, 0)$ is a group. Below, investigating the structure $(\mathbb{R}, +, 0)$, we will in fact investigate only the properties of a group.

Note that the element y of A3 depends on x. Note also that A3 does not tell us that $x + y = y + x$ for all x, $y \in \mathbb{R}$, it only tells us that it is so only for the specific pair x and y.

We also note that 0 is the only element that satisfies A2; indeed if 0_1 also satisfies A2, then $0 = 0 + 0_1 = 0_1$.

We now prove our first result:

Lemma 2.1.1 *Given $x \in \mathbb{R}$, the element y as in A3 is unique.*

Proof: Let $x \in \mathbb{R}$. Let y be as in A3. Let y_1 satisfy the equation $x + y_1 = 0$. We will show that $y = y_1$, proving more than the statement of the lemma. We start:

$$y \stackrel{A2}{=} y + 0 = y + (x + y_1) \stackrel{A1}{=} (y + x) + y_1 \stackrel{A3}{=} 0 + y_1 \stackrel{A2}{=} y_1.$$

Thus $y = y_1$. □

Since, given x, the element y that satisfies A3 is unique, we can name this element as a function of x. We will denote it by $-x$ and call it the **additive inverse** of x or just "**minus** x". Therefore, we have:

$$x + (-x) = (-x) + x = 0.$$

Since, by A2, 0 is its own inverse, we have $-0 = 0$.

As we have said, the proof of Lemma 2.1.1 proves more, namely the following:

Lemma 2.1.2 *If x, $y \in \mathbb{R}$ satisfy $x + y = 0$, then $y = -x$.*

We start to prove some number of "well-known" results:

Lemma 2.1.3 *For all $x \in \mathbb{R}$, $-(-x) = x$.*

Proof: It is clear from A3 that if y is the additive inverse of x, then x is the additive inverse of y. Thus x is the additive inverse of $-x$. Hence $x = -(-x)$. $\square$

Lemma 2.1.4 *If x, $y \in \mathbb{R}$ satisfy $x + y = 0$, then $x = -y$.*

Proof: Follows directly from lemmas 2.1.2 and 2.1.3. $\square$

Lemma 2.1.5 *For all x, $y \in \mathbb{R}$, $-(x + y) = (-y) + (-x)$.*

Proof: We compute directly: $(x+y)+((-y)+(-x)) = x+y+(-y)+(-x) = x + 0 + (-x) = x + (-x) = 0$. Here the first equality is the consequence of A1 (that states that the parentheses are useless).

Thus $(x+y)+((-y)+(-x)) = 0$. By Lemma 2.1.2, $(-y)+(-x) = -(x+y)$. $\square$

We should note that the lemma above does not state that $-(x+y) = (-x)+(-y)$. Although this equality holds in $\mathbb{R}$, we cannot prove it at this stage; to prove it we need Axiom A4, which is yet to be stated.

We now define the following terms:

$$\begin{aligned} x - y &:= x + (-y) \\ -x + y &:= (-x) + y \\ -x - y &:= (-x) + (-y) \end{aligned}$$

Lemma 2.1.6 *For all x, $y \in \mathbb{R}$,*

$$\begin{aligned} -(x - y) &= y - x \\ -(-x + y) &= -y + x \\ -(-x - y) &= y + x \end{aligned}$$

Proof: Left as an exercise. $\square$

The next lemma says that we can simplify from the left.

Lemma 2.1.7 (Left Cancellation) *For x, y, $z \in \mathbb{R}$ if $x + y = x + z$ then $y = z$.*

Proof: Add $-x$ to the left of both parts of the equality $x + y = x + z$, and using associativity, we get $y = z$. $\square$

Similarly we have,

Lemma 2.1.8 (Right Cancellation) *For x, y, $z \in \mathbb{R}$ if $y + x = z + x$ then $y = z$.*

Finally, we state our last axiom that involves only the addition.

A4. Commutativity of the Addition. *For any x, $y \in \mathbb{R}$, $x + y = y + x$.*

A set together with a binary operation, say $+$, and an element denoted 0 that satisfies the axioms A1, A2, A3 and A4 is called a **commutative** or an **abelian group**. Thus $(\mathbb{R}, +, 0)$ is an abelian group.

2.2 Axioms for Multiplication

Let $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. Since $1 \neq 0$, the element 1 is an element of $\mathbb{R}^*$. In this subsection, we will replace the symbols $\mathbb{R}$, $+$ and 0 of the above subsection, by $\mathbb{R}^*$, $\times$ and 1 respectively. For example, Axiom A1 will then read

M1. Multiplicative Associativity. *For any x, y, $z \in \mathbb{R}^*$, $x \times (y \times z) = (x \times y) \times z$.*

As we have said, we will prefer to write $x(yz) = (xy)z$ instead of $x \times (y \times z) = (x \times y) \times z$.

Axioms A2 and A3 take the following form:

M2. Multiplicative Identity Element. *For any $x \in \mathbb{R}^*$, $x1 = 1x = x$.*

M3. Multiplicative Inverse Element. *For any $x \in \mathbb{R}^*$ there is a $y \in \mathbb{R}^*$ such that $xy = yx = 1$.*

We accept M1, M2 and M3 as axioms. Thus $(\mathbb{R}^*, \times, 1)$ is a group.

All the results of the previous subsection will remain valid if we do the above replacements. Of course, the use of the axioms A1, A2, A3 in the proofs must be replaced by M1, M2, M3 respectively. That is what we will do now:

Lemma 2.2.1 *Given $x \in \mathbb{R}^*$, the element y as in M3 is unique.*

The proof of this lemma can be translated from the proof of Lemma 2.1.1 directly:

Proof: Let $x \in \mathbb{R}^*$. Let y be as in M3. Let y_1 satisfy the equation $xy_1 = 1$. We will show that $y = y_1$, proving more than the statement of the lemma. We start:

$$y \overset{M2}{=} y1 = y(xy_1) \overset{M1}{=} (yx)y_1 \overset{M3}{=} 1y_1 \overset{M2}{=} y_1.$$

Thus $y = y_1$. □

Since, given $x \in \mathbb{R}^*$, the element $y \in \mathbb{R}^*$ that satisfies M3 is unique, we can name this element as a function of x. We will denote it by x^{-1} and call it the **multiplicative inverse** of x or sometimes "x **inverse**". Therefore, we have:

$$xx^{-1} = x^{-1}x = 1.$$

Note that x^{-1} is defined only for $x \neq 0$. The term 0^{-1} will never be defined.

Since, by M2, 1 is its own inverse, we have $1^{-1} = 1$.

As we have noticed, the proof of Lemma 2.2.1 proves more, namely the following:

Lemma 2.2.2 *If* $x, y \in \mathbb{R}^*$ *satisfy* $xy = 1$*, then* $y = x^{-1}$*.*

Lemma 2.2.3 *For all* $x \in \mathbb{R}^*$*,* $(x^{-1})^{-1} = x$*.*

Proof: As in Lemma 2.1.3. □

Lemma 2.2.4 *If* $x, y \in \mathbb{R}^*$ *satisfy* $xy = 1$*, then* $x = y^{-1}$*.*

Proof: As in Lemma 2.1.4. □

Lemma 2.2.5 *For all* $x, y \in \mathbb{R}^*$*,* $(xy)^{-1} = y^{-1}x^{-1}$*.*

Proof: As in Lemma 2.1.5. □

We should note that the lemma above does not state that $(xy)^{-1} = x^{-1}y^{-1}$. Although this equality holds in $\mathbb{R}$, we cannot prove it at this stage; to prove it we need Axiom M4.

The next lemma says that we can simplify from the left and also from the right.

Lemma 2.2.6 (Cancellation) *Let* $x, y, z \in \mathbb{R}^*$*.*
i. If $xy = xz$ *then* $y = z$*.*
ii. If $yx = zx$ *then* $y = z$*.*

Proof: Left as an exercise. □

The lemma above is also valid if either y or z is zero (without x being zero), but we cannot prove it yet.

Finally, we state our last axiom that involves only multiplication.

M4. Commutativity of the Multiplication. *For any* $x, y \in \mathbb{R}^*$*,* $xy = yx$*.*

Thus $(\mathbb{R}^*, \times, 1)$ is an abelian group.

Sometimes, one writes x/y or $\frac{x}{y}$ instead of xy^{-1}.

2.3 Distributivity

In the first subsection, we stated the axioms that involve only the addition, and in the second subsection, the axioms that involve only the multiplication. Until now there is no relationship whatsoever between the addition and the multiplication. For the moment they appear to be two independent operations. Consequently, at this point we cannot prove any equality that involves both operation, e.g. the equalities $(-1)^{-1} = -1$ and $(-1)x = -x$ cannot be proven at this stage.

Below, we state an axiom that involves both addition and multiplication.

LD. Left Distributivity. *For all* x, y, $z \in \mathbb{R}$, $x(y+z) = xy + xz$.

Since the multiplication is commutative, we also have the partial right distributivity valid if x, y, $z \neq 0$ and $y + z \neq 0$:

$$(y+z)x \stackrel{M4}{=} x(y+z) \stackrel{D}{=} xy + xz \stackrel{M4}{=} yx + zx.$$

Lemma 2.3.1 *For all* $x \in \mathbb{R}$, $x0 = 0$.

Proof: Since $x0 + 0 \stackrel{A2}{=} x0 \stackrel{A2}{=} x(0+0) \stackrel{LD}{=} x0 + x0$, by Lemma 2.1.7, $0 = x0$. □

I do not know whether one can prove the right distributivity in its full generality or the equality "$0x = 0$" from the axioms above. It seems (but I may be wrong) that we need to add the right distributivity to the list of our axioms as well. This is what we will do now:

D. Distributivity. *For all* x, y, $z \in \mathbb{R}$,

$$x(y+z) = xy + xz \text{ and } (y+z)x = yx + zx.$$

Axiom D will be the only axiom relating the addition and the multiplication.

Lemma 2.3.2 *For all* $x \in \mathbb{R}$, $0x = 0$.

Proof: As in Lemma 2.3.1. □

It follows that M4 is valid for all x, $y \in \mathbb{R}$.

A set $\mathbb{R}$ together with two binary operations $+$ and $\times$ and constants 0 and 1 that satisfy the axioms A1, A2, A3, A4, M1, M2, M3, M4 and D is called a **field**. Thus $(\mathbb{R}, +, \times, 0, 1)$ is a field.

We investigate some other consequences of distributivity:

Lemma 2.3.3 *For all* x, $y \in \mathbb{R}$,

$$\begin{array}{rcl} (-x)y & = & -(xy) \\ x(-y) & = & -(xy) \\ (-x)(-y) & = & xy \end{array}$$

Proof: We compute directly: $0 = 0y \stackrel{A3}{=} (x + (-x))y \stackrel{D}{=} xy + (-x)y$ (the first equality is Lemma 2.3.2). Thus $(-x)y$ is the additive inverse of xy and so $(-x)y = -(xy)$. This is the first equality. The others are similar and are left as exercise. □

It follows that we can write $-xy$ for $(-x)y$ or $x(-y)$.

Corollary 2.3.4 *For all* $x \in \mathbb{R}$, $(-1)x = -x$.

Corollary 2.3.5 $(-1)^{-1} = -1$.

2.4 Axioms for the Order Relation

The first three axioms below involve only the binary relation "inequality" $<$. Very soon we will relate the inequality to the operations $+$ and $\times$.

O1. Transitivity. For all x, y, $z \in \mathbb{R}$, if $x < y$ and $y < z$ then $x < z$.

O2. Irreflexivity. For all $x \in \mathbb{R}$, it is not true that $x < x$, i.e. $x \not< x$.

O3. Total Order. For any x, $y \in \mathbb{R}$, either $x < y$ or $x = y$ or $y < x$.

Lemma 2.4.1 *For any x, $y \in \mathbb{R}$, only one of the relations*

$$x < y,\ x = y,\ y < x$$

hold.

Proof: Assume $x < y$ and $x = y$ hold. Then $x < x$, contradicting O2.
Assume $x < y$ and $y < x$ hold. Then by O1, $x < x$, contradicting O2.
Assume $x = y$ and $y < x$ hold. Then $x < x$, contradicting O2. □

Now we state the two axioms that relate the inequality with the two operations $+$ and $\times$:

OA. For all x, y, $z \in \mathbb{R}$, if $x < y$ then $x + z < y + z$.

OM. For all x, y, $z \in \mathbb{R}$, if $x < y$ and $0 < z$ then $xz < yz$.

A set $\mathbb{R}$ together with two binary operations $+$ and $\times$, two constants 0 and 1 and a binary relation $<$ that satisfies the axioms A1, A2, A3, A4, M1, M2, M3, M4, D, O1, O2, O3, OA and OM is called an **ordered field**. Thus $(\mathbb{R}, +, \times, 0, 1)$ is an ordered field. Below, we investigate the properties of ordered fields.

Lemma 2.4.2 *If $x < y$ then $-y < -x$.*

Proof: Adding $-x - y$ to both sides of the inequality $x < y$, by OA we get the result. □

Lemma 2.4.3 *If $0 < x < y$ then $0 < y^{-1} < x^{-1}$.*

Proof: Left as an exercise. □

Lemma 2.4.4 *If $x < 0$ and $y < 0$ then $0 < xy$.*

Proof: By Lemma 2.4.2, $0 < -x$ and $0 < -y$. Then, $0 \overset{2.3.2}{=} 0(-y) \overset{OM}{<} (-x)(-y) \overset{2.3.3}{=} xy$. □

For any x, we define x^2 to be xx.

Corollary 2.4.5 *For any $x \in \mathbb{R}$, $x^2 \geq 0$.*

Proof: Left as an exercise. □

If $x < 0$ we say that x is **strictly negative**, if $x > 0$ we say that x is **strictly positive**. If $x \leq 0$ we say that x is **nonpositive** and if $x \geq 0$ we say that x is **nonpositive**.

We let

$$\begin{array}{lllll} \mathbb{R}^{>0} & = & \{x \in \mathbb{R} : x > 0\} & & \\ \mathbb{R}^{\geq 0} & = & \{x \in \mathbb{R} : x \geq 0\} & = & \mathbb{R}{>}\, 0 \cup \{0\} \\ \mathbb{R}^{<0} & = & \{x \in \mathbb{R} : x < 0\} & = & -\mathbb{R}^{>0} \\ \mathbb{R}^{\leq 0} & = & \{x \in \mathbb{R} : x \leq 0\} & = & \mathbb{R}^{<0} \cup \{0\} = -\mathbb{R}^{\geq 0} \end{array}$$

To finish the axioms of real numbers, there is one more axiom left. This will be the subject of one of the later subsections.

Exercises and Examples.

i. The set $\{0, 1\}$ with the following addition

$$\begin{array}{lllll} 0+0 & = & 1+1 & = & 0 \\ 0+1 & = & 1+0 & = & 1 \end{array}$$

and multiplication defined by $0 \times x = x \times 0 = 0$ for $x = 0$, 1 and $1 \times 1 = 1$ satisfies all the axioms about addition and multiplication (A1, A2, A3, A4, M1, M2, M3, M4, D), but there is no order relation on $\{0, 1\}$ that satisfies the order axioms (O1, O2, O3, OA, OM).

ii. ¶ The set $\mathbb{Z}$ of integers, with the usual addition, multiplication and the order relation, satisfies all the axioms except M3.

iii. ¶ The set $\mathbb{Q}$ of rational numbers, with the usual addition, multiplication and the order relation, satisfies all the axioms. We will see that $\mathbb{Q}$ does not satisfy the Completeness Axiom that we will state in the next subsection.

We assume for the rest that we are in a structure that satisfies the axioms stated until now.

iv. Show that if $xy = 0$ then either x or y is zero.

v. Define 2 to be $1 + 1$. Show that $2x = x + x$.

vi. Show that the axioms above imply that $\mathbb{R}$ is infinite.

vii. Prove that $(x + y)^2 = x^2 + 2xy + y^2$ and that $x^2 - y^2 = (x - y)(x + y)$. (Recall that x^2 was defined to be xx).

viii. Prove that if $0 < x < y$ then $x^2 < y^2$.

ix. Prove that if $x^2 = y^2$ then either $x = y$ or $x = -y$.

x. Prove that if $x < y$ and $z < t$ then $x + z < y + t$.

xi. Prove that if $0 < x < y$ and $0 < z < t$ then $xz < yt$.

xii. Prove that if $x < y$, then $x < \frac{x+y}{2} < y$.

xiii. Define $|x|$ to be $\max(x, -x)$ (i.e. the largest of the two). For all x, $y \in \mathbb{R}$, show the following:

a) $|x| \geq 0$.

b) $|x| = 0$ if and only if $x = 0$.

c) $|-x| = |x|$.

d) $|x + y| \leq |x| + |y|$.

e) Conclude from (d) that $|x| - |y| \leq |x - y|$.

f) Show that $||x| - |y|| \leq |x - y|$. (Hint: Use (e)).

g) Show that $|xy| = |x||y|$.

The number $|x|$ is called the **absolute value** of x.

xiv. Let $x, y \in \mathbb{R}$. Show that

$$\max(x, y) = \frac{x + y + |x - y|}{2}\ , \qquad \min(x, y) = \frac{x + y - |x - y|}{2}.$$

xv. Show that $||x| - |y|| \leq |x - y|$ for any $x, y \in \mathbb{R}$.

xvi. Show that Example v on page 9 is a commutative group.

2.5 Totally Ordered Sets

Let X be a set together with a binary relation $<$ that satisfies the axioms O1, O2 and O3. We will call such a relation $<$ a **totally ordered set**, or a **linearly ordered set**, or a **chain**. Thus $\mathbb{R}$ is a totally ordered set. Thus $(\mathbb{R}, <)$ is a totally ordered set. We define the relations $x \leq y$, $x > y$, $x \geq y$, $x \not< y$ etc. as usual.

In a totally ordered set $(X, <)$ one can define **intervals** as follows (below a and b are elements of X):

$$\begin{array}{lcl}
(a, b) & = & \{x \in X : a < x < b\} \\
(a, b] & = & \{x \in X : a < x \leq b\} \\
[a, b) & = & \{x \in X : a \leq x < b\} \\
[a, b] & = & \{x \in X : a \leq x \leq b\} \\
(a, \infty) & = & \{x \in X : a < x\} \\
[a, \infty) & = & \{x \in X : a \leq x\} \\
(-\infty, a) & = & \{x \in X : x < a\} \\
(-\infty, a] & = & \{x \in X : x \leq a\} \\
(-\infty, \infty) & = & X
\end{array}$$

An element M of a totally ordered set X is called **maximal** if no element of X is greater than M. An element m of a poset X is called **minimal** if no element of X is less than m.

Let $(X, <)$ be a totally ordered set. Let $A \subseteq X$ be a subset of X. An element $x \in X$ is called an **upper bound** for A if $x \geq a$ for all $a \in A$.

An element $a \in X$ is called a **least upper bound** for A if,

i) a is an upper bound of A, and

ii) If b is another upper bound of A, $b \not< a$.

Let X be a poset. A subset of X that has an upper bound is said to be **bounded above**.

The terms **lower bound**, **greatest lower bound** and a set **bounded below** are defined similarly.

Lemma 2.5.1 *A subset of a totally ordered set which has a least upper bound (resp. a greatest lower bound) has a unique least upper bound (resp. greatest lower bound).*

Proof: Let X be a totally ordered set. Let $A \subseteq X$ be a subset which has a least upper bound, say x. Assume y is also a least upper bound for A. By definition $y \not< x$ and $x \not< y$. Then $x = y$ by O3. □

Thus if A is a subset of a totally ordered set whose least upper bound exists, then we can name this least upper bound as a function of X. We will use the notation $\text{lub}(A)$ or $\sup(A)$ for the least upper bound of A. We use the notations $\text{glb}(A)$ and $\inf(A)$ for the least upper bound.

Examples and Exercises.

i. Let $X = (\mathbb{R}, <)$. Let $A = (0, 1) \subseteq \mathbb{R}$. Then any number ≥ 1 is an upper bound for A. 1 is the only least upper bound of A. Note that $1 \notin A$.

ii. Let $X = (\mathbb{R}, <)$. Let $A = [0, 1] \subseteq \mathbb{R}$. Then any number ≥ 1 is an upper bound for A. 1 is the only least upper bound of A. Note that $1 \in A$.

iii. Let $X = (\mathbb{R}, <)$. Let $A = (0, \infty) \subseteq \mathbb{R}$. Then A has no upper bound. But it has a least upper bound.

iv. Let $X = (\mathbb{R}, <)$ and $A = \{\frac{x}{x+1} : x \in \mathbb{R} \text{ and } x \geq 1\}$. Show that 1 is the only least upper bound of A.

v. Let $X = (\mathbb{R}, <)$ and $A \subseteq \mathbb{R}$ any subset of $\mathbb{R}$. Define $-A = \{-a : a \in A\}$.

   i. Show that if x is an upper bound for A then $-x$ is a lower bound for $-A$.

   ii. Show that if x is the least upper bound for $A \subseteq \mathbb{R}$ then $-x$ is the greatest lower bound for $-A$.

vi. Let $(X, <)$ be a totally ordered set and $A \subseteq X$. Let

$$B := \{x \in X : x \text{ is an unpper bound for } A\}.$$

a) Assume that $\sup(A)$ exists. Show that $\inf(B)$ exists and $\inf(B) = \sup(A)$.

b) Assume that $\inf(B)$ exists. Show that $\sup(A)$ exists and $\sup(A) = \inf(B)$.

vii. Let X be a totally ordered set and $b \in X$. Show that b is the only least upper bound of $(-\infty, b]$, and also of $(-\infty, b)$.

viii. On $\mathbb{R} \times \mathbb{R}$ define the relation $\prec$ as follows $(x, y) \prec (x_1, y_1)$ by "either $y < y_1$, or $y = y_1$ and $x < x_1$".

a) Show that this is a total order (called **lexicographic ordering**).

b) Does every subset of this linear order which has an upper bound has a least upper bound?

ix. On $\mathbb{N} \times \mathbb{N}$ define the relation $\prec$ as above (lexicographic order).

a) Show that this is a total order.

b) Does every subset of this linear order which has an upper bound has a least upper bound?

x. Find a poset where the intersection of two intervals is not necessarily an interval.

2.6 Completeness Axiom

We now state the last axiom for $\mathbb{R}$. This axiom is different from the others in the sense that all the other axioms were about a property of one, two or at most three elements of $\mathbb{R}$. But this one is a statement about subsets of $\mathbb{R}$.

C. Completeness. *Any nonempty subset of* $\mathbb{R}$ *which is bounded above has a least upper bound.*

It follows from Lemma 2.5.1, that a subset X of $\mathbb{R}$ which has an upper bound has a unique least upper bound. We denote it by $\sup(X)$ or $\text{lub}(X)$. The least upper bound of a set is sometimes called the **supremum** of the set. Note that the supremum of a set may or may not be in the set.

This completes our list of axioms. From now on we fix a set $\mathbb{R}$ together with two binary operations $+$ and $\times$, two distinct constants $0 \in \mathbb{R}$ and $1 \in \mathbb{R}$ and a binary relation $<$ that satisfies the axioms A1, A2, A3, A4, M1, M2, M3, M4, D, O1, O2, O3, AO, MO and C stated above. (The existence of such a structure is proven in Math 112.)

Lemma 2.6.1 *Any nonempty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.*

Proof: Left as an exercise. Use Exercise v, 22 or Exercise i, 24 below. $\square$

The greatest lower bound of a (nonempty) set is denoted by $\inf(X)$ or $\mathrm{glb}(X)$. The greatest lower bound is sometimes called the **infimum** of the set. Note that the infimum of a set may or may not be in the set.

Exercises.

i. Show that if $X \subseteq \mathbb{R}$ has a least upper bound then the set $-X := \{-x : x \in X\}$ has a greatest lower bound and $\inf(-X) = -\sup(X)$.

ii. Suppose X, $Y \subseteq \mathbb{R}$ have least upper bounds. Show that the set $X+Y := \{x+y : x \in X,\, y \in Y\}$ has a least upper bound and that $\sup(X+Y) = \sup(X)+\sup(Y)$.

iii. Suppose X, $Y \subseteq \mathbb{R}^{\geq 0}$ have least upper bounds. Show that the set $XY := \{xy : x \in X,\, y \in Y\}$ has a least upper bound and $\sup(XY) = \sup(X)\sup(Y)$. Does the same equality hold for any two subsets of $\mathbb{R}$?

¶ The completeness axiom makes the difference between $\mathbb{Q}$ and $\mathbb{R}$. The equation $x^2 = 2$ has no solution in $\mathbb{Q}$ but has a solution in $\mathbb{R}$. This is what we now prove.

Theorem 2.6.2 *Let $a \in \mathbb{R}^{>0}$. Then there is an $x \in \mathbb{R}$ such that $x^2 = a$.*

Proof: Replacing a by $1/a$ if necessary, we may assume that $a \geq 1$. Let $A = \{x \in \mathbb{R}^{\geq 0} : x^2 \leq a\}$. For $x \in A$, we have $x^2 \leq a \leq a^2$. It follows that $0 \leq a^2 - x^2 = (a-x)(a+x)$, so $a \geq x$. We proved that a is an upper bound for A. Let $b = \mathrm{lub}(A)$. We will show that $b^2 = a$. Clearly $b \geq 1 > 0$.

Assume first that $b^2 < a$. Let $\epsilon = \min(\frac{a-b^2}{2b+1}, 1) > 0$. Then $(b+\epsilon)^2 = b^2 + 2b\epsilon + \epsilon^2 \leq b^2 + 2b\epsilon + \epsilon = b^2 + \epsilon(2b+1) \leq b^2 + (a-b^2) = a$. Hence $b+\epsilon \in A$. But this contradicts the fact that b is the least upper bound for A.

Assume now that $b^2 > a$. Let $\epsilon = \min(\frac{b^2-a}{2b}, b) > 0$. Now $(b-\epsilon)^2 = b^2 - 2b\epsilon + \epsilon^2 > b^2 - 2b\epsilon \geq b^2 - (b^2 - a) = a$. Thus $(b-\epsilon)^2 > a$. Let $x \in A$ be such that $b - \epsilon \leq x \leq b$. (There is such an x because $b-\epsilon$ is not an upper bound for A). Now we have $a < (b-\epsilon)^2 \leq x^2 \leq a$ (because $b - \epsilon \geq 0$), a contradiction.

It follows that $b^2 = a$. $\square$

Remark. Since every nonnegative real number has a square root, the order relation $<$ can be defined from $+$ and $\times$ as follows: for all x, $y \in \mathbb{R}$,

$$x < y \text{ if and only if } \exists z\, (z \neq 0 \wedge y = x + z^2).$$

Exercises.

i. Let $A \subseteq \mathbb{R}$ be a subset satisfying the following property: "For all a, $b \in A$ and $x \in \mathbb{R}$, if $a \leq x \leq b$ then $x \in A$". Show that A is an interval.

ii. Let x^3 mean $x \times x \times x$. Show that for any $x \in \mathbb{R}$ there is a unique $y \in \mathbb{R}$ such that $y^3 = x$.

Chapter 3

Other Number Sets

3.1 Natural Numbers and Induction

We say that a subset X of $\mathbb{R}$ is **inductive** if $0 \in X$ and if for all $x \in X$, $x+1$ is also in X. For example, the subsets $\mathbb{R}^{\geq 0}$, $\mathbb{R}$, $\mathbb{R} \setminus (0,1)$ are inductive sets. The set $\mathbb{R}^{>0}$ is not an inductive set.

Lemma 3.1.1 *An arbitrary intersection of inductive subsets is an inductive subset. The intersection of all the inductive subsets of* $\mathbb{R}$ *is the smallest inductive subset of* $\mathbb{R}$*.*

Proof: Trivial. $\square$

We let $\mathbb{N}$ denote the smallest inductive subset of $\mathbb{R}$. Thus $\mathbb{N}$ is the intersection of all the inductive subsets of $\mathbb{R}$.

The elements of $\mathbb{N}$ are called **natural numbers**.

Theorem 3.1.2 (Induction Principle (1)) *Let X be a subset of* $\mathbb{R}$*. Assume that $0 \in X$ and for any $x \in \mathbb{R}$, if $x \in X$ then $x+1 \in X$. Then $\mathbb{N} \subseteq X$.*

Proof: The statement says that X is inductive. Therefore the theorem follows directly from the definition of $\mathbb{N}$. $\square$

Suppose we want to prove a statement of the form "for all $x \in \mathbb{N}$, $\sigma(x)$". For this, it is enough to prove

i) $\sigma(0)$,

ii) If $\sigma(x)$ then $\sigma(x+1)$.

Indeed, assume we have proved (i) and (ii). let $X := \{x \in \mathbb{R} : \sigma(x)\}$. By (i), $0 \in X$. By (ii), if $x \in X$ then $x+1 \in X$. Thus, by the Induction Principle, $\mathbb{N} \subseteq X$. It follows that for all $x \in \mathbb{N}$, $\sigma(x)$.

Lemma 3.1.3 *i.* $\operatorname{lub}(\mathbb{N}) = 0$*, i.e. 0 is the least element of* $\mathbb{N}$*.*

ii. If $x \in \mathbb{N} \setminus \{0\}$ then $x - 1 \in \mathbb{N}$.

iii. $\mathbb{N}$ *is closed under addition and multiplication, i.e. if* $x, y \in \mathbb{N}$ *then* $x + y$, $xy \in \mathbb{N}$.

iv. If $0 < y < 1$ *then* $y \notin \mathbb{N}$.

v. Let $x \in \mathbb{N}$. *If* $x < y < x + 1$, *then* $y \notin \mathbb{N}$.

vi. Let $x, y \in \mathbb{N}$. *If* $x < y$ *then* $x + 1 \leq y$.

vii. Let $x, y \in \mathbb{N}$. *Assume* $y < x$. *Then* $x - y \in \mathbb{N}$.

viii. If $x, y \in \mathbb{N}$ *and* $y < x + 1$, *then either* $y = x$ *or* $y < x$.

Proof: i. Clearly $\mathbb{R}^{\geq 0}$ is an inductive set. But $\mathbb{N}$ is defined to be the smallest inductive subset of $\mathbb{R}$. Thus $\mathbb{N} \subseteq \mathbb{R}^{\geq 0}$. Since $0 \in \mathbb{N}$, it follows that 0 is the least element of $\mathbb{N}$.

ii. Assume that for $x \in \mathbb{N} \setminus \{0\}$, $x - 1 \notin \mathbb{N}$. Then the set $\mathbb{N} \setminus \{x\}$ is an inductive set (check carefully). Since $\mathbb{N}$ is the smallest inductive set, $\mathbb{N} \subseteq \mathbb{N} \setminus \{x\}$ and $x \notin \mathbb{N}$.

iii. Let $x, y \in \mathbb{N}$. We proceed by induction y to show that $x + y \in \mathbb{N}$, i.e. letting $\sigma(y)$ denote the statement $x + y \in \mathbb{N}$, we show that $\sigma(0)$ holds and that if $\sigma(y)$ holds then $\sigma(y + 1)$ holds. If $y = 0$ then $x + y = x + 0 = x \in \mathbb{N}$. Thus $\sigma(0)$ holds. Assume now $\sigma(y)$ holds, i.e. that $x + y \in \mathbb{N}$. Then $x + (y + 1) = (x + y) + 1$. Since $x + y \in \mathbb{N}$ by assumption, we also have $x + (y + 1) \in \mathbb{N}$. Thus $\sigma(y + 1)$ holds also. Therefore $\sigma(y)$ holds for all $\in \mathbb{N}$.

The proof for the multiplication is left as an exercise.

iv. The set $\mathbb{N} \setminus (0, y]$ is an inductive set as it can be shown easily. Thus $\mathbb{N} \subseteq \mathbb{N} \setminus (0, y]$ and $y \notin \mathbb{N}$.

v. We proceed by induction on x. The previous part gives us the case $x = 0$. Assume the statement holds for x and we proceed to show that the statement holds for $x + 1$. Let $x + 1 < y < (x + 1) + 1$, then $x < y - 1 < x + 1$. By the inductive hypothesis, $y - 1 \notin \mathbb{N}$. By part (i), either $y = 0$ or $y \notin \mathbb{N}$. Since $0 < x + 1 < y$, we cannot have $y = 0$. Thus $y \notin \mathbb{N}$.

vi. Assume not. Then $x < y < x + 1$, contradicting part (v).

vii. By induction on x. If $x = 0$, then the statement holds because there is no $y \in \mathbb{N}$ such that $y < x$. Assume the statement holds for x. We will show that it holds for $x + 1$. Let $y \in \mathbb{N}$ be such that $y < x + 1$. Then either $y < x$ or $y = x$. Now $(x + 1) - y = (x - y) + 1$. In case $y < x$, the induction hypothesis gives $x - y \in \mathbb{N}$ and so $(x + 1) - y \in \mathbb{N}$. In case $y = x$, we have $(x + 1) - y = 1 \in \mathbb{N}$.

viii. By (v), $y \leq x$. □

Lemma 3.1.4 *Any nonempty subset of* $\mathbb{N}$ *has a least element.*

Proof: Let $\emptyset \neq X \subseteq \mathbb{N}$. Assume X does not have a least element. We will prove that $X = \emptyset$, which is the same as proving that no element of $\mathbb{N}$ is in X. We will first show the following statement $\phi(n)$ for all n: "No natural number $m < n$ is in X".

Since there are no natural numbers < 0 the statement ϕ holds for 0.

Assume $\phi(n)$ holds. If $\phi(n + 1)$ were false, then n would be in X by Lemma 3.1.3.viii and it would be the smallest element of X, a contradiction.

Thus $\phi(n)$ holds for any n. Now if a natural number n were in X, since $\phi(n)$ holds, n would be the smallest element of X, a contradiction. □

There is a slightly more complicated version of the inductive principle that is very often used in mathematics:

Theorem 3.1.5 (Induction Principle (2)) *Let X be a subset of $\mathbb{N}$. Assume that for any $x \in \mathbb{N}$,*

$$(\forall y \in \mathbb{N}\ (y < x \rightarrow y \in X)) \rightarrow x \in X.$$

Then $X = \mathbb{N}$.

Proof: Assume not. Then $\mathbb{N} \setminus X \neq \emptyset$. Let x be the least element of $\mathbb{N} \setminus X$. Thus $\forall y \in \mathbb{N}\ (y < x \rightarrow y \in X)$. But then by hypothesis $x \in X$, a contradiction. $\square$

How does one use Induction Principle (2) in practice? Suppose we have statement $\sigma(x)$ to prove about natural numbers x. Given $x \in \mathbb{N}$, assuming $\sigma(y)$ holds for all natural numbers $y < x$, one proves that $\sigma(x)$ holds.This is enough to prove that $\sigma(x)$ holds for all $x \in \mathbb{N}$.

We immediately give some applications of the Induction Principles.

Theorem 3.1.6 (Archimedean Property) *Let $\epsilon \in \mathbb{R}^{>0}$ and $x \in \mathbb{R}$, then there is an $n \in \mathbb{N}$ such that $x < n\epsilon$.*

Proof: Assume not, i.e. assume that $n\epsilon \leq x$ for all $n \in \mathbb{N}$. Then the set $\mathbb{N}$ is bounded above by x/ϵ. Thus $\mathbb{N}$ has a least upper bound, say a. Hence $a - 1$ is not an upper bound for $\mathbb{N}$. It follows that there is an element $n \in \mathbb{N}$ that satisfies $a - 1 < n$. But this implies $a < n + 1$. Since $n + 1 \in \mathbb{N}$, this contradicts the fact that a is an upper bound for $\mathbb{N}$. $\square$

Lemma 3.1.7 *Any nonempty subset of $\mathbb{N}$ that has an upper bound contains its least upper bound.*

Proof: Let $\emptyset \neq A \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}$ that has an upper bound. Let x be the least upper bound of A. Since $x - 1$ is not an upper bound for A, there is an $a \in A$ such that $x - 1 < a \leq x$. By parts (iv) and (v) of Lemma 3.1.3.vii a is the largest element of A. $\square$

Theorem 3.1.8 (Integral Part) *For any $x \in \mathbb{R}^{\geq 0}$, there is a unique $n \in \mathbb{N}$ such that $n \leq x < n + 1\}$.*

Proof: Let $A = \{a \in \mathbb{N} : a \leq x\}$. Then $0 \in A$ and A is bounded above by x. By Lemma 3.1.7, A contains its least upper bound, say n. Thus $n \leq x < n + 1$. This proves the existence. Now we prove the uniqueness. Assume $m \in \mathbb{N}$ and $m \leq x < m + 1$. If $n < m$, then by Lemma 3.1.3.vii, $x < n + 1 \leq m \leq x$, a contradiction. Similarly $m \not< n$. Thus $n = m$. $\square$

Theorem 3.1.9 (Division) *For any n, $m \in \mathbb{N}$, $m \neq 0$ there are unique $q, r \in \mathbb{N}$ such that $n = mq + r$ and $0 \leq r < m$.*

Proof: We first prove the existence. We proceed by induction on n (Induction Principle 2). If $n < m$, then take $q = 0$ and $r = n$. Assume now $n \geq m$. By induction, there are q_1 and r_1 such that $n - m = mq_1 + r_1$ and $0 \leq r_1 < m$. Now $n = m(q_1 + 1) + r_1$. Take $q = q_1 + 1$ and $r = r_1$. This proves the existence.

We now prove the existence. Assume $n = mq + r = mq_1 + r_1$, $0 \leq r < m$ and $0 \leq r_1 < m$. Assume $q_1 > q$. Then $m > m - r_1 > r - r_1 = mq_1 - mq = m(q_1 - q) \geq m$. This is a contradiction. Similarly $q \not> q_1$. Hence $q_1 = q$. It follows immediately that $r = r_1$. □

Exercises.

i. Show that for any $n \in \mathbb{N} \setminus \{0\}$, $1 + 3 + \ldots + (2n - 1) = n^2$.

ii. Show that for any $n \in \mathbb{N} \setminus \{0\}$, $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

iii. Show that for any $n \in \mathbb{N}$,

$$\frac{1}{1}\frac{1}{3} + \frac{1}{3}\frac{1}{5} + \ldots + \frac{1}{2n-1}\frac{1}{2n+1} = \frac{n}{2n+1}.$$

iv. Show that for any $n \in \mathbb{N} \setminus \{0\}$, $1 + 3 + \ldots + (2n - 1) = n^2$.

v. Show that for any $n \in \mathbb{N} \setminus \{0\}$, $1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n+1)(2n+1)/6$.

vi. Show that for any $n \in \mathbb{N}$,

$$\frac{1}{1}\frac{1}{3} + \frac{1}{3}\frac{1}{5} + \ldots + \frac{1}{2n-1}\frac{1}{2n+1} = \frac{n}{2n+1}.$$

vii. Show that for any $n \in \mathbb{N} \setminus \{0\}$, $1^2 + 2^2 + \ldots + n^2 = \frac{n(2n+1)(n+2)}{6}$.

3.1.1 Exponentiation

Let $r \in \mathbb{R}$. For $n \in \mathbb{N}$, we define r^n, $n^{\mathbf{th}}$ **power of** r, as follows by induction on n:

$$r^0 = 1 \text{ if } r \neq 0$$
$$r^1 = r$$
$$r^{n+1} = r^n r$$

Note that 0^0 is not defined. We will leave it undefined. Note also that the previous definition of r^2 coincides with the one given above: $r^2 = r^{1+1} = r^1 r = rr$.

Proposition 3.1.10 *For $r \in \mathbb{R}$ and $n \in \mathbb{N}$ not both zero, we have,*
i. $(rs)^n = r^n s^n$.
ii. $r^n r^m = r^{n+m}$.
iii. $(r^n)^m = r^{nm}$.

Proof: Left as an exercise. $\square$

Theorem 3.1.11 *i. Let $r \in \mathbb{R}^{\geq 0}$ and $n \in \mathbb{N} \setminus \{0\}$. Then there is a unique $s \in \mathbb{R}$ such that $s^m = r$.*

ii. Let $r \in \mathbb{R}$ and let $n \in \mathbb{N}$ be odd. Then there is a unique $s \in \mathbb{R}$ such that $s^m = r$.

Proof: (ii) follows from (i). (i) is proved as in Theorem 2.6.2. Left as an exercise. $\square$

The number s is called the $m^{\mathbf{th}}$**-root** of r.

3.1.2 Factorial

For $n \in \mathbb{N}$, we define $n!$ by induction on n: Set $0! = 1$, $1! = 1$ and $(n+1)! = n!(n+1)$. This just means that $n! = 1 \times 2 \times \ldots \times n$.

Exercises.

i. Show that a set with n elements has $n!$ bijections.

ii. Find a formula that gives the number of injections from a set with n elements into a set with m elements.

iii. Prove that for $n \in \mathbb{N}$, the set $\{0, 1, \ldots, n-1\}$ has 2^n subsets. (Hint: You may proceed by induction on n).

iv. Show that $n! > 2^n$ for all n large enough.

v. Show that $(x-1)^n \geq x^n - nx^{n-1}$ for all $x > 1$. (Hint: By induction on n).

vi. Show that if $0 < x < 1$ and $n > 0$ is a natural number, then $(1-x)^n \leq 1 - nx + \frac{n(n-1)}{2}x^2$.

vii. Show that for any $n \in \mathbb{N} \setminus \{0\}$, $1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$.

viii. Show that for any $n \in \mathbb{N} \setminus \{0\}$, $1^4 + 2^4 + \ldots + n^4 = \frac{n(n+1)(6n^3+9n^2+n-1)}{30}$.

n **Choose** k. For n, $k \in \mathbb{N}$ and $k \leq n$, define

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!}.$$

Exercises.

i. Show that $\binom{n}{k} = \binom{n}{n-k}$.

ii. Show that $\binom{n}{0} = \binom{n}{n} = 1$.

iii. Show that $\binom{n}{1} = n$.

iv. Show that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

v. Deduce that $\binom{n}{k} \in \mathbb{N}$. (Hint: By induction on n).

vi. Show that for $n \in \mathbb{N}$ and $0 \leq k \leq n$, a set with n elements has $\binom{n}{k}$ subsets with k elements.

vii. Show that for $n \in \mathbb{N}$ and $k \in \mathbb{N}$ with $k \leq n$, a set with n elements has $\binom{n}{k}$ subsets with k elements.

viii. Show that for x, $y \in \mathbb{R}$ and $n \in \mathbb{N}$,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}.$$

(Hint: By induction on n).

ix. Show that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

x. Show that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$.

xi. Compute $(x+y+z)^3$ in terms of x, y and z.

xii. Compute $(x+y+z)^4$ in terms of x, y and z.

xiii. Show that for $x > 1$ and $n \in \mathbb{N}$, $(x-1)^n \geq x^n - nx^{n-1}$.

xiv. Show that for $x < 1$, $(1-x)^n \geq 1 - nx$.

xv. Show that for $n \in \mathbb{N} \setminus \{0\}$, $(1+\frac{1}{n})^n \leq (1+\frac{1}{n+1})^{n+1}$. (See Theorem 6.8.1).

3.1.3 Sequences

Let X be a set. A **sequence** in X is just a function $x : \mathbb{N} \longrightarrow X$. We let $x_n := x(n)$ and denote x – by listing its values – $(x_n)_n$ rather than by x. We can also write

$$x = (x_0, x_1, x_2, x_3, \ldots, x_n, \ldots).$$

If $X = \mathbb{R}$, we speak of a **real sequence**. For example $\left(\frac{1}{n+1}\right)_n$ is a real sequence. We can write this sequence more explicitly by listing its elements:

$$(1, 1/2, 1/3, 1/4, 1/5, \ldots, 1/(n+1), \ldots).$$

If we write a sequence such as $(1/n)_n$, we will assume implicitly that the sequence starts with $n = 1$ (since $1/n$ is undefined for $n = 0$). Thus, with this convention, the above sequence $(\frac{1}{n+1})_n$ may also be denoted by $(1/n)_n$. For example, the sequence $\left(\frac{1}{n(n-1)}\right)_n$ starts with $n = 2$ and its elements can be listed as

$$(1/2, 1/6, 1/12, 1/20, 1/30, \ldots, 1/n(n-1), \ldots).$$

A sequence $(x_n)_n$ is called **increasing** or **nondecreasing**, if $x_n \leq x_{n+1}$ for all $n \in \mathbb{N}$. A sequence $(x_n)_n$ is called **strictly increasing**, if $x_n < x_{n+1}$ for all $n \in \mathbb{N}$. The terms **decreasing**, **strictly decreasing** and **nonincreasing** are defined similarly.

Let $(x_n)_n$ be a sequence. Let $(k_n)_n$ be a strictly increasing sequence of natural numbers. Set $y_n = x_{k_n}$. Then we say that the sequence $(y_n)_n$ is a **subsequence** of the sequence $(x_n)_n$. For example, let $x_n = \frac{1}{n+1}$ and $k_n = 2n$. Then $y_n = x_{k_n} = x_{2n} = \frac{1}{2n+1}$. Thus the sequence $(y_n)_n$ is

$$(1, 1/3, 1/5, 1/7, \ldots, 1/(2n+1), \ldots).$$

If we take $k_0 = 1$ and $k_n = 2^n := 2 \times 2 \times \ldots \times 2$ (n times), then the subsequence $(y_n)_n$ becomes

$$(1/2, 1/3, 1/5, 1/9, 1/17, \ldots, 1/(2^n+1), \ldots).$$

We now prove an important consequence of the Completeness Axiom:

Theorem 3.1.12 (Nested Intervals Property) *Let $(a_n)_n$ and $(b_n)_n$ be two real sequences. Assume that for each n, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. Then $\cap_{n\in\mathbb{N}}[a_n, b_n] = [a, b]$ for some real numbers a and b. In fact $a = \sup\{a_n : n \in \mathbb{N}\}$ and $b = \inf_n\{a_n : n \in \mathbb{N}\}$.*

Proof: Since the set $\{a_n : n \in \mathbb{N}\}$ is bounded above by b_0, it has a least upper bound, say a. Similarly the set $\{b_n : n \in \mathbb{N}\}$ has a greatest lower bound, say b. I claim that $\cap_{n\in\mathbb{N}}[a_n, b_n] = [a, b]$.

If $x \geq a$, then $x \geq a_n$ for all n. Likewise, if $x \leq b$, then $x \leq b_n$ for all n. Hence, if $x \in [a, b]$, the $x \in [a_n, b_n]$ for all n.

Conversely, let $x \in \cap_{n\in\mathbb{N}}[a_n, b_n]$. Then $a_n \leq x \leq b_n$ for all n. Thus x is an upper bound for $\{a_n : n \in \mathbb{N}\}$ and a lower bound for $\{b_n : n \in \mathbb{N}\}$. Hence $a \leq x \leq b$. $\square$

Exercises.

i. Can you find an increasing sequence $(a_n)_n$ and a decreasing sequence $(b_n)_n \in \mathbb{N}$ of real numbers with $a_n < b_m$ for all $n, m \in \mathbb{N}$ such that $\bigcap_n [a_n, b_n) = \emptyset$?

ii. Prove Theorem 3.1.12 for $\mathbb{R}^n$ (with closed cubes or balls rather than closed intervals).

iii. Let G be a subset of $\mathbb{R}^*$ containing 1, closed under multiplication and inversion. Let $\mathrm{Seq}(\mathbb{R})$ be the set of sequences of $\mathbb{R}$. For $a = (a_n)_n$ and $b = (b_n)_n$ in $\mathrm{Seq}(\mathbb{R})$, set $a \equiv b$ if and only if for some $g \in G$, $a_n = gb_n$ eventually. Show that $\equiv$ is an equivalence relation on $\mathrm{Seq}(\mathbb{R})$. Find the equivalence classes when $G = \{1\}$, $G = \{1, -1\}$ and $G = \mathbb{R}^*$.

3.2 Integers and Rational Numbers

In $\mathbb{N}$, we can add and multiply any two numbers, but we cannot always subtract one number from another. We set $\mathbb{Z} = \{n - m : n, m \in \mathbb{N}\}$. Now in $\mathbb{Z}$ we can add, multiply and subtract any two numbers. The elements of $\mathbb{Z}$ are called **integers**.

In $\mathbb{Z}$, we can add, multiply and subtract any two numbers, but we cannot always divide one number to another. We set $\mathbb{Q} = \{n/m : n, m \in \mathbb{Z}, m \neq 0\}$. The elements of $\mathbb{Q}$ are called **rational numbers**. Now in $\mathbb{Q}$ we can add, multiply, subtract and divide any two numbers, with the only exception that we cannot divide a rational number by0.

We will not go into further detail about these number systems. We trust the reader in proving any elementary statement about numbers, for example the decomposition of integers into prime factors.

The structure $(\mathbb{Q}, +, \times, 0, 1, <)$ satisfies all the axioms A1-4, M1-4, O1-3, OA, OM. But it does not satisfy the Completeness Axiom C as the following lemma shows (compare with Theorem 2.6.2).

Lemma 3.2.1 *There is no $q \in \mathbb{Q}$ such that $q^2 = 2$.*

Proof: Assume not. Let $q \in \mathbb{Q}$ be such that $q^2 = 2$. Let $a, b \in \mathbb{Z}$ be such that $q = a/b$. Simplifying if necessary, we may choose a and b so that they are not both divisible by 2. From $(a/b)^2 = q^2 = 2$ we get $a^2 = 2b^2$. Thus a^2 is even. It follows that a is even. Let $a_1 \in \mathbb{Z}$ be such that $a = 2a_1$. Now $4a_1^2 = a^2 = 2b^2$ and $2a_1^2 = b^2$. Hence b is even as well, a contradiction. □

Theorem 3.2.2 *$\mathbb{Q}$ is* **dense** *in $\mathbb{R}$, i.e. for any real numbers $r < s$, there is a rational number q such that $r \leq q \leq s$.*

Proof: By Theorem 3.1.6, there is a natural number such that $1 < n(r - s)$. Now consider the set $A := \{m \in \mathbb{N} : m/n < s\}$. By Theorem 3.1.6 again, A is a bounded set. By Lemma 3.1.7, A has a maximal element, say m. Thus $m/n < s$ and $\frac{m+1}{n} \geq s$. We compute: $s \leq \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} < s + (r - s) = r$. □

3.2.1 Exponentiation

Let $r \in \mathbb{R}$. At page 30, we have defined r^n for $n \in \mathbb{N}$ (except for 0^0, which was left undefined). If $r \neq 0$, we can extend this definition to $\mathbb{Z}$ by $r^{-n} = (r^n)^{-1}$. Note that the previous definition of r^{-1} coincides with the one given above.

Proposition 3.2.3 *For $r \in \mathbb{R}$ and $n \in \mathbb{Z}$ not both zero, we have,*
i. $(rs)^n = r^n s^n$.
ii. $r^n r^m = r^{n+m}$.
iii. $(r^n)^m = r^{nm}$.

Proof: Left as an exercise. $\square$

If $r \in \mathbb{R}^{\geq 0}$ and $q \in \mathbb{Q}$, we can also define r^q as follows: Let $m \in \mathbb{N} \setminus \{0\}$, by Theorem 3.1.11, there is a unique $s \in \mathbb{R}$ such that $s^m = r$. Set $s = r^{1/m}$. Now for $n \in \mathbb{Z}$ and $m \in \mathbb{N} \setminus \{0\}$, define $r^{n/m}$ to be $(r^{1/m})^n$.

Proposition 3.2.4 *For $r \in \mathbb{R}^{\geq 0}$ and q, q_1, $q_2 \in \mathbb{Q}$, we have,*
i. $(rs)^q = r^q s^q$.
ii. $r^{q_1} r^{q_2} = r^{q_1+q_2}$.
iii. $(r^{q_1})^{q_2} = r^{q_1 q_2}$.

Proof: Left as an exercise. $\square$

Exercises

i. Show that if $q \in \mathbb{Q}$ is a square in $\mathbb{Q}$, then $2q$ is not a square in $\mathbb{Q}$.

ii. Let $a < b$ be real numbers. Show that for each $n \in \mathbb{N}$, there are rational numbers $a_n < b_n$ such that $\cap_n [a_n, b_n] = [a, b]$. Hint: See Theorem 3.2.2.

iii. Let $a < b$ be real numbers. Show that for each $n \in \mathbb{N}$, there are rational numbers $a_n < b_n$ such that $\cap_n (a_n, b_n) = [a, b]$. Hint: See the exercise above.

iv. Let $a < b$ be real numbers. Show that for each $n \in \mathbb{N}$, there are rational numbers $a_n < b_n$ such that $\cup_n [a_n, b_n] = (a, b)$. Show that if a and b are nonrational numbers (a_n and b_n are still rational numbers), then we can never have $\cup_n [a_n, b_n] = [a, b]$. Hint: See the exercise above.

v. For each $n \in \mathbb{N}$, let a_n and b_n be such that $a_{n+1} < a_n < b_n < b_{n+1}$. Show that $\cup_n (a_n, b_n)$ is an open interval (bounded or not).

vi. Show that for all rational number $q > 0$, there is a rational number x for which $0 < x^2 < q$.

vii. a) Show that if x and y are two nonnegative rational numbers whose sum is 1, then $a \leq ax + by \leq b$.

b) Let q be a rational number such that $a \leq q \leq b$. Show that there are two nonnegative rational numbers x and y such that $x + y = 1$ and $q = ax + by$.

c) Show that the numbers x and y of part b are unique.

viii. Show that for any $x \in \mathbb{R}$, there is a unique $n \in \mathbb{Z}$ such that $n \leq x < n+1\}$.

ix. Show that for any n, $m \in \mathbb{Z}$, $m \neq 0$ there are unique q, $r \in \mathbb{Z}$ such that $n = mq + r$ and $0 \leq r < m$.

x. Let A_ℓ and A_r be two nonempty subsets of $\mathbb{Q}$ such that

a) $A_\ell \cup A_r = \mathbb{Q}$.

b) $A_\ell \cap A_r = \emptyset$.

c) Any element of A_ℓ is less than any element of A_r.

Show that $\sup(A_\ell) = \inf(A_r)$.

3.3 Uniqueness of the Real Number System

We assumed without a proof that there was a structure $(\mathbb{R}, +, \times, <, 0, 1)$ that satisfies all our axioms. Assuming there is really such a structure, can there be other structures satisfying the axioms of real numbers? Of course! Just rename the elements of $\mathbb{R}$ and define the addition, the multiplication and the order accordingly to get another structure that satisfies the same axioms. For example, the structure $(\mathbb{R}', +', \times', <', 0', 1')$ defined by

$$\begin{array}{l} \mathbb{R}' = \{0\} \times \mathbb{R} \\ 0' = (0, 0) \\ 1' = (0, 1) \\ (0, r) +' (0, s) = (0, r + s) \\ (0, r) \times' (0, s) = (0, r \times s) \\ (0, r) <' (0, s) \iff r < s \end{array}$$

satisfies the axioms of $\mathbb{R}$.

One might argue that this structure we have just defined is not very different from the old one, that all we did was renaming the element r of $\mathbb{R}$ by $(0, r)$. Indeed... And that is all one can do as we will soon prove.

Note that in the above example, the map $f : \mathbb{R} \longrightarrow \mathbb{R}'$ defined by $f(r) = (0, r)$ is a bijection that has the following properties:

$$\begin{array}{l} f(0) = 0' \\ f(1) = 1' \\ f(r + s) = f(r) +' f(s) \\ f(r \times s) = f(r) \times' f(s) \\ r < s \iff f(r) <' f(s) \end{array}$$

for all r, $s \in \mathbb{R}$.

We will show that if $(\mathbb{R}', +', \times', <', 0', 1')$ is a structure that satisfies all the axioms of real numbers then there is a bijection $f : \mathbb{R} \longrightarrow \mathbb{R}'$ that satisfies the above properties. This means that $\mathbb{R}'$ is just $\mathbb{R}$ with its elements renamed: $r \in \mathbb{R}$ is named $f(r)$. Note also that all the theorems we have proved for $(\mathbb{R}, +, \times, <, 0, 1)$ are also valid for $(\mathbb{R}', +', \times', <', 0', 1')$. In particular $\mathbb{R}'$ has a smallest inductive set $\mathbb{N}'$, contains a dense subset $\mathbb{Q}'$ similar to $\mathbb{Q}$ in $\mathbb{R}$ (Theorem 3.2.2) etc.

Theorem 3.3.1 (Uniqueness of the Real Number System) *Let*

$$(\mathbb{R}', +', \times', <', 0', 1')$$

be a structure that satisfies all the axioms of the real numbers. Then there is a unique nonconstant map $f : \mathbb{R} \longrightarrow \mathbb{R}'$ such that for all r, $s \in \mathbb{R}$,

$$\begin{array}{lll} (1) & f(r+s) = f(r) +' f(s) & (f \text{ is additive}) \\ (2) & f(r \times s) = f(r) \times' f(s) & (f \text{ is multiplicative}) \end{array}$$

Furthermore such a map must be a bijection and must also satisfy

$$\begin{array}{lll} (3) & r < s \iff f(r) <' f(s) & (f \text{ is order preserving}) \\ (4) & f(0) = 0' & \\ (5) & f(1) = 1'. & \end{array}$$

Proof: 1. We first prove that a map that satisfies (1) and (2) must be a bijection that satisfies (3), (4), (5).

1a. $f(0) = 0'$**.** A map f that satisfies (1) must send 0 to $0'$, because $0' +' f(0) = f(0) = f(0+0) = f(0) +' f(0)$ and so by simplifying we get $f(0) = 0'$.

1b. $f(-x) = -f(x)$**.** (Here the sign $-$ on the right hand side stands for the additive inverse in $\mathbb{R}'$ for the binary operation $+'$). Indeed, $0' = f(0) = f(x + (-x)) = f(x) +' f(-x)$ and so $f(-x) = -f(x)$.

1c. f **is one to one.** Assume $f(x) = 0'$ for some $x \in \mathbb{R} \setminus \{0\}$. We will get a contradiction. We compute: $f(1) = f(x^{-1} \times x) = f(x^{-1}) \times' f(x) = f(x^{-1}) \times' 0' = 0'$ and so, for all $r \in \mathbb{R}$, $f(r) = f(r \times 1) = f(r) \times' f(1) = f(r) \times' 0' = 0'$, contradicting the fact that f is nonconstant. Thus $f(x) = 0'$ implies $x = 0$.

We can now show that f is one to one. Assume $f(x) = f(y)$. Then $0' = f(x) -' f(y) = f(x) +' (-f(y)) = f(x) +' f(-y) = f(x + (-y)) = f(x - y)$. By above $x - y = 0$, and $x = y$. Hence f is one to one.

1d. $f(1) = 1'$**.** Since $f(1) = f(1 \times 1) = f(1) \times' f(1)$, $f(1)$ is either $0'$ or $1'$. But since $f(0) = 0'$ and f is one to one, $f(1) = 1'$.

1e. $f(\mathbb{N}) = \mathbb{N}'$**.** We can show by induction on n that $f(n) \in \mathbb{N}'$. Thus $f(\mathbb{N}) \subseteq \mathbb{N}'$. It is also clear that $f(\mathbb{N})$ is an inductive subset of $\mathbb{N}'$. Hence $f(\mathbb{N}) = \mathbb{N}'$.

1f. $f(\mathbb{Q}) = \mathbb{Q}'$**.** By 1e and 1b, $f(\mathbb{Z}) = \mathbb{Z}'$. Let $n/m \in \mathbb{Q}$ with n, $m \in \mathbb{Z}$, $m \neq 0$. Then $f(m) \times' f(n/m) = f(m \times n/m) = f(n) \in f(\mathbb{Z}) = \mathbb{Z}'$. Since $f(m) \in f(\mathbb{Z})$ and since $f(m) \neq 0$ (because $m \neq 0$, see 1c), from this we get $f(n/m) \in \mathbb{Q}'$. Thus $f(\mathbb{Q}) \subseteq f(\mathbb{Q}')$. Since $f(\mathbb{Z}) = \mathbb{Z}'$, it is also easy to show that $f(\mathbb{Q}) = \mathbb{Q}'$.

1g. If $x < y$ **in** $\mathbb{R}$ **then** $f(x) <' f(y)$ **in** $\mathbb{R}'$**.** Note first that, in $\mathbb{R}$, $x \leq y$ if and only if $x + z^2 = y$ for some $z \in \mathbb{R}$. The same statement holds in $\mathbb{R}'$.

Assume x, $y \in \mathbb{R}$ are such that $x < y$. Then there is a $z \in \mathbb{R} \setminus \{0\}$ such that $y = x + z^2$. Therefore $f(y) = f(x) +' f(z)^2$ (here $f(z)^2$ is the squaring in $\mathbb{R}'$, i.e. $f(z)^2 = f(z) \times' f(z)$). Since $f(z) \neq 0'$, we get $f(x) <' f(y)$.

1h. f is onto. Let $r \in \mathbb{R}' \setminus \mathbb{Q}'$. Consider the sets

$$L'_r = \{q \in \mathbb{Q}' : q <' r\} = \mathbb{Q}' \cap (-\infty, r)$$

and

$$R'_r = \{q \in \mathbb{Q}' : r <' q\} = \mathbb{Q}' \cap (r, \infty).$$

The sets L'_r and R'_r partition $\mathbb{Q}'$. Since $f : \mathbb{Q} \longrightarrow \mathbb{Q}'$ is an order preserving bijection, $f^{-1}(L'_r)$ and $f^{-1}(L'_r)$ partition $\mathbb{Q}$ into two nonempty convex subsets (of $\mathbb{Q}$) and every element of $f^{-1}(L'_r)$ is strictly less than every element of $f^{-1}(R'_r)$. Then $\sup(f^{-1}(L'_r)) = \inf(f^{-1}(R'_r)$ (Exercise x, page 36). Let r be this number. Thus $r = \sup(f^{-1}(L'_r)) = \inf(f^{-1}(R'_r)$. We claim that $f(r) = r'$. If $f(r) < r'$, then (because $\mathbb{Q}'$ is dense in $\mathbb{R}$) there is a $q' \in \mathbb{Q}'$ such that $f(r) < q' < r'$. Thus $q' \in L'_r$. Let $q \in \mathbb{Q}$ be such that $f(q) = q'$. Thus $q \in f^{-1}(L'_r)$. So $q < r$, hence $q' = f(q) < f(r)$, a contradiction. Thus $f(r) \geq r'$. Similarly $f(r) \leq r'$. Therefore $f(r) = r'$.

2. Existence. We now prove the existence of the map f.

We define our map f first on $\mathbb{N}$. We let $f(0) = 0'$ and assuming $f(n)$ has been defined for $n \in \mathbb{N}$, we define $f(n+1)$ to be $f(n) +' 1'$. Thus by definition,

$$\begin{aligned} &f(0) = 0' \\ &f(n+1) = f(n) +' 1' \end{aligned}$$

By induction on n, one can prove that for all $n \in \mathbb{N}$, $f(n) \in \mathbb{N}'$. Also $f(\mathbb{N})$ is clearly an inductive subset of $\mathbb{R}$. Hence $f(\mathbb{N}) = \mathbb{N}'$.

We claim that $f : \mathbb{N} \longrightarrow \mathbb{R}'$ is one to one. Assuming $f(n) = f(m)$ for $n, m \in \mathbb{N}$, we will show that $n = m$. We proceed by induction on n. If $n = 0$, then $0' = f(0) = f(n) = f(m)$, therefore, by the very definition of f, m cannot be of the form $k + 1$ for some $k \in \mathbb{N}$, i.e. $m = 0'$. If $n = k + 1$ for some $k \in \mathbb{N}$, then $f(k) +' 1' = f(k+1) = f(n) = f(m)$, therefore $m \neq 0$ and so $m = \ell + 1$ for some $\ell \in \mathbb{N}$. Now $f(k) +' 1' = f(m) = f(\ell + 1) = f(\ell) +' 1'$ and so $f(k) = f(\ell)$. By induction $k = \ell$ and $n = k + 1 = \ell + 1 = m$, proving then $f : \mathbb{N} \longrightarrow \mathbb{R}$ is one to one.

We will denote the image of $n \in \mathbb{N}$ under f by n', i.e. we let $f(n) = n'$.

The proofs that for $n, m \in \mathbb{N}$, $f(n + m) = f(n) +' f(m)$ and $f(n \times m) = f(n) \times' f(m)$ are easy as well and are left as an exercise. From the additivity of f, it follows that f must preserve the order on $\mathbb{N}$.

Now consider the subset

$$\mathbb{Q}' := \{a/b : a \in \mathbb{N}' \text{ and } b \in \mathbb{N}' \setminus \{0\}$$

of $\mathbb{R}'$. Here a/b stands for $a \times' b^{-1}$ and b^{-1} denotes the multiplicative inverse of $b \in \mathbb{R}'$ with respect to $\times'$. We extend $f : \mathbb{N} \longrightarrow \mathbb{R}'$ to $\mathbb{Q}$ by setting $f(n/m) = f(n)/f(m) \in \mathbb{Q}' \subseteq \mathbb{R}'$. We should first check that this map is well-defined, meaning that $n/m = p/q$ for $n, m, p, q \in \mathbb{N}$ should imply $f(n)/f(m) = f(p)/f(q)$. Indeed, if $n/m = p/q$ for $n, m, p, q \in \mathbb{N}$, then $nq = mp$, so $f(n)f(q) = f(nq) = f(mp) = f(m)f(p)$ and $f(n)/f(m) = f(p)/f(q)$. Thus $f : \mathbb{Q} \longrightarrow \mathbb{Q}' \subseteq \mathbb{R}'$. We still denote by f this extended map.

$f : \mathbb{Q} \longrightarrow \mathbb{Q}'$ is certainly onto: If $a/b \in \mathbb{Q}'$ with $a, b \in \mathbb{N}$, then $f(n) = a$ and $f(m) = b$ for some $n, m \in \mathbb{N}$ and so $f(n/m) = f(n)/f(m) = a/b$.

It is also easy to check that $f : \mathbb{Q} \longrightarrow \mathbb{R}'$ is one to one, additive, multiplicative and order preserving.

We now extend f to $\mathbb{R}$. Let $r \in \mathbb{R}$. Consider the disjoint sets

$$L_r = \{q \in \mathbb{Q} : q \leq r\}$$

and

$$R_r = \{q \in \mathbb{Q} : r < q\}.$$

Note that the sets L_r and R_r partition $\mathbb{Q}$ and they do define r, as r is both $\sup L_r$ and $\inf R_r$. Consider the sets $f(L_r)$ and $f(R_r)$. Since $f : \mathbb{Q} \longrightarrow \mathbb{R}$ is order preserving, any element of $f(L_r)$ is strictly less than any element of $f(R_r)$. Also, since $f(\mathbb{Q}) = \mathbb{Q}'$, the sets $f(L_r)$ and $f(R_r)$ partition $\mathbb{Q}'$. Therefore $\sup f(L_r) = \inf f(R_r)$. We let $f(r) = \sup f(L_r) = \inf f(R_r)$.

It is a matter of writing to show that $f : \mathbb{R} \longrightarrow \mathbb{R}'$ satisfies all the required properties.

3. Uniqueness. Assume $f : \mathbb{R} \longrightarrow \mathbb{R}'$ and $g : \mathbb{R} \longrightarrow \mathbb{R}'$ are such maps. Then $g^{-1} \circ f : \mathbb{R} \longrightarrow \mathbb{R}$ is such a map as well. Therefore it is enough to show that a nonconstant map $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfying (1) and (2) (therefore also (3), (4) and (5)) is $Id_{\mathbb{R}}$. One can show quite easily (as above) that f is identity on $\mathbb{Q}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, it follows that $f = Id_{\mathbb{R}}$. $\square$

3.4 Complex Numbers

Let $\mathbb{C} = \mathbb{R} \times \mathbb{R}$. On $\mathbb{C}$ we define two operations called addition and multiplication as follows:

$$\begin{array}{ll} \text{Addition :} & (x, y) + (z, t) = (x + z, y + t) \\ \text{Multiplication :} & (x, y)(z, t) = (xz - yt, xt + yt) \end{array}$$

It is easy to check that the first nine axioms about the addition and multiplication of real numbers do hold in $\mathbb{C}$:

- A1, A2, A3, A4 (with $O_{\mathbb{C}} = (0, 0)$ as the additive identity and $(-x, -y)$ as the additive inverse of (x, y)),
- M1, M2, M3, M4 (with $1_{\mathbb{C}} := (1, 0)$ as the multiplicative identity and $(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2})$ as the multiplicative inverse of the nonzero element (x, y)) and
- D

hold. It can be checked that no order satisfying the axioms O1, O2, O3, OA and OM can be defined on $\mathbb{C}$ (See Exercise x, page 42). Thus $\mathbb{C}$ is a field which is not an ordered field.

The set $\mathbb{C}$ together with the addition and multiplication is called the **field of complex numbers**, each element of $\mathbb{C}$ is called a **complex number**.

The following property is easy to check

$$(*) \qquad (x,y) = (x,0) + (0,y) = (x,0) + (y,0)(0,1).$$

Let us consider the map $\alpha : \mathbb{R} \longrightarrow \mathbb{C}$ given by $\alpha(r) = (r,0)$. Then the following hold:

α is one to one
$\alpha(r+s) = \alpha(r) + \alpha(s)$
$\alpha(0) = 0_{\mathbb{C}}$
$\alpha(-r) = -\alpha(r)$
$\alpha(rs) = \alpha(r)\alpha(s)$
$\alpha(1) = 1_{\mathbb{C}}$
$\alpha(r^{-1}) = \alpha(r)^{-1}$

Thus the map α transports the structure $(\mathbb{R}, +, \times, 0, 1)$ onto the substructure $(\alpha(\mathbb{R}), +, \times, 0_{\mathbb{C}}, 1_{\mathbb{C}})$ of $(\mathbb{C}, +, \times, 0_{\mathbb{C}}, 1_{\mathbb{C}})$; in other words, the only difference between the two structures $(\mathbb{R}, +, \times, 0, 1)$ and $(\alpha(\mathbb{R}), +, \times, 0_{\mathbb{C}}, 1_{\mathbb{C}})$ is the names of the objects, what is called r in the first one is called $(r,0)$ in the second.

The property $(*)$ above now can be written as

$$(**) \qquad (x,y) = \alpha(x) + \alpha(y)(0,1)$$

We also note the following:

$$\alpha(r)(x,y) = (rx, ry).$$

From now on, we will identify $\mathbb{R}$ and its image $\alpha(\mathbb{R})$ via the map α, i.e. we will let $r = \alpha(r) = (r,0)$. Although $r \neq (r,0) = \alpha(r)$, we will make the identification $r = \alpha(r)$ for the sake of notational simplicity. In particular, we identify 0 and $0_{\mathbb{C}}$, as well as 1 with $1_{\mathbb{C}}$. In this way we will see $\mathbb{R}$ as a subset of $\mathbb{C}$. Thus the two properties above are now written as

$$(***) \qquad (x,y) = x + y(0,1)$$

$$r(x,y) = (rx, ry)$$

Let $i = (0,1)$. Note that

$$i^2 = ii = (0,1)(0,1) = (-1,0) = -1,$$

and that for any $(x,y) \in \mathbb{C}$, we have

$$(****) \qquad (x,y) = x + yi$$

From now on, we will represent a complex number z as $x + iy$ (or as $x + yi$) for x, $y \in \mathbb{R}$ rather than as the pair (x,y). Note that, given $z \in \mathbb{C}$, the real numbers x, y for which $z = x + iy$ are unique (this would not have been so if we assumed x and y were in $\mathbb{C}$ rather than in $\mathbb{R}$). The addition and the multiplication of complex numbers with this notation become:

$$\begin{array}{lcl}(x+yi)+(z+ti) & = & (x+z)+(y+t)i \\ (x+yi)(z+ti) & = & (xz-yt)+(xt+yz)i\end{array}$$

As is the custom, we will write $x - yi$ instead of $x + (-y)i$.

The inverse of a nonzero complex number is given by the formula

$$(x+yi)^{-1} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2}i.$$

Conjugation. We consider the following map $: \mathbb{C} \longrightarrow \mathbb{C}$ given by $\overline{x+yi} = x - yi$ (here $x, y \in \mathbb{R}$). For every $\alpha, \beta \in \mathbb{C}$, the following properties are easy to verify:

$$\begin{array}{lcl} \overline{\alpha+\beta} & = & \overline{\alpha}+\overline{\beta} \\ \overline{\alpha\beta} & = & \overline{\alpha}\,\overline{\beta} \\ \overline{\alpha^{-1}} & = & \overline{\alpha}^{-1} \end{array}$$

Also,

$$\overline{\alpha} = \alpha \text{ if and only if } \alpha \in \mathbb{R}.$$

The complex number $\overline{a}$ is called the **conjugate** of α.

Norm. If $\alpha = x + yi$ with $x, y \in \mathbb{R}$ then it is easy to check that

$$\alpha\overline{a} = x^2 + y^2 \in \mathbb{R}^{\geq 0}.$$

Thus we can take its square root. We define

$$|\alpha| := \sqrt{\alpha\overline{\alpha}} = \sqrt{x^2+y^2}.$$

Hence

$$|\alpha|^2 = x^2 + y^2.$$

The nonnegative real number $|\alpha|$ is called the **norm** —indexnorm of a complex number of α. Note that if $\alpha \in \mathbb{R} \subseteq \mathbb{C}$, then the norm of α is equal to the absolute value of α, so that the two meanings that we have given to the notation $|\alpha|$ coincide.

For all $\alpha, \beta \in \mathbb{C}$, we have the following properties:

$$\begin{array}{ll} P_1 & |\alpha| \geq 0 \text{ and } |\alpha| = 0 \text{ if and only if } \alpha = 0 \\ P_2 & |\alpha\beta| = |\alpha|\,|\beta| \end{array}$$

Lemma 3.4.1 *For all $\alpha, \beta \in \mathbb{C}$,*

i. $|\alpha+\beta| \leq |\alpha| + |\beta|$,

ii. $||\alpha| - |\beta|| \leq |\alpha - \beta|$.

Proof: i. On the one hand, $|\alpha+\beta|^2 = (\alpha+\beta)\overline{\alpha+\beta} = (\alpha+\beta)(\overline{\alpha}+\overline{\beta}) = \alpha\overline{a} + \alpha\overline{\beta} + \overline{a}\beta + \beta\overline{\beta} = |\alpha|^2 + \alpha\overline{\beta} + \overline{a}\beta + |\beta|^2$.

On the other hand, $(|\alpha| + |\beta|)^2 = |\alpha|^2 + 2|\alpha||\beta| + |\beta|^2$.

Thus we need to prove that $\alpha\overline{\beta} + \overline{\alpha}\beta \leq 2|\alpha||\beta|$, or that $\alpha\overline{\beta} + \overline{\alpha}\beta \leq 2\sqrt{\alpha\overline{\alpha}\beta\overline{\beta}}$. Setting $\gamma = \alpha\overline{\beta}$, this means that it is enough to prove that $\gamma + \overline{\gamma} \leq 2\sqrt{\gamma\overline{\gamma}}$ for all $\gamma \in \mathbb{C}$. Set $\gamma = x + yi$ where $x, y \in \mathbb{R}$. Then $\gamma + \overline{\gamma} = 2x$ and $\sqrt{\gamma\overline{\gamma}} = \sqrt{x^2+y^2}$. Thus the statement "$\gamma + \overline{\gamma} \leq 2\sqrt{\gamma\overline{\gamma}}$ for all $\gamma \in \mathbb{C}$" is equivalent to the statement

"$x \leq \sqrt{x^2+y^2}$ for all $x, y \in \mathbb{R}$. We will prove this last inequality. Since $x^2 \leq x^2+y^2$, taking the square roots of both sides, we infer $|x| \leq \sqrt{x^2+y^2}$. Thus $x \leq |x| \leq \sqrt{x^2+y^2}$.

ii. From part (i) we have $|\alpha| = |\beta+(\alpha-\beta)| \leq |\beta|+|\alpha-\beta|$. Thus $|\alpha|-|\beta| \leq |\alpha-\beta|$.

Again from part (i), we have $|\beta| = |\alpha+(\beta-\alpha)| \leq |\alpha|+|\beta-\alpha| = |\alpha|+|\alpha-\beta|$. Thus $|\beta|-|\alpha| \leq |\alpha-\beta|$.

These two inequalities mean exactly that $||\alpha|-|\beta|| \leq |\alpha-\beta|$. □

Exercises.

i. Show that $(\frac{\sqrt{3}}{2}+\frac{1}{2}i)^3 = i$.

ii. Show that $(\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}i)^2 = i$.

iii. Show that for any complex number α there is a polynomial $p(X) = aX^2+bX+c \in \mathbb{R}[X]$ such that $p(\alpha) = 0$. (Note: a, b and c should be real numbers).

iv. Show that for every $\alpha \in \mathbb{C}$ there is a $\beta \in \mathbb{C}$ such that $\beta^2 = \alpha$.

v. Let $\alpha, \beta, \gamma \in \mathbb{C}$. Assume that not both α and β are zero. Show that the equation $\alpha x^2+\beta x+\gamma = 0$ has a solution in $\mathbb{C}$. Show that this equation has at most two solutions in $\mathbb{C}$. (Hint: Recall the quadratic formula and its proof).

vi. Show that for every $\alpha \in \mathbb{C}$ and every $n \in \mathbb{N} \setminus \{0\}$, there is a $\beta \in \mathbb{C}$ such that $\beta^{2^n} = \alpha$.

vii. Let $a_0, a_1, \ldots, a_n \in \mathbb{R}$. Show that if $\alpha \in \mathbb{C}$ is a solution of $a_0+a_1x+\ldots+a_nx^n$ then $\overline{\alpha}$ is also a solution of this equation.

viii. Show that there is a bijection between $\{\alpha \in \mathbb{C} : |\alpha| > 1\}$ and $\{\alpha \in \mathbb{C} : 0 < |\alpha| < 1\}$.

ix. For $\alpha, \beta \in \mathbb{C}$, define the **distance,** $d(\alpha,\beta)$ by $d(\alpha,\beta) = |\alpha-\beta|$. Show the following:

i. $d(\alpha,\beta) = 0$ if and only if $\alpha = \beta$,

ii. $d(\alpha,\beta) = d(\beta,\alpha)$,

iii. $d(\alpha,\beta) \leq d(\alpha,\gamma)+d(\gamma,\beta)$.

x. Show that no order satisfying the axioms O1, O2, O3, OA and OM can be defined on $\mathbb{C}$ (Hint: See Corollary 2.4.5).

Chapter 4

Real Vector Spaces

Let $\mathbb{R}^n$ denote the cartesian product of n copies of $\mathbb{R}$, i.e.

$$\mathbb{R}^n = \{(r_1, \dots, r_n) : r_i \in \mathbb{R} \text{ for } i = 1, \dots, n\}.$$

The elements of $\mathbb{R}^n$ will be called **vectors**. We will denote a vector by $\vec{x}$, $\vec{v}$, $\vec{a}$ etc. To denote the vectors we will use the following convention as a rule: $\vec{x} = (x_1, \dots, x_n)$, $\vec{v} = (v_1, \dots, v_n)$, $\vec{a} = (a_1, \dots, a_n)$.

The vector $(0, \dots, 0)$ will be denoted by $\vec{0}$.

Given $\vec{v} = (v_1, \dots, v_n)$, we let $-\vec{v} = (-v_1, \dots, -v_n)$.

Given $\vec{v} = (v_1, \dots, v_n)$ and $\vec{w} = (w_1, \dots, w_n)$, we define the sum $\vec{v} + \vec{w}$ of $\vec{v}$ and $\vec{w}$ as

$$\vec{v} + \vec{w} = (v_1 + w_1, \dots, v_n + w_n).$$

Given $\vec{v} = (v_1, \dots, v_n)$ and $r \in \mathbb{R}$, we define the **scalar multiplication** of $r \in \mathbb{R}$ with the vector $\vec{v}$ as

$$r\vec{v} = (rv_1, \dots, rv_n).$$

With these definitions, letting $V = \mathbb{R}^n$, the following hold:

A1. Additive Associativity. *For any $\vec{x}$, $\vec{y}$, $\vec{z} \in V$, $\vec{x} + (\vec{y} + \vec{z}) = (\vec{x} + \vec{y}) + \vec{z}$.*

A2. Additive Identity Element. *There is a vector $\vec{0}$ such that for any $\vec{x} \in V$, $\vec{x} + \vec{0} = \vec{0} + \vec{x} = \vec{x}$.*

A3. Additive Inverse Element. *For any $\vec{x} \in V$ there is a $\vec{y} \in V$ (namely $-\vec{x}$) such that $\vec{x} + \vec{y} = \vec{y} + \vec{x} = \vec{0}$.*

A4. Commutativity of the Addition. *For any $\vec{x}$, $\vec{y} \in V$, $\vec{x} + \vec{y} = \vec{y} + \vec{x}$.*

B1. Associativity of the Scalar Multiplication. *For any r, $s \in \mathbb{R}$ and $\vec{x} \in V$, $r(s\vec{x}) = (rs)\vec{x}$.*

B2. Distributivity of the Scalar Multiplication 1. *For any* r, $s \in \mathbb{R}$ *and* $\vec{x} \in V$, $(r+s)\vec{x} = r\vec{x} + s\vec{x}$.

B3. Distributivity of the Scalar Multiplication 2. *For any* $r \in \mathbb{R}$ *and* $\vec{x}$, $\vec{y} \in V$, $r(\vec{x}+\vec{y}) = r\vec{x} + r\vec{y}$.

B4. Identity. *For any* $\vec{x} \in V$, $1\vec{x} = \vec{x}$.

Note that the properties A1, A2, A3, A4 are the same as in section 2.1. Thus $(\mathbb{R}^n, +, \vec{0})$ is a commutative group. A set V on which a binary operation $+$ and a "**scalar multiplication**" $\mathbb{R} \times V \longrightarrow V$ that sends a pair $(r, \vec{v})$ of $\mathbb{R} \times V$ to an element of V (denoted by $r\vec{v}$) satisfying the properties A1-A4 and B1-B4 is called a **vector space over** $\mathbb{R}$ or a **real vector space**. More precisely, a real vector space is a triple $(V, +, \mathbb{R} \times V \longrightarrow V)$ satisfying the properties above.

If in the above axioms $\mathbb{R}$ is replaced by a field F, the resulting structure is called a **vector space over** F. Note that any vector space over $\mathbb{R}$ is also a vector space over the field $\mathbb{Q}$ of rational numbers. But in this monograph, we will only need real vector spaces.

As with the real numbers, one can show that the element $\vec{0}$ of a vector space satisfying A2 is unique. One can also show that, given $\vec{x} \in V$, the element $\vec{y}$ that satisfies A3 is unique. We set $\vec{y} = -\vec{x}$. Of course all the consequences of Axioms A1-A4 investigated in Section 2.1 hold. For example, we have $\vec{(x)} = -(-\vec{x})$.

Examples.

i. Let $V = \{(x, y, z, t) \in \mathbb{R}^4 : x - 2y + 3t = 0\}$. Then V is a vector space with the usual componentwise addition and scalar multiplication.

ii. $\mathbb{R}$ itself is a real vector space.

iii. The set $\mathbb{C}$ of complex numbers is a vector space over $\mathbb{R}$.

iv. The singleton set $\{a\}$ is a vector space over $\mathbb{R}$ if we define $a + a = a$ and $ra = a$ for any $r \in \mathbb{R}$. We will denote a by 0 of course.

v. Let $\mathbb{R}[x]$ be the set of real polynomials in x, i.e.

$$\mathbb{R}[x] = \{a_0 + a_1x + \ldots + a_nx^n : a_i \in \mathbb{R}\}.$$

(We will avoid the mathematical definition of the polynomials. Keep in mind only the fact two polynomials $a_0 + a_1x + \ldots + a_nx^n$ and $b_0 + b_1x + \ldots + b_mx^m$ are equal if and only if $n = m$ and $a_i = b_i$ for all $i = 0, \ldots, n$. The precise mathematical definition of polynomials is given in Math 211). We define the addition of two polynomials

$$p(x) = a_0 + a_1x + \ldots + a_nx^n$$

and

$$q(x) = b_0 + b_1x + \ldots + b_mx^m$$

as

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_k + b_k)x^k$$

where $k = \max(n, m)$ and $a_i = 0$ in case $i > n$ and $b_j = 0$ in case $m > j$. For example

$$(1 - 2x^2 + x^3) + (3 + x + 2x^2 + 3x^5 + x^6) = 4 + x + x^3 + 3x^5 + x^6.$$

And we define the scalar multiplication of a real number r with a polynomial $a_0 + a_1x + \ldots + a_nx^n$ as

$$r(a_0 + a_1x + \ldots + a_nx^n) = ra_0 + ra_1x + \ldots + ra_nx^n.$$

With these definitions, it is easy to check that $\mathbb{R}[x]$ satisfies A1-A4 and B1-B4, thus is a real vector space.

2. ¶ Consider the set $\mathbb{R}(x) := \{p/q : p \in \mathbb{R}[x],\ q \in \mathbb{R}[x] \setminus \{0\}\}$ with the convention that $p/q = p_1/q_1$ if and only if $pq_1 = p_1q$. We define the addition and scalar multiplication as expected:

$$\frac{p}{q} + \frac{p_1}{q_1} = \frac{pq_1 + p_1q}{qq_1}$$
$$r\frac{p}{q} = \frac{rp}{q}$$

for p, q, p_1, $q_1 \in \mathbb{R}[x]$ and $r \in \mathbb{R}$.

With these definitions, it is easy to check that $\mathbb{R}(x)$ satisfies A1-A4 and B1-B4, thus is a real vector space. (You may want to check that $\mathbb{R}(x)$ is a field).

vi. Let X be any set and V any real vector space. Let $\mathrm{Func}(X, V)$ be the set of all functions from the set X into V. For two functions f, $g \in \mathrm{Func}(X, V)$ and a real number $r \in \mathbb{R}$, we define the functions $f + g$, $rf \in \mathrm{Func}(X, V)$ as follows

$$(f + g)(x) = f(x) + g(x),$$
$$(rf)(x) = r \cdot f(x),$$

for all $x \in X$. With these definitions the set $\mathrm{Func}(X, V)$ becomes a real vector space.

The element $\vec{0}$ of $\mathrm{Func}(X, V)$ corresponds to the function that sends every element of X to $\vec{0} \in V$.

Given $f \in \mathrm{Func}(X, V)$, the element $-f \in \mathrm{Func}(X, V)$ that satisfies A3 is defined by $(-f)(x) = -f(x)$.

vii. Let V be a vector space. Consider the set $\mathrm{Seq}(V)$ of sequences from V. Thus – by definition – an element $v \in \mathrm{Seq}(V)$ is of the form $(v_n)_{n\in\mathbb{N}}$ where $v_n \in V$ for all $n \in \mathbb{N}$. For two sequences $v = (v_n)_n$ and $w = (w_n)_n$ and a real number r, set $v + w = (v_n + w_n)_n$ and $rv = (rv_n)_n$. Then $\mathrm{Seq}(V)$ becomes a vector space. The zero element $\vec{0}$ of $\mathrm{Seq}(V)$ is the **zero sequence** that consists of zero vectors and $-(v_n)_n = (-v_n)_n$.

viii. Let V be a vector space. Consider the set $\mathrm{Seq}_f(V)$ of sequences from V which are zero after a while. Thus – by definition – an element $v \in \mathrm{Seq}(V)$ is of the form $(v_n)_{n\in\mathbb{N}}$ where $v_n \in V$ for all $n \in \mathbb{N}$ and for which $v_n = \vec{0}$ for n large enough, i.e. for $n \geq N$ for certain N (that depends on v). For two such sequences $v = (v_n)_n$ and $w = (w_n)_n$ and a real number r, set $v + w = (v_n + w_n)_n$ and $rv = (rv_n)_n$. Then $\mathrm{Seq}_f(V)$ becomes a vector space. The zero element $\vec{0}$ is the zero sequence of zero vectors and $-(v_n)_n = (-v_n)_n$.

Exercises. Suggestion: First do Exercise i.

i. Let V be a real vector space. Let W be a subset of V. Suppose that W is closed under addition and scalar multiplication, i.e. suppose that $w + w' \in W$ for all w, $w' \in W$ and $rw \in W$ for all $r \in \mathbb{R}$ and $w \in W$. Show that W is a vector space. Such a subset is called a **subspace** of V.

ii. Are the following sets vector space under the usual addition and scalar multiplication?

$$\begin{array}{l}
\{(x,y,z) \in \mathbb{R}^3 : x + y + z = 0\} \\
\{(x,y,z) \in \mathbb{R}^3 : x + y + z = 1\} \\
\{(x,y,z) \in \mathbb{R}^3 : xyz = 0\} \\
\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 0\} \\
\{(x,y,z) \in \mathbb{R}^3 : x^3 + y^3 + z^3 = 0\} \\
\{(x,y,z) \in \mathbb{R}^3 : xyz \geq 0\} \\
\{(x,y,z) \in \mathbb{R}^3 : x \in \mathbb{Q}\}
\end{array}$$

iii. Which of the following are not vector spaces over $\mathbb{R}$ (with the componentwise addition and scalar multiplication) and why?

$$\begin{array}{l}
V_1 = \{(x,y,z) \in \mathbb{R}^3 : xy \geq 0\} \\
V_2 = \{(x,y,z) \in \mathbb{R}^3 : 3x - 2y + z = 0\} \\
V_3 = \{(x,y,z) \in \mathbb{R}^3 : xyz \in \mathbb{Q}\} \\
V_4 = \{(x,y) \in \mathbb{R}^3 : x + y \geq 0\} \\
V_5 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 0\} \\
V_6 = \{(x,y) \in \mathbb{C}^2 : x^2 + y^2 = 0\}
\end{array}$$

iv. Are the following sets vector space under the usual addition of functions and scalar multiplication of a function with a real number?

$$\begin{array}{l}
\{f \in \mathrm{Func}(X,\mathbb{R}) : f(x) = 0\} \text{ where } x \text{ is a fixed element of } X \\
\{f \in \mathrm{Func}(\mathbb{R},\mathbb{R}) : f(x) = 0 \text{ for any } x \in (0,1)\} \\
\{f \in \mathrm{Func}(\mathbb{R},\mathbb{R}) : f(0) \geq 0\}
\end{array}$$

v. Let V and W be two real vector spaces. Consider the set $\mathrm{Hom}(V,W)$ of functions $f : V \longrightarrow W$ such that $f(v_1 + v_2) = f(v_1) + f(v_2)$ and $f(rv) = rf(v)$ for all v, $v_1, v_2 \in V$, $r \in \mathbb{R}$ Show that $\mathrm{Hom}(V,W)$ is vector space under the usual addition of functions and scalar multiplication (see Example vi, page 45).

vi. ¶ Show that $\mathbb{R}(x)$ is a field. Find all orders on $\mathbb{R}(x)$ that makes it an ordered field.

Chapter 5

Metric Spaces

5.1 Examples.

Real Numbers and Absolute Value. Let us consider the function $d : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^{\geq 0}$ defined by $d(x, y) = |x - y|$. This function has the following properties: For all x, y, $z \in \mathbb{R}$,

i. $d(x, y) = 0$ if and only if $x = y$,

ii. $d(x, y) = d(y, x)$,

iii. $d(x, y) \leq d(x, z) + d(z, y)$.

We will see several examples of sets X together with a function $d : X \times X \longrightarrow \mathbb{R}^{\geq 0}$ satisfying the three properties above.

Euclidean Spaces. For $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ define

$$d(\vec{x}, \vec{y}) = \sqrt{|x_1 - y_1|^2 + \ldots + |x_n - y_n|^2}.$$

We will show that d satisfies the properties i, ii, iii stated above. The first two are immediate. It will take us some time to prove the third equality.

For two vectors $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$, define their **scalar product** as

$$\vec{x}\vec{y} = \sum_{i=1}^{n} x_i y_i.$$

We have for all $\vec{x}$, $\vec{y}$, $\vec{z} \in \mathbb{R}^n$ and α, $\beta \in \mathbb{R}$,

$$\begin{array}{ll} C1. & (\alpha\vec{x} + \beta\vec{y})\vec{z} = \alpha\vec{x}\vec{z} + \beta\vec{y}\vec{z} \\ C2. & \vec{x}\vec{y} = \vec{y}\vec{x} \\ C3. & \vec{x}(\alpha\vec{y} + \beta\vec{z}) = \alpha\vec{x}\vec{y} + \beta\vec{x}\vec{z} \end{array}$$

(C1 and C2 are direct consequences of the definition. C3 follows from C1 and C2.) Note that $\vec{x}\vec{x} \geq 0$, so we can take its square root. Let $\|\vec{x}\| = (\vec{x}\vec{x})^{1/2}$. It is easy to show from the definition that, for all $\vec{x} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we have

$$\begin{array}{ll} C4. & \|\vec{x}\| \geq 0 \text{ ,and } \|\vec{x}\| = 0 \text{ if and only if } \vec{x} = 0 \\ C5. & \|\alpha\vec{x}\| = |\alpha|\|\vec{x}\| \end{array}$$

Now we show that for all $\vec{x} \in \mathbb{R}^n$ we have

$$C6. \quad |\vec{x}\vec{y}| \leq \|\vec{x}\| \, \|\vec{y}\|$$

Indeed, let $\vec{z} = \alpha\vec{x} - \beta\vec{y}$ where $\alpha = \vec{x}\vec{y}$ and $\beta = \|\vec{x}\|^2$. Use $\|\vec{z}\|^2 \geq 0$ to prove C6. Details: $0 \leq (\alpha\vec{x} - \beta\vec{y})(\alpha\vec{x} - \beta\vec{y}) = \alpha^2\vec{x}\vec{x} - 2\alpha\beta\vec{x}\vec{y} + \beta^2\vec{y}\vec{y} = \alpha^2\|\vec{x}\|^2 - 2\alpha\beta\vec{x}\vec{y} + \beta^2\|\vec{y}\|^2 = (\vec{x}\vec{y})^2\|\vec{x}\|^2 - 2(\vec{x}\vec{y})^2\|\vec{x}\|^2 + \|\vec{x}\|^4\|\vec{y}\|^2 = -(\vec{x}\vec{y})^2\|\vec{x}\|^2 + \|\vec{x}\|^4\|\vec{y}\|^2$. Hence $(\vec{x}\vec{y})^2\|\vec{x}\|^2 \leq \|\vec{x}\|^4\|\vec{y}\|^2$. Simplifying we get $(\vec{x}\vec{y})^2 \leq \|\vec{x}\|^2\|\vec{y}\|^2$. Taking the square roots we get $\vec{x}\vec{y} \leq \|\vec{x}\| \, \|\vec{y}\|$. This proves C6.

Now we prove

$$C7. \quad \|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

For this we use C6 and the fact that

$$\|\vec{x} + \vec{y}\|^2 = \|\vec{x}\|^2 + 2\vec{x}\vec{y} + \|\vec{y}\|^2.$$

Details: $\|\vec{x}+\vec{y}\|^2 = (\vec{x}+\vec{y})(\vec{x}+\vec{y}) = \|\vec{x}\|^2 + 2\vec{x}\vec{y} + \|\vec{y}\|^2 \leq \|\vec{x}\|^2 + 2|\vec{x}\vec{y}| + \|\vec{y}\|^2 \overset{A6}{\leq} \|\vec{x}\|^2 + 2\|\vec{x}\| \, \|\vec{y}\| + \|\vec{y}\|^2 = (\|\vec{x}\| + \|\vec{y}\|)^2$. Thus $\|\vec{x}+\vec{y}\|^2 \leq (\|\vec{x}\| + \|\vec{y}\|)^2$. Taking the square roots we finally obtain C7.

We now remark that $d(\vec{x}, \vec{y}) = \|\vec{x}-\vec{y}\|^{1/2}$ and we prove the third property (iii) above: $d(\vec{x}, \vec{z}) = \|\vec{x}-\vec{z}\|^{1/2} = (\|(\vec{x}-\vec{y})+(\vec{y}-\vec{z})\|)^{1/2} \overset{C7}{=} (\|\vec{x}-\vec{y}\|+\|\vec{y}-\vec{z}\|)^{1/2} \leq \|\vec{x}-\vec{y}\|^{1/2} + \|\vec{y}-\vec{z}\|^{1/2} = d(\vec{x}, \vec{y}) + d(\vec{y}, \vec{z})$.

Thus we have proved the following result.

Theorem 5.1.1 *For $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\vec{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ define*

$$d(\vec{x}, \vec{y}) = \sqrt{|x_1 - y_1|^2 + \ldots + |x_n - y_n|^2}.$$

Then the couple $(\mathbb{R}^n, d)$ satisfies the properties i, ii, iii above.

5.2 Definition and Further Examples

A set X together with a function $d : X \times X \longrightarrow \mathbb{R}^{\geq 0}$ is said to be a **metric space** if for all x, y, $z \in X$,

MS1. $d(x, y) = 0$ if and only if $x = y$.

MS2. $d(x, y) = d(y, x)$.

MS3. (**Triangular Inequality.**) $d(x, y) \leq d(x, z) + d(z, y)$.

The function d is called a **metric** or a **distance function** on X.

We will give several examples of metric spaces.

Euclidean Metric. By Theorem 5.1.1, $(\mathbb{R}^n, d)$ where

$$d(\vec{x}, \vec{y}) = \sqrt{|x_1 - y_1|^2 + \ldots + |x_n - y_n|^2}$$

is a metric space, called the **Euclidean metric space**, or sometimes the **usual metric** on $\mathbb{R}^n$.

Metric on a Product. The following theorem will be proven much later (Lemma 14.5.4), when we will give a sense to the term x^r for $x \in \mathbb{R}^{\geq 0}$ and $r \in \mathbb{R} \setminus \{0\}$. In any event, the theorem below makes sense for $p \in \mathbb{N}$ even at this point.

Theorem 5.2.1 *For $i = 1, \ldots, n$, let (X_i, δ_i) be a metric space. Let $X = X_1 \times \ldots \times X_n$. Let $p \geq 1$ be any real number. For x, $y \in X$ let $d_p(x, y) = (\sum_{i=1}^n \delta_i(x_i, y_i)^p)^{1/p}$. Then (X, d_p) is a metric space.*

If we take $X_i = \mathbb{R}$ and $p = 2$ in the theorem above with $d_i(x, y) = |x - y|$, we get the Euclidean metric on $\mathbb{R}^n$.

Unless stated otherwise, when we speak about the product of finitely metric spaces, we will always take $p = 2$.

Induced Metric. Any subset of a metric space is a metric space with the same metric, called the **induced metric**. More precisely we define the induced metric as follows: Let (X, d) be a metric space. Let $Y \subseteq X$. Then $(Y, d_{|Y \times Y})$ is a metric space. We say that the metric of Y is **induced** from that of X.

Sup Metric. For $i = 1, \ldots, n$, let (X_i, d_i) be a metric space and let $X = X_1 \times \ldots \times X_n$. For x, $y \in X$, set

$$d_\infty(x, y) = \max\{d_i(x_i, y_i) : i = 1, \ldots, n\}.$$

Then (X, d_∞) is a metric space as it can be checked easily.

Discrete Metric. Let X be any set. For x, $y \in X$, set

$$d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

Then (X, d) is a metric space. This metric on the set X is called the **discrete metric**.

p-adic Metric. Let $X = \mathbb{Z}$ and p a positive (prime) integer. For distinct x, $y \in \mathbb{Z}$, set $d(x, y) = 1/p^n$ if p^n divides $x - y$ but p^{n-1} does not divide $x - y$. Set $d(x, x) = 0$. Then (X, d) satisfies the properties i, ii and iii stated above. In fact it satisfies the stronger triangular inequality $d(x, y) \leq \max\{d(x, z), d(z, y)\}$. Such a metric is called **ultrametric**. Even further $d(x, y) = \max\{d(x, z), d(z, y)\}$ if $d(x, z) \neq d(y, z)$. It follows that any triangle in this space is isosceles.

Metric on Sequences. Let X be any set. Consider the set $\mathrm{Seq}(X)$ of sequences from X. Thus an element $x \in \mathrm{Seq}(X)$ is of the form $(x_n)_{n\in\mathbb{N}}$ where $x_n \in X$ for all $n \in \mathbb{N}$. For two distinct sequences $x = (x_n)_n$ and $y = (y_n)_n$, set $d(x, y) = 1/2^n$ if n is the first natural number where $x_n \neq y_n$. Then $(\mathrm{Seq}(X), d)$ is a metric space. Note that, in this metric, the maximum distance between two sequences is 1.

We end this subsection with a simple but useful calculation.

Proposition 5.2.2 *In a metric space (X, d), for all x, y, $z \in X$, we have $|d(x, z) - d(z, y)| \le d(x, y)$.*

Proof: We have to show that $d(x, z) - d(z, y) \le d(x, y)$ and $d(z, y) - d(x, z) \le d(x, y)$.

We have $d(x, z) \le d(x, y) + d(y, z) = d(x, y) + d(z, y)$, so the first inequality follows. The second one is similar and is left as an exercise. □

Exercises.

i. Show that the discrete metric is ultrametric.

ii. Show that the metric on the set of sequences defined above is an ultrametric.

5.3 Normed Real Vector Spaces and Banach Spaces and Algebras

Most of what one can say about $\mathbb{R}$ can be said mot à mot about $\mathbb{R}^n$ and $\mathbb{C}$, and most of what one can say about $\mathbb{R}^n$ and $\mathbb{C}$ can be said more generally (mot à mot again) for Banach spaces and algebras, concepts that generalize $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{C}$ and $\mathbb{C}^n$ and that will be defined in this section. Since the proofs are exactly the same, we prefer treat the most general case. Readers who will be psychologically affected by this generality may prefer to read this treatise by considering only the cases of $\mathbb{R}$ and $\mathbb{R}^n$, and sometimes $\mathbb{C}$.

A **normed real vector space** is a vector space V together with a map $|\ | : V \longrightarrow \mathbb{R}$ such that

NVS1. For all $v \in V$, $|v| \ge 0$.

NVS2. For all $v \in V$, $|v| = 0$ if and only if $v = 0$.

NVS3. For all $v \in V$ and $r \in \mathbb{R}$, $|rv| = |r||v|$.

NVS4. For all v, $w \in V$, $|v + w| \le |v| + |w|$.

Examples and Exercises.

i. The Euclidean space $\mathbb{R}^n$ with the usual norm is a normed space.

ii. Consider the set $\oplus_\omega \mathbb{R}$ of real sequences $(r_n)_n$ whose terms are all zero except for finitely many of them. It is clear that $\oplus_\omega \mathbb{R}$ is a real vector space. Define $|(r_n)_n| = \sum_{n=0}^{\infty} r_n^2$. The reader should show that this turns $\oplus_\omega \mathbb{R}$ into a normed vector space.

iii. Let X be a set and $(V, |\ |)$ a normed vector space. Consider the set $\mathcal{B}(X, V)$ of functions from X into Y which are bounded. Thus, a function $f : X \longrightarrow V$ is in $\mathcal{B}(X, V)$ if and only if there is a real number M such that for all $x \in X$, $|f(x)| < M$. Show that $\mathcal{B}(X, V)$ is a vector space.

For $f \in \mathcal{B}(X, V)$, define $|f|$ to be the minimum of the numbers M as above. Thus
$$|f| = \sup\{|f(x)| : x \in X\}.$$

Show that the vector space $\mathcal{B}(X, V)$ together with $|\ |$ defined as above is a normed vector space.

Suppose $V = \mathbb{R}$. Show that if $f, g \in \mathcal{B}(X, \mathbb{R})$, then $f \cdot g \in \mathcal{B}(X, \mathbb{R})$ and that $|f \cdot g| = |f| \cdot |g|$.

iv. Let X be a set and $(V, |\ |)$ a normed vector space. Consider the set $\mathcal{F}(X, V)$ of functions from X into Y. Show that $\mathcal{F}(X, V)$ is a vector space.

For $f \in \mathcal{F}(X, V)$, define
$$|f| = \inf\{\sup\{|f(x)| : x \in X\}, 1\}.$$

Show that the vector space $\mathcal{F}(X, V)$ together with $|\ |$ defined as above is a normed vector space.

Suppose $V = \mathbb{R}$. Is it true that if $f, g \in \mathcal{B}(X, \mathbb{R})$, then $f \cdot g \in \mathcal{F}(X, \mathbb{R})$ and that $|f \cdot g| = |f| \cdot |g|$.

Proposition 5.3.1 *Let $(V, |\ |)$ be a normed real vector space. The map $d(v, w) = |v - w|$ defines a metric on V.*

Proof: Easy. □

A normed real metric space which is complete with respect to the metric defined above is called a **Banach space**.

Define **Banach algebra**.

Problem. Suppose V is a normed real vector space and $W \leq V$ a subspace. Can you make V/W into a normed vector space in a natural way?

Exercises.

i. Show that if V is a Banach space, then so is $\mathcal{B}(X, V)$.

ii. Show that if V is a Banach space, then so is $\mathcal{F}(X, V)$.

5.4 Open Subsets of a Metric Space

Let (X,d) be a metric space, $a \in X$ and $r \in \mathbb{R}$. Then the **open ball** with **center** a and **radius** r is defined as

$$B(a,r) = \{x \in X : d(a,x) < r\}.$$

The **circle** with **center** a and **radius** r is defined as

$$B(a,r) = \{x \in X : d(a,x) = r\}.$$

An arbitrary union of open balls in a metric space is called on **open** subset.

Proposition 5.4.1 *A subset U of a metric space is open if and only if for every $a \in U$ there is an $r \in \mathbb{R}^{>0}$ such that $B(a,r) \subseteq U$.*

Proof: ($\Rightarrow$) Let $a \in U$. Since U is a union of open balls, there is an open ball in U that contains a. Set $a \in B(b,s) \subseteq U$. Let $r = s - d(a,b)$. Then $r > 0$. We claim that $B(a,r) \subseteq B(b,s)$. Indeed, let $x \in B(a,r)$. Then $d(x,b) \leq d(x,a) + d(a,b) < r + d(a,b) = r + (s-r) = s$. Thus $x \in B(b,s)$. This proves the claim. Now $B(a,r) \subseteq B(b,s) \subseteq U$.

($\Leftarrow$) Suppose that for any $a \in U$, there is an $r_a \in \mathbb{R}^{>0}$ such that $B(a,r_a) \subseteq U$. Since $r_a > 0$, $a \in B(a,r_a)$. It follows that $U = \bigcup_{a \in U} B(a,r_a)$. Hence U, being a union of open balls, is open. □

Lemma 5.4.2 *In a metric space, the intersection of two open balls is an open subset.*

Proof: Let $B(a,r)$ and $B(b,s)$ be two open subsets of a metric space (X,d). To show that $B(a,r) \cap B(b,s)$ is open, we will use Proposition 5.4.1. Let $c \in B(a,r) \cap B(b,s)$. Then $r - d(c,a) > 0$ and $s - d(c,b) > 0$. Let

$$\epsilon = \min(r - d(c,a), s - d(c,b)).$$

Then $\epsilon > 0$. We claim that $B(c,\epsilon) \subseteq B(a,r) \cap B(b,s)$. Let $x \in B(c,\epsilon)$. Then $d(a,x) \leq d(a,c) + d(c,x) < d(a,c) + \epsilon \leq d(a,c) + (r - d(a,c)) = r$. Thus $x \in B(a,r)$. Similarly $x \in B(b,s)$. □

Proposition 5.4.3 *Let (X,d) be a metric space. Then,*
i. $\emptyset$ and X are open subsets.
ii. An arbitrary union of open subsets is open.
iii. A finite intersection of open subsets is an open subset.

Proof: i. If $a \in X$, then the ball with center a and radius 0 is the emptyset. Hence the emptyset is open (being an open ball). The whole space X is the union of all the open balls of radius, i.e. $X = \bigcup |x \in XB(x,1)$. Thus X is open.

ii. This is clear by definition of an open subset.

iii. Let U and V be two open subsets. Thus U and V are unions of open balls. Write $U = \bigcup_{a\in A} B(a, r_a)$ and $V = \bigcup_{b\in B} B(b, s_b)$ where A and B are subsets of X and r_a and s_b are real numbers. Now

$$U \cap V = \bigcup_{a\in A} B(a, r_a) \cap \bigcup_{b\in B} B(b, s_b) = \bigcup_{a\in A,\, b\in B} (B(a, r_a) \cap B(b, s_b)).$$

Thus, by part ii, it is enough to show that $B(a, r_a) \cap B(b, s_b)$ is open. But this is the content of Lemma 5.4.2. □

Exercises.

i. Describe $B(\overrightarrow{0}, 1)$ in the three metric spaces $(\mathbb{R}^n, d_1)$, $(\mathbb{R}^n, d_2)$ and $(\mathbb{R}^n, d_\infty)$ defined above.

ii. Consider $\mathbb{R}^n$ with one of the distances d_p $(p \geq 1)$ or d_∞ defined above. Show that any open subset of $\mathbb{R}^n$ is a countable union of open balls.

iii. An **isometry** between two metric spaces (X, d_1) and (Y, d_2) is a bijection $f : X \longrightarrow Y$ such that $d_1(x_1, x_2) = d_2(f(x_1), f(x_2))$. Find all isometries of $\mathbb{R}$ into $\mathbb{R}$.

iv. Find all isometries of $\mathbb{R}^2$ into $\mathbb{R}^2$.

v. Show that the metric spaces $(\mathbb{R}^n, d_p)$ and $(\mathbb{R}^n, d_\infty)$ defined above all have the same open subsets.

vi. Let $X = \{1/2^n : n \in \mathbb{N}\} \subseteq \mathbb{R}$. Consider X with the natural metric (the induced metric). Show that any subset of X is open.

vii. Let $X = \{1/2^n : n \in \mathbb{N}\} \cup \{0\} \subseteq \mathbb{R}$. Consider X with the natural metric (the induced metric). Show that open subsets of X are the cofinite subsets[1] of X and the ones that do not contain 0.

viii. Show that a closed ball $\overline{B}(a, r) = \{x \in \mathbb{R}^n : d(a, x) \leq r\}$ is not open in the Euclidean metric unless $r < 0$.

ix. Let (X, d) be a metric space. Show that $d_1(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a metric. Show that the open subsets in both metrics are the same.

x. Show that $\mathbb{R}$ cannot be the union of two nonempty open subsets (for the usual metric).

xi. Show that $\mathbb{R}^n$ cannot be the union of two nonempty open subsets (for the usual metric).

[1] A subset Y of X is called **cofinite** if $X \setminus Y$ is finite.

Chapter 6

Sequences and Limits

6.1 Definition

Let (X, d) be a metric space, $(x_n)_{n\in\mathbb{N}}$ a sequence in X and $x \in X$. We say that the sequence $(x_n)_{n\in\mathbb{N}}$ **converges** to x if for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for all natural numbers $n > N$. We then write $\lim_{n\to\infty} x_n = x$. We call x the **limit** of the sequence $(x_n)_n$. When the limit of a sequence $(x_n)_n$ exists we say that the sequence $(x_n)_n$ is **convergent** or that it converges, otherwise we say that the sequence is **divergent** or that it diverges.

Note that, above, we used the symbol "∞" without defining it. Above we only defined "$\lim_{x\to\infty} x_n = x$" as if it were one single word. We will never define the symbol ∞, there is no such an object in mathematics.

Examples and Exercises.

i. A sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = x$ for all $n \in \mathbb{N}$ converges to x. Such a sequence is called a **constant sequence**.

ii. A sequence $(x_n)_{n\in\mathbb{N}}$ where $x_n = x$ for all n greater than a certain $n_\circ$ converges to x. Such a sequence is called an **eventually constant sequence**.

iii. If we delete finitely many elements from a sequence or add finitely many elements to a sequence, its divergence and convergence remains unaltered, and its limit (if it exists) does not change.

iv. Let $x_n = n$. Then the sequence $(x_n)_n$ does not converge in the usual metric of $\mathbb{R}$.

v. Let $x_n = (-1)^n$. Then the sequence $(x_n)_n$ does not converge in the usual metric of $\mathbb{R}$.

vi. Let X be a metric space. Let $a_1, \dots, a_k \in X$. Let $(x_n)_n$ be a sequence of X such that for each n, $x_n = a_i$ for some $i = 1, \dots k$. Show that the sequence $(x_n)_n$ converges if and only if it is eventually constant.

vii. Let $x_n = 0$ if n is not a power of 2 and $x_{2^n} = 1$. Then the sequence $(x_n)_n$ does not converge in the usual metric of $\mathbb{R}$.

viii. Consider $\mathbb{Z}$ with the 2-adic metric and set $x_n = 2^n$. Then the sequence $(x_n)_n$ converges to 0.

ix. Consider a set X with its discrete metric. Then a sequence $(x_n)_n$ converges to some element in X if and only if the sequence $(x_n)_n$ is eventually constant.

x. Show that if in Theorem 3.1.12, $\lim_{n\to\infty}(b_n - a_n) = 0$, then $\cap_{n\in\mathbb{N}}[a_n, b_n]$ is a singleton set.

Remarks.

i. In the definition of convergence, ϵ should be thought as a small (but positive) real number. The integer N depends on ϵ. The smaller ϵ is, the larger N should be.

ii. In the definition of convergence, we could replace $d(x_n, x) < \epsilon$ by $d(x_n, x) \leq \epsilon$, i.e. the sequence $(x_n)_{n\in\mathbb{N}}$ **converges** to x if and only if for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $d(x_n, x) \leq \epsilon$ for all natural numbers $n > N$.

iii. In the definition of convergence, since $\mathbb{Q}$ is dense in $\mathbb{R}$, we could take ϵ to range over $\mathbb{Q}$ instead of over $\mathbb{R}$.

iv. The convergence of a sequence depends strongly on the metric. It is possible that a sequences converges to some element for a certain metric, but that this same sequence diverges for some other metric.

v. The notation $\lim_{x\to\infty} x_n = x$ suggests that the limit of a sequence is unique when it exists. This is indeed the case and will be proven right now.

Theorem 6.1.1 *In a metric space, the limit of a sequence, if it exists, is unique.*

Proof: Let (X, d) be a metric space, $(x_n)_n$ a sequence from X and $a, b \in X$. Suppose that a and b are limits of the sequence $(x_n)_n$. We will show that $a = b$ by showing that $d(a, b) < \epsilon$ for any $\epsilon > 0$.

Let $\epsilon > 0$. Since a is a limit of $(x_n)_n$ there is an $N_\circ$ such that $d(x_n, a) < \epsilon/2$ for all $n > N_\circ$. Similarly there is an N_1 such that $d(x_n, b) < \epsilon/2$ for all $n > N_1$. Let $N = \max(N_\circ, N_1) + 1$. Then $d(x_N, a) < \epsilon/2$ and $d(x_N, b) < \epsilon/2$. Hence $d(a, b) \leq d(a, x_N) + d(x_N, b) < \epsilon/2 + \epsilon/2 = \epsilon$. □

Lemma 6.1.2 *In a metric space,* $\lim_{n\to\infty} x_n = x$ *if and only if* $\lim_{n\to\infty} d(x_n, x) = 0$.

Proof: This is a triviality. □

A subset of a metric space is said to be **bounded** if it is contained in a ball. For example $\mathbb{Z}$ is not bounded in $\mathbb{R}$ (with its usual metric). But the set $\{1/2^n : n \in \mathbb{N}\}$ is bounded. A sequence $(x_n)_n$ is called **bounded** if the set $\{x_n : n \in \mathbb{N}\}$ is bounded.

Theorem 6.1.3 *A convergent sequence is bounded.*

Proof: Let $(x_n)_n$ be a sequence converging to some a. Take $\epsilon = 1$ in the definition of the convergence. Thus there exists an N such that $d(x_n, a) < 1$ for any $n > N$. Let $r = \max(d(x_1, a), \ldots, d(x_N, a), 1) + 1$. Then the sequence $(x_n)_n$ is entirely in $B(a, r)$. □

Exercises

i. Show that $\lim_{n\to\infty} 1/n = 0$ in the usual metric of $\mathbb{R}$.

ii. For a natural number n we define $\nu(n) = \sup\{m \in \mathbb{N} : 2^m \leq n\}$. Show that $\lim_{n\to\infty} \nu(n)/n = 0$.

iii. Let $x_n = 0$ if n is not a power of 2 and $x_{2^n} = 1/n$. Show that the sequence $(x_n)_n$ converges to 0 in the usual metric of $\mathbb{R}$.

iv. Show that a sequence of integers converges in the usual metric of $\mathbb{R}$ if and only if it is eventually constant.

v. Let $x_n = \sum_{k=0}^{n} (-1)^{n-k} \begin{pmatrix} n \\ k \end{pmatrix} 2^k$. Does the sequence $(x_n)_n$ converge?

vi. Let $(x_n)_n$ be a real sequence. Assume that there is an $r > 0$ such that for all $n \neq m$, $|x_n - x_m| > r$. Show that the sequence $(x_n)_n$ diverges.

vii. Let (X, d) be a metric space, $(x_n)_n$ a sequence in X and $x \in X$. Show that $\lim_{n\to\infty} x_n = x$ if and only if $\lim_{n\to\infty} d(x_n, x) = 0$.

viii. Let $(a_n)_n$ be a sequence of nonnegative real numbers. Suppose that the sequence $(a_n^2)_n$ converges to a. Show that the sequence $(a_n)_n$ converges to $\sqrt{a}$. Does this hold for sequences which are not nonnegative?

ix. Let X be the set of sequences of zeroes and ones. For two distinct elements $x = (x_n)_n$ and $y = (y_n)_n$ of X define $d(x, y) = 1/2^n$ if n is the least natural number for which $x_n \neq y_n$. Let $d(x, x) = 0$. Then (X, d) is a metric space. Let χ_n be the element of X whose first n coordinates are 1, the rest is 0. Show that $\lim_{n\to\infty} \chi_n$ exists. Find the limit.

x. Let $X_\circ$ be the set of sequences of zeroes and ones with only finitely many ones. Consider $X_\circ$ as a metric space with the metric defined above. Let χ_n be the element of X considered above. Note that $\chi_n \in X_\circ$. Show that $\lim_{n\to\infty} \chi_n$ does not exist in $X_\circ$ (but it exists in X as the above exercise shows).

xi. Let $X = \mathbb{Q}$ with the 3-adic distance. Let $x_n = 1+3+3^2+\ldots+3^n$. Show that $(x_n)_n$ is convergent in $\mathbb{Q}$.

xii. Let $X_\circ = \mathbb{Z}$ with the 3-adic distance. Let $x_n = 1+3+3^2+\ldots+3^n \in X_\circ$. Show that $(x_n)_n$ is not convergent in $\mathbb{Z}$ (but it is in $\mathbb{Q}$).

xiii. Let A be a bounded subset of a metric space (X, d). Show that for any $b \in X$ there is a ball centered at b that contains A.

xiv. Let A and B bounded subsets of $\mathbb{R}^n$ with its usual Euclidean metric. Let

$$A + B = \{a + b : a \in A, b \in B\}$$

and

$$AB = \{ab : a \in A, b \in B\}.$$

Show that $A + B$ and AB are bounded subsets of $\mathbb{R}^n$.

xv. Let A be a bounded subset of $\mathbb{R}$ (with its usual metric) that does not contain 0. Let

$$A^{-1} = \{a^{-1}; a \in A\}.$$

a) Show that A^{-1} is not necessarily bounded.

b) Assume there is an $r > 0$ such that $B(0, r) \cap A = \emptyset$. Show that A^{-1} is bounded.

c) Conversely show that if A^{-1} is bounded then there is an $r > 0$ such that $B(0, r) \cap A = \emptyset$.

xvi. Let $(x_n)_n$ be a sequence of $\mathbb{R}$ (with its usual metric). Show that $(x_n)_n$ is bounded if and only if there is a real number r such that $|x_n| \le r$ for all $n \in \mathbb{N}$.

xvii. Show that any real number is the limit of a rational sequence.

6.2 Examples of Convergence in $\mathbb{R}$ and $\mathbb{C}$

The most important cases for us are the case of $\mathbb{R}$ and $\mathbb{C}$ with their usual metric $d(x, y) = |x - y|$. Translating the definition of convergence of a sequence to this case, we obtain the following special case: A sequence $(x_n)_n$ of real or complex numbers converges to $x \in \mathbb{R}$ if and only if for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_n - x| < \epsilon$ for all natural numbers $n > N$. We know that the limit x, when it exists, is unique.

In this subsection, we prove the convergence and divergence of some important sequences: $(1/n)_{n=1}^{\infty}$, $(\alpha^n)_n$ and $\alpha^n/n!$ for any r. In the meantime we prove a criterion for convergence (Sandwich Lemma).

The following result will be handy.

Theorem 6.2.1 *Let* $\alpha_n \in \mathbb{C}$. *Then* $\lim_{n\to\infty} \alpha_n = 0$ *if and if* $\lim_{n\to\infty} |\alpha_n| = 0$.

Proof: This is an immediate consequence of the definition. $\square$

6.2.1 The Sequence $(1/n)_n$

Let us start by showing that $\lim_{n\to\infty} 1/n = 0$.

Lemma 6.2.2 $\lim_{n\to\infty} 1/n = 0$.

Proof: Let $\epsilon > 0$ be fixed. By Archimedean Property of the real numbers (Theorem 3.1.6), there is a natural number N such that $N\epsilon > 1$. Then $1/N < \epsilon$. Now for all $n > N$, $|1/n - 0| = 1/n < 1/N < \epsilon$. □

Exercises.

i. Prove that $\lim_{n\to\infty} \left(\frac{n+3}{n^2+n-5}\right) = 0$.

ii. Prove that $\lim_{n\to\infty} \left(\frac{n+3}{n^2+n-5}\right)^n = 0$.

iii. Prove that $\lim_{n\to\infty} \left(\frac{n+3}{n^2+n-5}\right)^{\frac{n-1}{3n+2}} = 0$.

iv. Prove that $\lim_{n\to\infty} \left(\frac{n+3}{n^2+n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$.

6.2.2 The Sequence $(\alpha^n)_n$

Proposition 6.2.3 *If $r \in (-1, 1)$ then the sequence $(r^n)_n$ converges to 0. If $r \notin (-1, 1]$ the sequence $(r^n)_n$ diverges.*

Proof: We first claim that if $s > -1$, then for all natural numbers n, $(1+s)^n \geq 1+ns$. We proceed by induction on n to prove the claim. The claim clearly holds for $n = 0$. We now assume the claim holds for n and prove it for $n + 1$. Since $1 + s > 0$, $(1 + s)^{n+1} = (1 + s)^n(1 + s) \geq (1 + ns)(1 + s) = 1 + (1 + n)s + ns^2 \geq 1 + (1 + n)s$. This proves the claim.

We now return to the proof of the proposition. Assume $r \in (-1, 1)$. Since for $r = 0$ the statement is clear, we may assume that $r \neq 0$. Let $\epsilon > 0$ be a real number. We have to show that there exists a natural number N such that for all $n > N$, $|r|^n = |r^n - 0| < \epsilon$. We may therefore assume that $r \geq 0$. Thus $r > 0$. Let $s = 1/r - 1$. Note that $r = \frac{1}{1+s}$. It is also easy to check that $s > 0$. Thus by the claim $(1 + s)^n \geq 1 + ns$. Let $N > 0$ be a natural number such that $1/s \leq N\epsilon$ (Theorem 3.1.6). Now for all $n > N$,

$$|r^n| = r^n = \left(\frac{1}{1+s}\right)^n \leq \frac{1}{1+ns} < \frac{1}{1+Ns} < \frac{1}{Ns} \leq \epsilon.$$

This proves the first part of the proposition.

Claim. Let $r > 1$. Then the sequence $(r^n)_n$ is unbounded.

Proof of the Claim. By the first part of the proposition the sequence $(1/r^n)_n$ converges to 0. Hence for all $\epsilon > 0$ there is an N such that for all

$n > N$, $1/r^n < \epsilon$, i.e. $r^n > 1/\epsilon$. Thus $(r^n)_n$ is unbounded. This proves the claim.

Now by Theorem 6.1.3, $(r^n)_n$ is divergent if $r > 1$. The case $r \leq -1$ is left as an exercise. □

Corollary 6.2.4 *If $r > 1$ then the sequence $(r^n)_n$ is unbounded.*

Corollary 6.2.5 *Let $\alpha \in \mathbb{C}$. Then the sequence $(\alpha^n)_n$ converges to 0 if $|\alpha| < 1$, converges to 1 if $\alpha = 1$, diverges otherwise.*

Proof: If $|\alpha| < 1$ then the result follows from Theorem 6.2.1 and Proposition 6.2.3. If $|\alpha| = 1$ then the statement is clear. Assume $|\alpha| > 1$. Then the sequence $(|\alpha^n|)_n$ is unbounded by Corollary 6.2.4. By Theorem 6.1.3, $(\alpha^n)_n$ is divergent. □

Exercises.

i. Find $\lim_{n\to\infty} \left(\frac{1}{n}\right)^n$.

ii. Find $\lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n$.

6.2.3 The Sequence $(\alpha^n/n!)_n$

Proposition 6.2.6 *For any $r \in \mathbb{R}$, the sequence $(r^n/n!)_n$ converges to 0.*

Proof: We may assume that $r > 0$. Let $\epsilon > 0$. Let $x_n = r^n/n!$. Let $N_\circ$ be a natural number such that $r/N_\circ < 1$ (Theorem 3.1.6). We claim that for each natural number k,

$$x_{N_\circ+k} \leq x_{N_\circ} \left(\frac{r}{N_\circ + 1}\right)^k.$$

We prove this by induction on k. The claim is clear for $k = 0$. Assume it holds for k. Then

$$x_{N_\circ+k+1} = \frac{r^{N_\circ+k+1}}{(N_\circ + k + 1)!} = \frac{r^{N_\circ+k}}{(N_\circ + k)!}\,\frac{r}{N_\circ + k + 1} = x_{N_\circ+k}\frac{r}{N_\circ + k + 1}$$

$$\leq x_{N_\circ} \left(\frac{r}{N_\circ + 1}\right)^k \frac{r}{N_\circ + k + 1} \leq x_{N_\circ} \left(\frac{r}{N_\circ + 1}\right)^{k+1}.$$

This proves the claim.

Since $0 < \frac{r}{N_\circ+1} < 1$, by Proposition 6.2.3, the sequence $((\frac{r}{N_\circ+1})^k)_k$ converges to 0. Therefore there exists an N_1 such that for $k \geq N_1$, $(\frac{r}{N_\circ+1})^k < \epsilon/x_{N_\circ}$.

Thus for $n > N_\circ + N_1$, by letting $k = n - N_\circ - 1$, we see that $x_n < \epsilon$. This proves the proposition. □

Corollary 6.2.7 *For any $\alpha \in \mathbb{C}$, the sequence $(\alpha^n/n!)_n$ converges to 0.*

Proof: Follows from Theorem 6.2.1 and Proposition 6.2.6. □

6.3 Convergence and the Order

We first prove a result that will show that a sequence that is squeezed between two sequences converging to the same number converges.

Lemma 6.3.1 (Sandwich Lemma) *Let $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$ be three real sequences. Suppose that $\lim_{n\to\infty} x_n$ and $\lim_{n\to\infty} z_n$ exist and are equal. Suppose also that $x_n \leq y_n \leq z_n$ eventually (i.e. for $n > N_\circ$ for some $N_\circ$). Then $\lim_{n\to\infty} y_n$ exists and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$.*

Proof: Let a be the common limit of $(x_n)_n$ and $(z_n)_n$. Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = a$ there is an N_1 such that for $n > N_1$, $|x_n - a| < \epsilon/3$. Since $\lim_{n\to\infty} z_n = a$ there is an N_2 such that for $n > N_2$, $|z_n - a| < \epsilon/3$. Let $N = \max(N_\circ, N_1, N_2)$. Then for $n > N$, we have $|y_n - a| \leq |y_n - z_n| + |z_n - a| \leq (z_n - x_n) + |z_n - a| \leq (z_n - a) + (a - x_n) + |z_n - a| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$. □

Corollary 6.3.2 $\lim_{n\to\infty} 1/n^2 = 0$.

Proof: Follows from Lemmas 6.2.2 and 6.3.1. □
Second proof: Let $\epsilon > 0$. Let $N > 1$ be such that $1/2 < N\epsilon$. Then for all $n > N$, $|1/n^2 - 0| = 1/n^2 \leq 1/2n < 1/2N < \epsilon$. □

Corollary 6.3.3 *Let $(x_n)_n$ be a converging sequences and $r \in \mathbb{R}$. Assume that $x_n \geq r$. Then $\lim_{n\to\infty} x_n \geq r$.*

Note that we could well have $x_n > r$ and $\lim_{n\to\infty} x_n = r$, for example take $x_n = r + 1/n$.

Corollary 6.3.4 *Let $(x_n)_n$ and $(y_n)_n$ be two converging sequences. Suppose that $x_n \leq y_n$ for almost all n. Then $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n$.*

Exercises.

i. Find $\lim_{n\to\infty} \left(\frac{1}{n}\right)^n$.

ii. Show that $\lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{n}\right)^n = 0$.

iii. Show that $\lim_{n\to\infty}(\sqrt{n^2 - n} - n) = 1/2$.

iv. Let $(x_n)_n$ be a convergent sequence in $\mathbb{C}$ (or a Banach space X), $r > 0$ and assume that $|x_n| \leq r$ for all n. Show that $|\lim_{n\to\infty} x_n| \leq r$.

v. Let $(x_n)_n$ be a convergent complex sequence. Show that $\lim_{n\to\infty} |x_n| = |\lim_{n\to\infty} x_n|$. (**Solution:** Let x be the limit. Since $||a| - |b|| \leq |a - b|$ for all a, $b \in \mathbb{C}$, $||x_n| - |x|| \leq |x_n - x|$. Thus it is enough to show that the limit of the right hand side is 0, which is obvious.)

6.4 Convergence and the Four Operations

In this subsection we study the behavior of the convergence of real sequences under several operations.

We first show that a converging sequence of $\mathbb{R}$ that does not converge to an element a is "**bounded away from** a". We will need this result later in this subsection.

Proposition 6.4.1 *Let X be a metric space and $a \in X$. Let $(x_n)_n$ be a converging sequence of X that does not converge a. Then there is an $r > 0$ such that $B(a,r) \cap \{x_n : n \in \mathbb{N}\}$ is finite.*

Proof: Let b be the limit of $(x_n)_n$. Let $r = d(a,b)/2$. Then $r > 0$. Since b is the limit of $(x_n)_n$, there exists an N such that $d(x_n, b) < r$ for all $n > N$. Thus, for $n > N$, $d(x_n, a) \geq |d(x_n, b) - d(a, b)| = |d(x_n, b) - 2r| = 2r - d(x_n, b) > r$. (We used Lemma 5.2.2 in the first inequality). Hence $B(a,r) \cap \{x_n : n \in \mathbb{N}\} \subseteq \{x_1, \ldots, x_N\}$. □

Now we analyze the relationship between the four operations on $\mathbb{R}$ and the convergence of sequences.

Theorem 6.4.2 *Let $(x_n)_n$ and $(y_n)_n$ be sequences converging to a and b respectively. Then the sequences $(x_n + y_n)_n$ and $(x_n y_n)_n$ converge to $a+b$ and ab respectively. In particular, for $r \in \mathbb{R}$ the sequence $(rx_n)_n$ converges to ra.*

Proof: $(+)$. Let $\epsilon > 0$. Since the sequence $(x_n)_n$ converges to a, there is an $N_\circ$ such that $|x_n - a| < \epsilon/2$. Since the sequence $(y_n)_n$ converges to b, there is an N_1 such that $|x_n - a| < \epsilon/2$. Let $N = \max(N_\circ, N_1)$. Then for $n > N$, $|(x_n + y_n) - (a+b)| = |(x_n - a) + (y_n - b)| \leq |x_n - a| + |y_n - b| < \epsilon/2 + \epsilon/2 = \epsilon$. Thus $(x_n + y_n)_n$ converges to $a + b$.

$(\times)$. Assume first $b = 0$. We have to show that $\lim |n \to \infty x_n y_n = 0$. Let $\epsilon > 0$. By Exercise xvi, page 60, there exists an $r > 0$ such that $|x_n| \leq r$ for all n. Since $(y_n)_n$ converges to 0, there exists an N such that for all $n > N$, $|y_n| = |y_n - 0| < \frac{\epsilon}{r}$. Now for $n > N$, we have $|x_n y_n - 0| = |x_n y_n| = |x_n||y_n| < r\frac{\epsilon}{r} = \epsilon$. Thus the sequence $(x_n y_n)_n$ converges to 0.

Assume next $b \neq 0$. Let $\epsilon > 0$. Since $b \neq 0$ and $(x_n)_n$ converges to a,, there exists an $N_\circ$ such that $|x_n - a| \leq \frac{\epsilon}{2|b|}$ for all $n > N_\circ$. Since $(x_n)_n$ is a convergent sequence, it is bounded by Theorem 6.1.3. By Exercise xvi, page 60, there is an $M > 0$ such that $|x_n| < M$ for all $n \in \mathbb{N}$. Since $(y_n)_n$ converges to b, there exists an N_1 such that for $n > N_1$, $|y_n - b| < \frac{\epsilon}{2M}$. Let $N = \max(N_\circ, N_1)$. For $n > N$, we have $|x_n y_n - ab| = |x_n(y_n - b) + (x_n - a)b| \leq |x_n(y_n - b)| + |(x_n - a)b| \leq |x_n||y_n - b| + |x_n - a||b| \leq M\frac{\epsilon}{2M} + \frac{\epsilon}{2|b|}|b| = \epsilon$. Thus the sequence $(x_n y_n)_n$ converges to ab.

For the last part: Take $y_n = r$ for all n apply the above part. □

Corollary 6.4.3 *Let $(x_i)_i$ and $(y_i)_i$ be sequences converging to a and b respectively. Then the sequence $(x_i - y_i)_i$ converges to $a - b$.*

Corollary 6.4.4 *Let $(x_i)_i$ be a sequence converging to a. Let $k \in \mathbb{N}\setminus\{0\}$. Then the sequence $(x_i^k)_i$ converges to a^k.*

Corollary 6.4.5 *Let $(x_i)_i$ and $(y_i)_i$ be sequences converging to a and b respectively. Let α, $\beta \in \mathbb{C}$. Then the sequence $(\alpha x_i + \beta y_i)_i$ converges to $\alpha a - \beta b$.*

Corollary 6.4.6 *Let $(x_i)_i$ be a sequence converging to a. Let $a_\circ, a_1, \ldots, a_k \in \mathbb{C}$. Then the sequence $(a_\circ + a_1 x_i + \ldots + a_k x_i^k)_i$ converges to $a_\circ + a_1 a + \ldots + a_k a^k$.*

If $a_\circ, a_1, \ldots, a_k \in \mathbb{C}$, then a term of the form $a_0 + a_1 x + \ldots + a_k x^k$ is called a (complex) **polynomial**. The numbers $a_\circ, a_1, \ldots, a_k$ are called its **coefficients**. If $a_k \neq 0$, then the natural number k is called the **degree** of the polynomial $p(x)$. A polynomial $p(x) = a_0 + a_1 x + \ldots + a_k x^k$ gives rise to a function from $\mathbb{C}$ into $\mathbb{C}$ whose value at $\alpha \in \mathbb{C}$ is $p(r) := a_0 + a_1\alpha + \ldots + a_k \alpha^k$. Thus the corollary above can be translated as follows:

Corollary 6.4.7 *Let $(x_n)_n$ be a sequence converging to α. Let $p(x) \in \mathbb{R}[x]$. Then the sequence $(p(x_n))_n$ converges to $p(\alpha)$.*

In general, if the sequence $(x_n)_n$ is convergent and $x_n \neq 0$ for all n, then the sequence $(x_n^{-1})_n$ is not necessarily convergent. This is what happens if we take $x_n = 1/n$. However if the limit of $(x_n)_n$ is not zero, then the convergence of the sequence $(x_n^{-1})_n$ is guaranteed as the following theorem shows:

Theorem 6.4.8 *Let $(x_n)_n$ be a sequence of $\mathbb{C}$ converging to a nonzero number α. Assume $x_n \neq 0$ for all n. Then the sequence $(x_n^{-1})_n$ converges to α^{-1}.*

Proof: Let $\epsilon > 0$. By Proposition 6.4.1, there is an $r > 0$ such that $B(0, r) \cap \{x_n : n \in \mathbb{N}\}$ is finite. Let $N_\circ$ be such that $|x_n| \geq r$ for all $n > N_\circ$. Since a and r are nonzero and since the sequence $(x_n)_n$ converges to α, there exists an N_1 such that $|x_n - \alpha| < \epsilon$ for all $n > N_1$. Let $N = \max(N_\circ, N_1)$. Now for $n > N$ we have,

$$|x_n^{-1} - a^{-1}| = \frac{|x_n - a|}{|a||x_n|} \leq \frac{|x_n - a|}{|a|r} < \frac{\epsilon|a|r}{|a|r} = \epsilon.$$

Thus the sequence $(x_n^{-1})_n$ converges to α^{-1}. □

Corollary 6.4.9 *Let $(x_n)_n$ and $(y_n)_n$ be sequences converging to α and β respectively. Assume $\beta \neq 0$ and that $y_n \neq 0$ for all n. Then the sequence $(x_n/y_n)_n$ converges to α/β.*

Corollary 6.4.10 *Let $p(x)$ and $q(x) \neq 0$ be two polynomials of degree d and e respectively. Then the sequence $(p(n)/q(n))_n$ converges if and only if $e \geq f$. If $e > f$, then the limit is zero. If $e = f$ then the limit is α/β where α and β are the leading coefficients of $p(x)$ and $q(x)$ respectively.*

Corollary 6.4.11 *Assume $\lim_{n\to\infty} a_n$ exists and is nonzero. Then the sequence $(a_n/a_{n+1})_n$ converges to 1.*

The sequence $(a_n/a_{n+1})_n$ may not converge if $\lim_{n\to\infty} a_n = 0$. For example, choose

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1/n^2 & \text{if } n \text{ is odd} \end{cases}$$

Clearly $\lim_{n\to\infty} a_n = 0$, but

$$\frac{a_n}{a_{n+1}} = \begin{cases} \frac{(n+1)^2}{n} & \text{if } n \text{ is even} \\ \\ \frac{n+1}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

And the subsequence $\frac{(n+1)^2}{n}$ diverges to ∞, although the subsequence $\frac{n+1}{n^2}$ converges to 0.

Corollary 6.4.12 *Let $(x_n)_n$ be a sequence converging to α. Let $p(x), q(x) \in \mathbb{C}[x]$. Assume that $q(a) \neq 0$. Then the sequence $(p(x_n)/q(x_n)_n$ converges to $p(\alpha)/q(\alpha)$.*

Exercises.

i. Find the following limits and prove your result using only the definition of convergence.
   a. $\lim_{n\to\infty} \frac{2n-5}{5n+2}$
   b. $\lim_{n\to\infty} \frac{2n-5}{-n+2}$
   c. $\lim_{n\to\infty} \frac{2n-5}{-n^2+2}$

ii. Let $-1 < r \leq 1$. Show that the sequence $(r^n/n)_n$ converges to 0.

iii. Show that
   a. $\lim_{n\to\infty} \frac{n+3}{n^2+3n+2} = 0$.
   b. $\lim_{n\to\infty} \left(\frac{n+3}{n^2+n-5}\right)^n = 0$.
   c. $\lim_{n\to\infty} \left(\frac{n+3}{n^2+n-5}\right)^{\frac{n-1}{3n+2}} = 0$.
   d. $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$.

iv. Show that if a, $b > 0$ then $\lim_{n\to\infty}(a^n + b^n)^{\frac{1}{n}} = \max(a, b)$.

v. Let $(a_n)_n$ and $(b_n)_n$ be two real sequences that converge to two different numbers in the usual metric. Show that $\{a_n : n \in \mathbb{N}\} \cap \{b_n : n \in \mathbb{N}\}$ is finite.

vi. Show that the set of convergent sequences of $\mathbb{R}$ (in the usual metric) is a real vector space. (For the addition of sequences and scalar multiplication of a sequence by a real number, see Example vii, page 45).

vii. Show that the set of sequences of $\mathbb{R}$ (in the usual metric) that converges 0 is a real vector space.

viii. Show that the set of bounded sequences of $\mathbb{R}$ (in the usual metric) is a real vector space.

6.5 More On Sequences

Lemma 6.5.1 *A subsequence of a convergent sequence is also convergent; furthermore the limits are equal.*

Proof: Let $(x_n)_n$ converge to x and let $(y_i)_i = (x_{n_i})_i$ be a subsequence of $(x_n)_n$. We will show that $\lim_{i\to\infty} y_i = x$ also. Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, there is an N such that for all $i > N$, $d(x_i, x) < \epsilon$. Since the sequence $(n_i)_i$ is strictly increasing, $n_i \geq i$ (one can prove this by induction on i). Thus for all $i > N$, $n_i > N$ also and hence $d(y_i, x) = d(x_{n_i}, x) < \epsilon$. □

Limit in the Topological Language. We can translate the definition of a limit of a sequence in a language that involves only open subsets (rather than the metric). This is what we will do now:

Theorem 6.5.2 *Let (X, d) be a metric space, $(x_n)_n$ a sequence from X and $x \in X$. Then $\lim_{n\to\infty} x_n = x$ if and only if for any open subset U containing x, the sequence $(x_n)_n$ is eventually in U, i.e. there is an N such that $x_n \in U$ for all $n > N$.*

Proof: ($\Rightarrow$) Let U be an open subset containing x. By Proposition 5.4.1, there is an $\epsilon > 0$ such that $B(x, \epsilon) \subseteq U$. Since $\lim_{n\to\infty} x_n = x$, there is an N such that for all $n > N$, $d(x_n, x) < \epsilon$, i.e. $x_n \in B(x, \epsilon) \subseteq U$.

($\Leftarrow$) Let $\epsilon > 0$. Then $B(x, \epsilon)$ is open. Thus, by hypothesis, there is an N such that for all $n > N$, $x_n \in B(x, \epsilon)$, i.e. $d(x_n, x) < \epsilon$. Hence $\lim_{n\to\infty} x_n = x$ by definition of limits. □

Exercises.

i. Let $(a_n)_n$ be a convergent sequence of real numbers.

  a. Does the sequence $(a_{2n})_n$ converge necessarily?

  b. Assume $a_n \neq 0$. Does the sequence $(a_n/a_{n+1})_n$ converge necessarily?

ii. Let $x_{n,m} = \frac{n}{n+m}$. Find $\lim_{n\to\infty} \lim_{m\to\infty} x_{n,m}$ and $\lim_{m\to\infty} \lim_{n\to\infty} x_{n,m}$.

iii. Let $(x_n)_n$ and $(y_n)_n$ be two sequences. Define

$$z_n = \begin{cases} x_{n/2} & \text{if } n \text{ is even} \\ y_{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

Show that $(z_n)_n$ converges if and only if $(x_n)_n$ and $(y_n)_n$ both converge to the same number.

iv. Let $(x_n)_n$ be a real sequence and $(y_n)_n$ a complex sequence. Let $y \in \mathbb{C}$. Assume that $\lim_{n\to\infty} x_n = 0$ and $|y_n - y| \le x_n$ for all n. Show that $\lim_{n\to\infty} y_n = y$.

v. Let $(x_n)_n$ be a convergent sequence of real numbers. Let

$$y_n = \frac{x_1 + \ldots + x_n}{n}.$$

Show that $\lim_{n\to\infty} y_n$ exists and is equal to $\lim_{n\to\infty} x_n$.

6.6 Cauchy Sequences

The elements of a convergent sequence, since they approach to a fixed element, should also approach to each other. A sequence that has this property is called a **Cauchy sequence**. We formalize this as follows: A sequence $(x_n)_n$ is called a **Cauchy sequence** if for any $\epsilon > 0$ there is an N such that $d(x_n, x_m) < \epsilon$ for all n, $m > N$. We formalize this by writing $\lim_{n,m\to\infty} d(x_n, x_m) = 0$.

We first prove what we have said in a mathematical way:

Theorem 6.6.1 *A convergent sequence is a Cauchy sequence.*

Proof: Let $(x_n)_n$ be a sequence converging to x. Let $\epsilon > 0$. Since $\lim_{n\to\infty} x_n = x$, there is an N such that $d(x_n, x) < \epsilon/2$ for all $n > N$. Now if n, $m > N$, then $d(x_n, x_m) \le d(x_n, x) + d(x, x_m) < \epsilon/2 + \epsilon/2 = \epsilon$. □

On the other hand, a Cauchy sequence need not be convergent (see Exercises x, page 59 and xii, page 60). But we have the following:

Proposition 6.6.2 *A Cauchy sequence that has a convergent subsequence is convergent. Furthermore the limits are the same.*

Proof: Let $(x_n)_n$ be a Cauchy sequence. Let $(y_i)_i = (x_{n_i})_i$ be a convergent subsequence of $(x_n)_n$. Say $\lim_{i\to\infty} y_i = x$. We will show that $\lim_{n\to\infty} x_n = x$. Let $\epsilon > 0$. Since $(x_n)_n$ is a Cauchy sequence, there is an $N_\circ$ such that if n, $m > N_\circ$, then $d(x_n, x_m) < \epsilon/2$. Also, since $\lim_{i\to\infty} y_i = x$, there is an N_1 such that if $i > N_1$ then $d(y_i, x) < \epsilon/2$. Let $N = \max(N_\circ, N_1)$. Then for all $i > N$, we have $n_i \ge i > N$ (because the sequence $(n_i)_i$ is strictly increasing) and so $d(x_i, x) \le d(x_i, y_i) + d(y_i, x) = d(x_i, x_{n_i}) + d(y_i, x) < \epsilon/2 + \epsilon/2 = \epsilon$. □

Proposition 6.6.3 *A Cauchy sequence is bounded.*

Proof: Let $(x_n)_n$ be Cauchy sequence. Take $\epsilon = 1$ in the definition. Thus there exists an N such that for all n, $m > N$, $d(x_n, x_m) < 1$. In particular, for all $n > N$, $d(x_{N+1}, x_n) < 1$. Let

$$r = \max(1, d(x_0, x_{N+1}) + 1, \ldots, d(x_N, x_{N+1}) + 1).$$

Then $\{x_n : n \in \mathbb{N}\} \subseteq B(x_{N+1}, r)$. □

Exercises.

i. Let $(a_n)_n$ and $(b_n)_n$ be two Cauchy sequences such that the set $\{a_n : n \in \mathbb{N}\} \cap \{b_n : n \in \mathbb{N}\}$ is infinite. Show that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

ii. Give an example of each of the following, or argue that such a request is impossible.

a) A Cauchy sequence that is not monotone.

b) A monotone sequence that is not Cauchy.

c) A Cauchy sequence with a divergent subsequence.

d) An unbounded sequence with a Cauchy subsequence.

iii. Let $X_1, \ldots, X_m$ be metric spaces. Let $X = X_1 \times \ldots \times X_n$ be the product space. Let $(x_n)_n$ be a sequence in X. Set $x_n = (x_{n1}, \ldots, x_{nm})$. Then $(x_n)_n$ is Cauchy if and only if $x_{ni})_n$ is Cauchy for all $i = 1, \ldots, m$.

iv. Let $x_1 = 1$, $x_2 = 2$ and $x_n = (x_{n-1} + x_{n-2})/2$ for $n > 2$.

a. Show that $1 \leq x_n \leq 2$ for all n. (Hint: By induction on n).

b. Show that $|x_n - x_{n+1}| = 1/2^{n-1}$ for all n. (Hint: By induction on n).

c. Show that if $m > n$ then $x_n - x_m < 1/2^{n-2}$ for all n. (Hint: Use part c).

d. Show that $(x_n)_n$ is a Cauchy sequence.

e. Find its limit. Note that the sequence is formed by taking the arithmetic mean of the previous two terms. It can be guessed that the limit is at the two thirds of the way from x1 = 1 to x2 = 2, i.e. it is 5/3. This can be shown by some elementary linear algebra. But there is an easier way: Note that part b can be sharpened into $x_n - x_{n+1} = (-1)^n/2^{n-1}$, thus $x_{n+1} - 1 = x_{n+1} - x_1 = (x_{n+1} - x_n) + ... + (x_2 - x_1) = 1 - 1/2 + 1/4 - ... + (-1)^n 1/2^{n-1}$. It follows that the limit x, that we know it exists from the previous question, is equal to the infinite sum $x = 1 + 1 - 1/2 + 1/4 - \ldots + (-1)^n 1/2^{n-1} + \ldots$ Now just remarque that 2x = 2 + 2-1 + 1/2-1/4 + $\ldots$ = 3 + (1/2-1/4 + 1/8 ...) = 3 + (2-x) and so $x = 5/3$.

v. We say that a sequence $(x_n)_n$ is **contractive** if there is a constant c, $0 < c < 1$, such that $|x_{n+2} - x_{n+1} \leq c|x_{n+1} - x_n|$ for all n. Show that every contractive sequence is convergent. Hint: $|x_{n+1} - x_n| \leq c^n|x_1 - x_0|$. Thus we can estimate $|x_n - x_m|$ as is done above and show that such a sequence is Cauchy.

6.7 Convergence of Real Cauchy Sequences

The purpose of this subsection is to prove the following important theorem:

Theorem 6.7.1 *Every real Cauchy sequence is convergent.*

Recall that a sequence $(x_n)_n$ of real numbers is called **nondecreasing** (resp. nonincreasing) if $x_n \leq x_m$ (resp. $x_n \geq x_m$) for all $n \leq m$. A nonincreasing or nondecreasing sequence is called a **monotone sequence**.

To prove this theorem, we first prove that every monotone and bounded sequence converges.

Theorem 6.7.2 (Monotone Convergence Theorem) *A monotone and bounded sequence of* $\mathbb{R}$ *converges.*

Proof: Let $(x_n)_n$ be an increasing sequence of real numbers which is bounded above. Then the set $\{x_n : n \in \mathbb{N}\}$ has a least upper bound ℓ. We claim that $\lim_{x \longrightarrow \infty} x_n = \ell$. Let $\epsilon > 0$. Since $\ell - \epsilon$ is not an upper bound, there is an N such that $\ell - \epsilon < x_N$. Since the sequence $(x_n)_n$ is increasing, $\ell - \epsilon < x_n$ for all $n \geq N$. Since ℓ is an upper bound, $\ell - \epsilon < x_n \leq \ell$ for all $n \geq N$. Thus $|\ell - x_n| \leq \epsilon$ for all $n \geq N$.

The second part follows easily from the first. □

Theorem 6.7.3 *Every real sequence has either an increasing or a nonincreasing subsequence.*

Proof: We will choose an increasing sequence $(n_i)_i$ such that given any i, either $x_{n_j} > x_{n_i}$ for all $j > i$ or either $x_{n_j} \leq x_{n_i}$ for all $j > i$.

Let $n_\circ = 0$.

Consider the sets

$$G_0 = \{n > 0 : x_n > x_0\}$$

and

$$L_0 = \{n > 0 : x_n \leq x_0\}.$$

Either G_0 or L_0 is infinite. If G_0 is infinite, consider only the subsequence that consists of x_0 and the x_n's for which $n \in G_0$, deleting the rest. If G_0 is finite, consider only the subsequence that consists of x_0 and the x_n's for which $n \in L_0$, deleting the rest.

Renaming the elements, assume that $(y_n)_n$ is this new sequence. In this new sequence y_0 has the following property: "Either $y_n > y_0$ for all $n > 0$ or $y_n \leq y_0$ for all $n > 0$".

Now consider y_1 and the sets

$$G_1 = \{n > 1 : y_n > y_1\}$$

and

$$L_1 = \{n > 1 : y_n \leq y_1\}.$$

Either G_1 or L_1 is infinite. If G_1 is infinite, consider only the subsequence that consists of y_0, y_1 and the y_n's for which $n \in G_1$, deleting the rest. If G_1 is finite, consider only the subsequence that consists of y_0, y_1 and the y_n's for which $n \in L_1$, deleting the rest.

Renaming the elements, assume that $(y_n)_n$ is this new sequence. In this new sequence y_0 has the following property: "Either $y_n > y_0$ for all $n > 0$ or $y_n \leq y_0$

for all $n > 0$" and y_1 has the following property: "Either $y_n > y_1$ for all $n > 1$ or $y_n \leq y_1$ for all $n > 1$".

Next we consider y_2 and the sets

$$G_2 = \{n > 2 : y_n > y_2\}$$

and

$$L_2 = \{n > 2 : y_n \leq y_2\}.$$

Either G_2 or L_2 is infinite. If G_2 is infinite, consider only the subsequence that consists of y_0, y_1, y_2 and the y_n's for which $n \in G_2$, deleting the rest. If G_2 is finite, consider only the subsequence that consists of y_0, y_1, y_2 and the y_n's for which $n \in L_2$, deleting the rest.

Continuing this way, we find a subsequence, say $(z_n)_n$, of the original sequence $(x_n)_n$ such that for all n, either for all $m > n$, $x_m > x_n$ or for all $m > n$, $x_m \leq x_n$.

We call n red if the first case occurs, i.e. if for all $m > n$, $x_m > x_n$. We call n green if the second case occurs, i.e. if for all $m > n$, $x_m \leq x_n$.

Thus we gave a color, either red or green, to each natural number. If there are infinitely many red numbers, then the sequence $(z_n)_{n \text{ is red}}$ is increasing. If there are infinitely many green numbers, then the sequence $(z_n)_{n \text{ is green}}$ is nonincreasing. □

Proof of Theorem 6.7.1. Let $(x_n)_n$ be a real Cauchy sequence. By Proposition 6.6.3, $(x_n)_n$ is bounded. By Theorem 6.7.3, $(x_n)_n$ has a monotone subsequence, say $(y_n)_n$. The subsequence $(y_n)_n$ is still bounded. By Theorem 6.7.2, $(y_n)_n$ converges. By Proposition 6.6.2, $(x_n)_n$ converges. □

Theorem 6.7.1 can also be proven as a consequence of the following important result:

Theorem 6.7.4 (Bolzano-Weierstrass Theorem) *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof: Let $(x_m)_m$ be a bounded sequence in $\mathbb{R}$. Choose a, $b \in \mathbb{R}$ such that $x_m \in [a, b]$ for all m. We will choose a Cauchy subsequence (x_{m_i}). Let $m_0 = 0$. Separate the interval $[a, b]$ into two parts $[a, \frac{a+b}{2}]$ and $[\frac{a+b}{2}, b]$. One of these two smaller intervals contains x_m for infinitely many m. Let m_1 to be the smallest index not equal to m_0 which is in this smaller interval that contains x_m for infinitely many m. Now do the same with this smaller interval. Thus the sequence $y_i = x_{m_i}$ has the property that for all $i > j$, $d(y_i, y_j) \leq (b-a)/2^j$.

By the Theorem of Nested Intervals 3.1.12, the intersection of these intervals of length $(b-a)/2^j$ is nonempty. If y is in the intersection, then $\lim_{i\to\infty} y_i = y$. □

Note that this theorem is false in general metric space: In a discrete metric space a sequence of distinct elements does not have a Cauchy subsequence at all.

Second Proof of Theorem 6.7.1. Let $(x_n)_n$ be a Cauchy sequence in $\mathbb{R}^n$. By Proposition 6.6.3, $(x_n)_n$ is bounded. By Theorem 6.7.4, $(x_n)_n$ has a convergent subsequence. By Proposition 6.6.2, $(x_n)_n$ converges. □

Exercises.

i. Let $x_n = 1/1^2 + 1/2^2 + 1/3^2 + \ldots + 1/n^2$.

   a. Show that for all integers $n \geq 1$, $x_n \leq 2 - 1/n < 2$.

   b. Conclude that the sequence $(x_n)_n$ converges.

   c. Show that for an integer n large enough, $n^2 \leq 2^n$.

   d. Conclude that the sequence $(x_n)_n$ is bounded above by $1 + 1/4 + 1/9 + 1/8$ $(= 107/72)$.

   We will see later that this sequence converges to $\pi^2/6$.

ii. Show that the sequence $\sqrt{2}$, $\sqrt{2\sqrt{2}}$, $\sqrt{2\sqrt{2\sqrt{2}}}$ converges. Find its limit.

iii. Discuss the convergence of the sequence $\sqrt{a}$, $\sqrt{a\sqrt{a}}$, $\sqrt{a\sqrt{a\sqrt{a}}}$ for $a \imath \mathbb{R}^{\geq 0}$.

iv. Show that if $k \geq 2$ is an integer, the sequence $(x_n)_n$ where $x_n = 1/1^k + 1/2^k + 1/3^k + \ldots + 1/n^k$ is Cauchy and hence converges.

v. Let $x_0 = x$ and $x_{n+1} = \frac{1}{4 - x_n}$.

   a) Assume that the sequence converges. Show that the limit is either $2 + \sqrt{3}$ or $2 - \sqrt{3}$.

   b) Show that if $2 - \sqrt{3} \leq x_n \leq 2 + \sqrt{3}$ then $2 - \sqrt{3} \leq x_{n+1} \leq 2 + \sqrt{3}$.

   c) Assume $x \in [2 - \sqrt{2}, 2 + \sqrt{2}]$. Show that $(x_n)_n$ is decreasing. Conclude that $\lim_{n\to\infty} x_n = 2 - \sqrt{2}$.

   d) Discuss the convergence and the divergence of $(x_n)_n$ for other values of x.

vi. Discuss the convergence of the sequence $(x_n)_n$ defined by $x_0 = x$ and $x_{n+1} = 4 - 1/x_n$ in terms of x.

vii. Let $x_1 = x$ and $x_{n+1} = \sqrt{2x_n}$.

   a) Write the first four terms of x_n when $x = \sqrt{2}$.

   b) Show that if the sequence $(x_n)_n$ converges, then, it must converge to either 0 or 2.

   c) Discuss the convergence of the sequence $(x_n)_n$ according to the values of x.

viii. Let $x_1 = x$ and $x_{n+1} = x_n/2 + 1/x_n$. Discuss the convergence of the sequence $(x_n)_n$ according to the values of x.

ix. (Existence of Square Roots) Let $x_1 = 2$ and $x_{n+1} = (x_n + 2/x_n)/2$. Show that $x_n^2 \geq 2$ for all n. Use this to prove that the sequence $(x_n)_n$ is decreasing. Show that $\lim_{n\to\infty} x_n = \sqrt{2}$. Modify the sequence so that it converges to $\sqrt{c}$.

x. Prove the Bolzano-Weierstrass Theorem for $\mathbb{R}^n$. Deduce that every Cauchy sequence of $\mathbb{R}^n$ converges.

6.8 Convergence of Some Sequences

6.8.1 The Sequence $((1+1/n)^n)_n$

Proposition 6.8.1 *The sequence $((1+1/n)^n)_n$ converges to some real number between 2 and 3.*

Proof: We will show that the sequence $((1+1/n)^n)_n$ is bounded and increasing. Theorem 6.7.2 will then prove the theorem. We proceed in several steps.

Claim 1. *For all natural numbers $n > 0$ and all real numbers $x > -1$, $(1+x)^n \geq 1 + nx$. In particular if $y < 1$ then $(1-y)^n \geq 1 - ny$*

For $n = 0$, this is clear. (Note that we need $x \neq -1$ for this). Assume the inequality holds for n. Then we have $(1+x)^{n+1} = (1+x)^n(1+x) \leq (1+nx)(1+x) = 1 + (n+1)x + nx^2 \leq 1 + (n+1)x$. (The second step is by the induction hypothesis and for this step we also need the fact that $1 + x > 0$, which we know it holds).

Claim 2. $2 \leq (1+1/n)^n \leq 3$ *for all natural numbers $n > 0$.*

Replacing x by $1/n$ in the first claim, we get $2 \leq (1+1/n)^n$ for all natural numbers $n > 0$. Now we show that $(1+1/n)^n \leq 3$, for all natural numbers $n > 0$. We compute carefully. For $n \geq 1$ we have,

$$\begin{array}{rcl}
(1+1/n)^n & = & \sum_{i=1}^n \begin{pmatrix} n \\ i \end{pmatrix} \frac{1}{n^i} \\
& = & 1 + \sum_{i=1}^n \frac{n(n-1)\ldots(n-i+1)}{1\cdot 2\cdot\ldots\cdot i}\frac{1}{n^i} \\
& = & 1 + \sum_{i=1}^n \frac{n}{n}\frac{n-1}{n}\cdots\frac{n-i+1}{n}\frac{1}{1\cdot 2\cdot\ldots\cdot i} \\
& = & 1 + \sum_{i=1}^n (1-\frac{1}{n})\ldots(1-\frac{i-1}{n})\frac{1}{1\cdot 2\cdot\ldots\cdot i} \\
& < & 1 + \sum_{i=1}^n \frac{1}{1\cdot 2\cdot\ldots\cdot i} \\
& \leq & \sum_{i=1}^n \frac{1}{2^{i-1}} \\
& = & 1 + \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} \\
& = & 1 + 2(1-\frac{1}{2^n}) \\
& < & 3
\end{array}$$

Claim 3. $n \in \mathbb{N} \setminus \{0\}$, $(1+\frac{1}{n})^n \leq (1+\frac{1}{n+1})^{n+1}$.

Since $(1+\frac{1}{n+1})^{n+1} = (1+\frac{1}{n}-\frac{1}{n}+\frac{1}{n+1})^{n+1} = (1+\frac{1}{n}-\frac{1}{n(n+1)})^{n+1}$, we have to show that $(1+\frac{1}{n}-\frac{1}{n(n+1)})^{n+1} \geq (1+\frac{1}{n})^n$. By setting $a = 1 + 1/n$, this is equivalent to the statement $(a - \frac{1}{n(n+1)})^{n+1} \geq a^n$, i.e. to the statement

$(1 - \frac{1}{an(n+1)})^{n+1} \geq 1/a$, i.e. to the statement $(1 - \frac{1}{(n+1)^2})^{n+1} \geq \frac{n}{n+1}$. By Claim 1, if $n \geq 1$, $(1 - \frac{1}{(n+1)^2})^{n+1} \geq 1 - \frac{1}{n+1} = \frac{n}{n+1}$. This proves Claim 3.

Now apply Theorem 6.7.2. □

Exercises.

i. We have seen in this paragraph that the sequence given $((1 + 1/n)^n)_n$ converges to a real number > 2. Let e be this limit. Do the following sequences converge? If so find their limit.

a) $\lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^n$.

b) $\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^n$.

c) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{2n}$.

d) $\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{3n}$.

e) $\lim_{n\to\infty} \left(1 + \frac{1}{3n}\right)^{2n}$.

6.8.2 The Sequences $(2^{1/n})_n$ and $(n^{1/n})_n$

Proposition 6.8.2 $\lim_{n\to\infty} 2^{1/n} = 1$.

Proof: Clearly $2^{1/n} > 1$. It is also clear that the sequence $(2^{1/n})_n$ is decreasing. Therefore it has a limit, say ℓ. Then $\ell = \lim_{n\to\infty} 2^{1/n} = \lim_{n\to\infty} 2^{1/2n} = \ell^{1/2}$. Hence $\ell = 1$. □

Proposition 6.8.3 $\lim_{n\to\infty} n^{1/n} = 1$.

Proof: Clearly $n^{1/n} \geq 1$. We will show that the sequence is nonincreasing for $n \geq 3$, i.e. we will show that $(n+1)^{1/(n+1)} \leq n^{1/n}$, which is equivalent to $(n+1)^n \leq n^{n+1}$ and to $(1 + 1/n)^n \leq n$. But we have seen in the proof of Theorem 6.8.1 that $(1 + 1/n)^n \leq 3$ for all n. Thus $(1 + 1/n)^n \leq n$ for $n \geq 3$.

Therefore $\lim_{n\to\infty} n^{1/n}$ exists, say ℓ. We have, $1 \leq \ell = \lim_{n\to\infty}(2n)^{1/2n} = \lim_{n\to\infty} 2^{1/2n} n^{1/2n} = 1 \cdot \sqrt{\ell}$. (We used Proposition 6.8.2 in the last equality). Thus $\ell = 1$. □

6.8.3 The Sequence $(nx^n)_n$

Proposition 6.8.4 $\lim_{n\to\infty} nx^n = 0$ *for* $|x| < 1$. *The sequence diverges otherwise.*

Proof: Since $\lim_{n\to\infty} \frac{n}{n+1} = 1$, there is an N such that $|x| < \frac{n}{n+1}$ for all $n > N$. Now for $n > N$, $(n+1)|x|^{n+1} = (n+1)|x| \cdot x^n < nx^n$. Hence the sequence $(nx^n)_n$ is decreasing after a while. Therefore the sequence converges, say to a. Assume $a \neq 0$. Then by Corollary 6.4.11, $1 = \lim_{n\to\infty} \frac{nx^n}{(n+1)x^{n+1}} = \lim_{n\to\infty} \frac{n}{n+1} \frac{1}{x} = 1/x$, a contradiction. Thus $\lim_{n\to\infty} nx^n = 0$. The rest is left as an exercise. □

Exercises.

i. Let $(x_n)_n$ be a sequence such that $x_n \neq 0$ for every $n \in \mathbb{N}$ and $|x_{n+1}/x_n| \leq r$ for some fixed $r \in (0,1)$. Show that $\lim_{n\to\infty} x_n = 0$.

ii. Let $x_1 = 1$, $x_2 = 2$ and

$$x_n = \frac{x_{n-1} + x_{n-2}}{2}$$

for $n > 2$.

a) Show that $1 \leq x_n \leq 2$ for all n.

b) Show that $|x_n - x_{n+1}| = 1/2^{n-1}$ for all n.

c) Show that if $m > n$ then $|x_n - x_m| < 1/2^{n-2}$ for all n.

d) Show that $(x_n)_n$ is a Cauchy sequence.

e) Find its limit. **Answer:** Note that the sequence is formed by taking the arithmetic mean of the previous two terms. It can be guessed that the limit is at the two thirds of the way from $x_1 = 1$ to $x_2 = 2$, i.e. it is $5/3$. This can be shown by some elementary linear algebra. But there is an easier way: Note that part b can be sharpened into $x_n - x_{n+1} = (-1)^n/2^{n-1}$, thus $x_{n+1} - 1 = x_{n+1} - x_1 = (x_{n+1} - x_n) + \ldots + (x_2 - x_1) = 1 - 1/2 + 1/4 \ldots + (-1)^{n-1}/2^{n-1}$. It follows that the limit x, that we know it exists from the previous question, is equal to $\lim_{n\to\infty} 2 - 1/2 + 1/4 \ldots + (-1)^{n-1}/2^{n-1}$. Now we just remarque that $2x = 2\lim_{n\to\infty} 2 - 1/2 + 1/4 - \ldots + (-1)^{n-1}/2^{n-1} = \lim_{n\to\infty} 4 - 1 + 1/2 - \ldots + (-1)^{n-1}/2^{n-2} = \lim_{n\to\infty} 4 - (1 - 1/2 - \ldots + (-1)^n/2^{n-2}) = \lim_{n\to\infty} 5 - (2 - 1/2 - \ldots + (-1)^n/2^{n-2}) = 5 - x$, so $3x = 5$ and $x = 5/3$.

iii. Let $x_0 = 1$, $x_{n+1} = 1 + 1/x_n$. Show that $(x_n)_n$ is convergent.

iv. For $k \in \mathbb{N}$ and $x \in \mathbb{R}$, discuss the convergence of the sequence $(n^k x^n)_n$.

v. Let $f_0 = f_1 = 1$, $f_{n+2} = f_n + f_{n+1}$. Show that the sequence $(\frac{f_{n+1}}{f_n})_n$ is convergent. (It converges to $\frac{2}{1+\sqrt{5}}$).

vi. We say that a sequence $(x_n)_n$ is **contractive** if there is a constant $c \in (0,1)$ such that $|x_{n+2} - x_{n+1}| \leq c|x_{n+1} - x_n|$ for all n. Show that every contractive sequence is convergent. **Hint:** Note first that $|x_{n+1} - x_n| \leq c^n|x_1 - x_0|$. Now estimate $|x_n - x_m|$ and show that such a sequence is Cauchy.

vii. Let $0 < a_0 = a < b_0 = b$. Define $a_{n+1} = (2a_nb_n)/(a_n + b_n)$, $b_{n+1} = (a_n + b_n)/2$. Show that $(a_n)_n$ and $(b_n)_n$ are both convergent. Show that their limits are equal. (Hint: Use mean inequalities.)

viii. Let $a_0 = a$, $b_0 = b, c_0 = c$. Define $a_{n+1} = (b_n + c_n)/2$, $b_{n+1} = (c_n + a_n)/2$, $c_{n+1} = (a_n + b_n)/2$. Show that $(a_n)_n$, $(b_n)_n$ and $(c_n)_n$ are convergent. Compute their limits.(Hint: First take $a = b = 0$, $c = 1$.)

6.9 Divergence to Infinity

Let $(x_n)_n$ be a real sequence. We say that the sequence $(x_n)_n$ **diverges to infinity** if for all real numbers A there is an N such that for all $n > N$, $x_n > A$. Here, A should be regarded as a very large number, as large as it can be. In this case one writes $\lim_{n\to\infty} x_n = \infty$.

Similarly we say that the sequence $(x_n)_n$ **diverges to minus infinity** if for all real numbers A there is an N such that for all $n > N$, $x_n < A$. Here, A should be thought as a very small negative number. In this case, we write $\lim_{n\to\infty} x_n = -\infty$

If an increasing sequence $(a_n)_n$ is bounded then $\lim_{n\to\infty} x_n \in \mathbb{R}$ (Theorem 6.7.2), if it is unbounded, then $\lim_{n\to\infty} x_n = \infty$ (Exercise vi, page 77). Similarly for decreasing sequences.

Note that until now, we have not defined "infinity", and we will never really define it. Above, we only defined the phrase "$\lim_{n\to\infty} x_n = \infty$" as if it were one single entity.

Now we let ∞ and $-\infty$ to be two new distinct symbols. We will never give any meaning to these symbols. Their only property is that they are new symbols. We let $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty, -\infty\}$, we extend the operations and the order relation defined on $\mathbb{R}$ to $\overline{\mathbb{R}}$ partially as follows:

$$
\begin{array}{ll}
-\infty < r < \infty & \text{for all } r \in \mathbb{R} \\
r + \infty = \infty + r = \infty & \text{for all } r \in (-\infty, \infty] \\
r + (-\infty) = r - \infty = -\infty + r = -\infty & \text{for all } r \in [-\infty, \infty)\} \\
-(-\infty) = \infty & \\
r\infty = \infty r = \infty & \text{for all } r \in (0, \infty] \\
r\infty = \infty r = -\infty & \text{for all } r \in [-\infty, 0) \\
r(-\infty) = (-\infty)r = -\infty & \text{for all } r \in (0, \infty] \\
r(-\infty) = (-\infty)r = \infty & \text{for all } r \in [-\infty, 0) \\
r/\infty = r/(-\infty) = 0 & \text{for all } r \in \mathbb{R} \\
\pm\infty/r = \pm\infty(1/r) & \text{for all } r \in \mathbb{R}^* \\
r^{\infty} = \infty & \text{for all } r \in (1, \infty] \\
r^{\infty} = 0 & \text{for all } r \in (-1, 1) \\
r^{-\infty} = 0 & \text{for all } r \in (1, \infty] \\
r^{-\infty} = \infty & \text{for all } r \in (0, 1) \\
r^{-\infty} = -\infty & \text{for all } r \in (-1, 0)
\end{array}
$$

Note that the following terms are not defined:

$(-\infty) + \infty$
$\infty + (-\infty)$
∞/∞
$\infty/(-\infty)$
$(-\infty)/\infty$
$(-\infty)/-\infty$
$\infty/0$
$-\infty/0$
∞^0
$(-\infty)^0$
0^∞
$(\pm 1)^\infty$

Theorem 6.9.1 *Let $(x_n)_n$ and $(y_n)_n$ be two sequences in $\mathbb{R}$ such that $\lim_{n\to\infty} x_n \in \overline{\mathbb{R}}$ and $\lim_{n\to\infty} y_n \in \overline{\mathbb{R}}$. Let $*$ be any of the five basic arithmetic operations (addition, substraction, multiplication, division and power). If $\lim_{n\to\infty} x_n * \lim_{n\to\infty} y_n$ is defined, then $\lim_{n\to\infty} x_n * y_n = \lim_{n\to\infty} x_n * \lim_{n\to\infty} y_n$.*

Proof: Left as an exercise. $\square$

Exercises.

i. Show that the sequences $(n)_n$ and $(n^2)_n$ diverge to infinity.

ii. Show that the sequence $((-1)^n n)_n$ does not converge or diverge neither to ∞ or to $-\infty$.

iii. Show that $\lim_{n\to\infty}(3/2 + 1/n)^n = \infty$.

iv. Find the following limits and prove your result using only the definition:

a) $\lim_{n\to\infty} \frac{2n^2-5}{n+2}$.

b) $\lim_{n\to\infty} \frac{2n^2-5}{-n+2}$.

c) $\lim_{n\to\infty} \frac{2n^3-5}{n^2-n+2}$.

d) $\lim_{n\to\infty} \frac{2n^2-1}{n^2-n-1}^n$.

e) $\lim_{n\to\infty} \left(2 + \frac{1}{n}\right)^n$.

v. Show that if $\lim_{n\to\infty} x_n = \infty$ then $\lim_{n\to\infty} -x_n = -\infty$.

vi. Show that an increasing and unbounded sequence converges to ∞.

vii. Let $p(x)$ and $q(x) \neq 0$ be two polynomials of degree d and e respectively and with leading coefficients a and b respectively. Then the sequence $(p(n)/q(n))_n$ diverges to infinity if and only if $e < f$ and $a/b > 0$. And the sequence $(p(n)/q(n))_n$ diverges to $-\infty$ if and only if $e < f$ and $a/b < 0$.

viii. Show that if $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = a \in \mathbb{R}$ then $\lim_{n\to\infty} x_n + y_n = \infty$.

ix. Show that if $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n = \infty$ then $\lim_{n\to\infty} x_n + y_n = \infty$.

x. Show that if $\lim_{n\to\infty} x_n = \infty$ then $\lim_{n\to\infty} x_n^{-1} = 0$.

xi. Show that there are sequences $(x_n)_n$ such that $\lim_{n\to\infty} x_n = 0$ but $\lim_{n\to\infty} x_n^{-1} \neq \pm\infty$.

xii. Show that if $\lim_{n\to\infty} x_n = \infty$ and $\lim_{n\to\infty} y_n > 0$ then $\lim_{n\to\infty} x_n y_n = \infty$.

xiii. Show that if $\lim_{n\to\infty} x_n = -\infty$ and $\lim_{n\to\infty} y_n = -\infty$ then $\lim_{n\to\infty} x_n y_n = \infty$.

xiv. Let $x_n = 1 + 1/2 + \ldots + 1/n$. Show that $\lim_{n\to\infty} x_n = \infty$. (See Theorem 7.1.3).

6.10 Limit Superior and Inferior

Let $(a_n)_n$ be a sequence of real numbers. We let

$$\overline{\lim}_n a_n = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, \ldots\} \in \overline{\mathbb{R}}.$$

Since the sequence $(\sup\{a_n, a_{n+1}, \ldots\})_n$ is decreasing, $\overline{\lim}_n a_n$ is either a real number or is $\pm\infty$. $\overline{\lim}_n a_n$ is called the **limit superior** of the sequence $(a_n)_n$.

We also define

$$\underline{\lim}_n a_n = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, \ldots\} \in \overline{\mathbb{R}}.$$

Since the sequence $\inf\{a_n, a_{n+1}, \ldots\}$ is increasing, $\underline{\lim}_n a_n$ is either a real number or is $\pm\infty$. $\underline{\lim}_n a_n$ is called the **limit inferior** of the sequence $(a_n)_n$.

We always have $\underline{\lim}_n a_n \leq \overline{\lim}_n a_n$.

Lemma 6.10.1 *i. $\overline{\lim}_n a_n \neq \infty$ if and only if the sequence $(a_n)_n$ has an upper bound.*

ii. $\overline{\lim}_n a_n = \infty$ if and only if the sequence $(a_n)_n$ is unbounded by above.

iii. If $\overline{\lim}_n a_n = r \in \mathbb{R}$ then for all $\epsilon > 0$ there is an N such that for all $n > N$, $a_n < r + \epsilon$.

Proof: (i) and (ii) are clear We prove (iii). Let $\epsilon > 0$. Let N be such that for all $n > N$, $|\sup\{a_n, a_{n+1}, \ldots\} - r| < \epsilon$. Thus $-\epsilon < \sup\{a_{N+1}, a_{N+2}, \ldots\} - r < \epsilon$ and so $\sup\{a_{N+1}, a_{N+2}, \ldots\} < r + \epsilon$. Thus $a_n < r + \epsilon$ for all $n > N$. □

We leave to the reader the task of stating and proving the analogue of the above lemma for the limit inferior.

Theorem 6.10.2 *Let $(a_n)_n$ be a sequence of real numbers. Then $\lim_{n\to\infty} a_n$ exists if and only if $\overline{\lim}_n a_n$ and $\underline{\lim}_n a_n$ are real numbers and are equal. In this case, $\underline{\lim}_n a_n = \lim_{n\to\infty} a_n = \overline{\lim}_n a_n$.*

Proof: This follows from Lemma 6.10.1.iii and its analogue for the limit inferior. □

Exercises.

i. Suppose $(u_n)_n$ and $(v_n)_n$ are positive sequences. Let $U = \overline{\lim}_{n\longrightarrow\infty} u_n$ and $V = \lim_{n\longrightarrow\infty} v_n$. Assume that if one of U and V is ∞, then the other is nonzero. Then $L = \overline{\lim}_{n\longrightarrow\infty} u_n v_n = UV$.

6.11 Complete Metric Spaces

A metric space is called **complete** if its Cauchy sequences converge. Thus $\mathbb{Q}$ is not complete. Theorem 6.7.1 says that $\mathbb{R}$ is complete with its usual metric. We can generalize this to the Euclidean spaces.

Theorem 6.11.1 *The Euclidean space $\mathbb{R}^n$ is complete.*

To prove this we only need to prove the following.

Theorem 6.11.2 *Let $x_k = (x_{k,1}, \ldots, x_{k,n}) \in \mathbb{R}^n$ with $\mathbb{R}^n$ considered with its Euclidean metric. Then*

i) $(x_k)_k$ is a Cauchy sequence if and only if $(x_{k,i})_k$ is a Cauchy sequence for all $i = 1, \ldots, n$.

ii) $(x_k)_k$ is convergent if and only if $(x_{k,i})_k$ is convergent for all $i = 1, \ldots, n$. And in this case

$$\lim_{k\to\infty} x_k = (\lim_{k\to\infty} x_{k,1}, \ldots, \lim_{k\to\infty} x_{k,n}).$$

Proof: i. Assume $(x_k)_k$ is a Cauchy sequence. Let $i = 1, \ldots, n$. Let $\epsilon > 0$. Let N be such that for all $k, \ell > N$, $d(x_k, x_\ell) < \epsilon$. Then for all $k, \ell > N$,

$$|x_{k,i} - x_{\ell,i}| \leq \sqrt{(x_{k,1} - x_{\ell,1})^2 + \ldots (x_{k,n} - x_{\ell,n})^2} = d(x_k, x_\ell) < \epsilon.$$

Thus $(x_{k,i})_k$ is a Cauchy sequence.

Conversely, assume that $(x_{k,i})_k$ is a Cauchy sequence for all $i = 1, \ldots, m$. Let $\epsilon > 0$. Let N_i be such that for all $k, \ell > N_i$, $|x_{k,i} - x_{\ell,i}| < \epsilon/n$. Let $N = \max(N_1, \ldots, N_n)$. Now for all $k, \ell > N$,

$$\begin{aligned} d(x_k, x_\ell) &= \sqrt{(x_{k,1} - x_{\ell,1})^2 + \ldots (x_{k,1} - x_{\ell,1})^2} \\ &\leq |x_{k,1} - x_{\ell,1}| + \ldots + |x_{k,n} - x_{\ell,n}| < \epsilon/n + \ldots + \epsilon/n = \epsilon. \end{aligned}$$

ii. The proof is similar and we leave it as an exercise. □

One can generalize this theorem easily:

Theorem 6.11.3 *For $i = 1, \dots, n$, let (X_i, d_i) be a metric space. Consider $X = X_1 \times \ldots \times X_n$ with the product metric. Let $x_k = (x_{k,1}, \dots, x_{k,n}) \in X$. Then*

i) $(x_k)_k$ is a Cauchy sequence if and only if $(x_{k,i})_k$ is a Cauchy sequence for all $i = 1, \dots, n$.

ii) $(x_k)_k$ is convergent if and only if $(x_{k,i})_k$ is convergent for all $i = 1, \dots, n$. And in this case

$$\lim_{k\to\infty} x_k = (\lim_{k\to\infty} x_{k,1}, \dots, \lim_{k\to\infty} x_{k,n}).$$

Proof: Left as an exercise. □

Exercises.

i. Show that $\lim_{n\to\infty} nx^n = 0$ if $0 \le x < 1$.

Corollary 6.11.4 *Let $(x_n)_n$ and $(y_n)_n$ be two converging sequences in $\mathbb{R}^m$. Then $(x_n + y_n)_n$ and $(x_n y_n)_n$ are converging sequences and*

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n$$

and

$$\lim_{n\to\infty} (x_n y_n) = \lim_{n\to\infty} x_n \lim_{n\to\infty} y_n.$$

Corollary 6.11.5 *Let $(x_n)_n$ be a converging sequence in $\mathbb{R}^m$. Then any subsequence of $(x_n)_n$ is convergent and it converges to the same limit.*

Corollary 6.11.6 *Product of finitely many complete metric spaces is complete.*

Corollary 6.11.7 *The metric space of complex numbers $\mathbb{C}$ is complete.*

Exercises.

i. Let X be any set. Show that the space $\mathrm{Seq}(X)$ defined on page 52 is complete.

6.12 Completion of a Metric Space

In this subsection, given a metric space X, we will find "the smallest" complete metric space containing X.

Let (X, d) be a metric space. Consider the set $\mathcal{C}(X)$ of Cauchy sequences of X. On $\mathcal{C}(X)$ define the following equivalence relation:

$$(x_n)_n \equiv (y_n)_n \iff \lim_{n\to\infty} d(x_n, y_n) = 0.$$

Lemma 6.12.1 *The relation $\equiv$ is an equivalence relation on the set $\mathcal{C}(X)$.*

Proof: Trivial. $\square$

Set $\overline{X} = \mathcal{C}(X)/\equiv$. For $(x_n)_n \in \mathcal{C}(X)$, let $\overline{(x_n)_n} \in \overline{X}$ be its class.

We will turn the set $\overline{X}$ into a metric space. To do this we need the following.

Lemma 6.12.2 *i. Let $(x_n)_n$ and $(y_n)_n$ be two Cauchy sequences in X. Then $\lim_{n\to\infty} d(x_n, y_n)$ exists.*

ii. Let $(x_n)_n$, $(x'_n)_n$, $(y_n)_n$, $(y'_n)_n$ be four Cauchy sequences in X. If $(x_n)_n \equiv (x'_n)_n$ and $(y_n)_n \equiv (y'_n)_n$, then $\lim_{n\to\infty} d(x_n, y_n) = \lim_{n\to\infty} d(x'_n, y'_n)$.

Proof: We first note the following:

$$d(x_n, y_n) \le d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n),$$

implying

$$d(x_n, y_n) - d(x_m, y_m) \le d(x_n, x_m) + d(y_m, y_n).$$

Exchanging n and m, we get

$$|d(x_n, y_n) - d(x_m, y_m)| \le d(x_n, x_m) + d(y_m, y_n).$$

i. It is enough to show that the real sequence $(d(x_n, y_n))_n$ is Cauchy. Let $\epsilon > 0$. Let N be large enough so that for n, $m > N$, $d(x_n, x_m) < \epsilon/2$ and $d(y_m, y_n) < \epsilon/2$. By the above inequality for n, $m > N$, $|d(x_n, y_n) - d(x_m, y_m)| > \epsilon$. $\square$.

ii. By assumption $\lim_{n\to\infty} d(x_n, x'_n) = \lim_{n\to\infty} d(y_n, y'_n) = 0$. So the second part also follows from the inequality proven in the beginning. $\square$

The lemma above says that we are allowed to define a map $\overline{d}$ from $\overline{X} \times \overline{X}$ into $\mathbb{R}^{\ge 0}$ by the rule,

$$\overline{d}(\overline{(x_n)_n}, \overline{(y_n)_n}) = \lim_{n\to\infty} d(x_n, y_n).$$

Lemma 6.12.3 $(\overline{X}, \overline{d})$ *is a metric space.*

Proof: Suppose $\overline{d}(\overline{(x_n)_n}, \overline{(y_n)_n}) = 0$, i.e. $\lim_{n\to\infty} d(x_n, y_n) = 0$. By definition $\overline{(x_n)_n} = \overline{(y_n)_n}$.

Clearly $\overline{d}(\overline{(x_n)_n}, \overline{(y_n)_n}) = \overline{d}(\overline{(y_n)_n}, \overline{(x_n)_n})$.

It remains to prove the triangular inequality:

$$\begin{array}{rcl} \overline{d}(\overline{(x_n)_n}, \overline{(y_n)_n}) & = & \lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty}(d(x_n, z_n) + d(x_n, z_n)) \\ & = & \lim_{n\to\infty} d(x_n, z_n) + \lim_{n\to\infty} d(x_n, z_n) \\ & = & \overline{d}(\overline{(x_n)_n}, \overline{(z_n)_n}) + \overline{d}(\overline{(z_n)_n}, \overline{(y_n)_n}) \end{array}.$$

$\square$

Lemma 6.12.4 *The map i that sends $x \in X$ to the class $\overline{(x)_n}$ of the constant sequence $(x)_n$ is a continuous embedding of X into $\overline{X}$. Furthermore $\overline{d}(i(x), i(y)) = d(x, y)$.*

Proof: If $\overline{(x)_n} = \overline{(y)_n}$, then $\lim_{n\to\infty} d(x,y) = 0$, i.e. $d(x,y) = 0$, hence $x = y$, proving that i is one to one.

For the last equality: $\overline{d}(i(x), i(y)) = lim_{n\to\infty} d(x,y) = d(x,y)$.

Let us now show that i is continuous. Let $(x_k)_k$ be a sequence from X converging to a. Let us show that the class of constant sequences $\overline{(x_k)_n}$ converge to the class of the constant sequence $\overline{(a)_n}$ when k goes to infinity. By Lemma 6.1.2, it is enough to show that $\lim_{n\to\infty} d(\overline{(x_k)_n}, \overline{(a)_n}) = 0$. We have: $\lim_{k\to\infty} d(\overline{(x_k)_n}, \overline{(a)_n}) = \lim_{k\to\infty} d(x_k, a) = 0$. This proves it. □

Lemma 6.12.5 $(\overline{X}, \overline{d})$ *is a complete metric space.*

Proof:

From now on we identify X with its image $i(X)$ and we assume that X is a subset of $\overline{X}$.

Lemma 6.12.6 X *is dense in* $\overline{X}$.

Lemma 6.12.7 *Let* Y *be a metric space. Let* $f : X \longrightarrow Y$ *be a continuous map. Then there is a unique continuous extension* $\overline{f} : \overline{X} \longrightarrow \overline{Y}$ *of* f.

Lemma 6.12.8 *Uniqueness of* $\overline{X}$.

6.13 Supplementary Problems

i. One tosses a coin. Each head adds one point, each tail adds two points. Let $p(n)$ be the probability of reaching the integer n. Show that $\lim_{n\to\infty} p(n) = 2/3$.

ii. **Fibonacci Sequence.** Let $f_0 = 0$, $f_1 = 1$ and define $f_{n+2} = f_{n+1} + f_n$. The sequence $(f_n)_n$ is called the Fibonacci sequence. Show that $\lim_{n\to\infty} f_n/f_{n+1} = \frac{2}{1+\sqrt{5}}$.

iii. Let $(a_n)_n$ be a bounded real sequence with the property that every converging subsequence converges to the same limit. Show that the sequence converges.

iv. Let $(a_n)_n$ be a bounded real sequence. Let

$$S = \{x \in \mathbb{R} : x < a_n : \text{ for infinitely many } n\}.$$

Show that there exists a subsequence converging to $\sup(S)$. Deduce the Bolzano-Weierstrass Theorem from this.

Chapter 7

Series

7.1 Definition and Examples

DO IT IN BANACH SPACES

Let $(a_i)_i$ be a sequence of complex numbers. (For most of what follows we can also assume that $(a_i)_i$ is a sequence in a normed vector space). Let $s_n := \sum_{i=0}^{n} a_i$. Assume that the sequence $(s_n)_n$ converges in $\mathbb{C}$. Then we write

$$\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i$$

and we say that the **series** $\sum_{i=0}^{\infty} a_i$ **converges** to $\lim_{n\to\infty} \sum_{i=0}^{n} a_i$ or that its sum is $\lim_{n\to\infty} \sum_{i=0}^{n} a_i$. Else, we say that the series $\sum_{i=0}^{\infty} a_i$ is **divergent**.

The numbers $s_n := \sum_{i=0}^{n} a_i$ are called the **partial sums** of the series $\sum_{i=0}^{\infty} a_i$.

Although we defined a series for complex numbers, the reader is welcome to consider only real series, i.e. series where $a_i \in \mathbb{R}$ for all i.

If $\sum_{i=0}^{\infty} a_i$ is a real series and $\lim_{n\to\infty} \sum_{i=0}^{n} a_i = \infty$, we say that the series $\sum_{i=0}^{\infty} a_i$ diverges to ∞ and we write $\sum_{i=0}^{\infty} a_i = \infty$. Similarly for $-\infty$.

Whenever we write $\sum_{i=0}^{\infty} a_i = \infty$, we will assume that $a_i \in \mathbb{R}$ even if this is not explicitly stated.

Examples.

i. If $a_i = 1$ for all i, then $\sum_{i=0}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=0}^{n} a_i = \lim_{n\to\infty} n = \infty$. In general, if $(a_i)_i$ is a constant sequence of real numbers, then the series $\sum_{i=0}^{\infty} a_i$ converges if and only if $a_i = 0$. If the sequence $(a_i)_i$ is a real constant nonzero sequence, then $\sum_{i=0}^{\infty} a_i$ diverges to $\pm\infty$ depending on the sign of a_i.

ii. If $a_i = (-1)^i$ for all i, then $\sum_{i=0}^{\infty} a_i$ does not exist (is not even $\pm\infty$).

Theorem 7.1.1 *The set of sequence $(a_i)_i$ such that $\sum_i a_i$ converges form a vector space. Furthermore we have,*

$$\sum_i (\alpha a_i + \beta b_i) = \alpha \sum_i a_i + \beta \sum b_i.$$

The equality persists if $\sum_i a_i$ and $\sum_i b_i$ are in $\mathbb{R} \cup \{\infty, -\infty\}$ if the operations in the equality are allowed.

Proof: Easy. □

Remarks.

i. Regrouping the terms of a series may change the convergence of the series. For example consider the series $\sum_{i=0}^{\infty} a_i$ where $a_i = (-1)^i$. If we let $b_i = a_{2i} + a_{2i+1}$, then $\sum_{n=0}^{\infty} b_i = 0$ because each $b_i = 0$. Thus a divergent sequence may turn into a convergent sequence.

   On the other hand regrouping the terms of a convergent series does not change its convergence and the sums are equal. For example, the terms of the alternating series $\sum_{i=1}^{\infty} (-1)^i/i$ that we will show is convergent can be regrouped two by two to get that the series $\sum_{i=1}^{\infty} 1/(2i+1)(2i+2)$ converges to the same limit.

   As we have seen we cannot dissociate a series without altering its sum (unless the terms are positive).

ii. A **rearrangement** of the terms of the series may change the convergence of the series (unless the terms are positive, see Theorem 7.3.2). Consider the series

$$\sum_i (-1)^i/i = 1 - 1/2 + 1/3 - 1/4 + 1/5 - \ldots.$$

   We will see later that this series converges to a nonzero number. Now shuffle the terms as follows:

$$(1-1/2-1/4)+(1/3-1/6-1/8)+\ldots+(1/(2n+1)-1/2(2n+1)-1/2(2n+2))+\ldots$$

   Regrouping a positive term with the next term which follows it, we get

$$1/2 - 1/4 + 1/6 - 1/8 + \ldots + 1/2(2n+1) - 1/2(2n+2)) + \ldots$$

   and this is half of the initial series.

iii. On the other hand, if $\sum_{i=0}^{\infty} a_i$ converges, then if we regroup the terms without changing the order, then the sum

$$(a_1 + a_2 + \ldots + a_{n_1}) + (a_{n_1+1} + \ldots + a_{n_2}) + (a_{n_2+1} + \ldots + a_{n_3}) + \ldots,$$

remains unaltered. This may be called the associativity of series (see Exercise v, page 86). But we cannot dissociate: For example

$$(1 + (-1)) + (1 + (-1)) + \ldots = 0,$$

but

$$1 + (-1) + 1 + (-1) + \ldots$$

is not convergent.

iv. The ordering of the terms of the series is important and in some cases the sum may change. For example, summing first the terms with even subscripts a_{2i} and then the terms with odd subscripts a_{2i+1} and then adding these two sums may change the sum. We give an example in page 89.

The series $\sum_{i=0}^{\infty} r^i$ is called **geometric series**. This will be our first example of a convergent series.

Theorem 7.1.2 (Geometric Series) *Let $\alpha \in \mathbb{C}$. Then series $\sum_{i=0}^{\infty} \alpha^i$ is convergent if and only if $|\alpha| < 1$. In this case*

$$\sum_{i=0}^{\infty} \alpha^i = \frac{1}{1-\alpha}.$$

Proof: If $|\alpha| \geq 1$, then $\lim_{n\to\infty} \alpha^n \neq 0$, so that $\sum_{i=0}^{\infty} \alpha^i$ cannot be convergent. Assume $|\alpha| < 1$. Let $s_n = \sum_{i=0}^{n} \alpha^i$. Then $(1-\alpha)s_n = s_n - \alpha s_n = 1 - \alpha^{n+1}$ and so $s_n = \frac{1-\alpha^{n+1}}{1-\alpha}$. Therefore by Proposition 6.2.3, $\sum_{i=1}^{\infty} \alpha^i = \frac{1}{1-\alpha}$. □

The series $\sum_{i=1}^{\infty} 1/i$ is called **harmonic series**.

Theorem 7.1.3 (Harmonic Series) $\sum_{i=1}^{\infty} 1/i = \infty$.

Proof: For any natural number k and for the 2^{k-1} many natural numbers $i \in [2^{k-1}, 2^k)$, we have $1/i > 1/2^k$, so that $\sum_{i=2^{k-1}}^{2^k-1} 1/i > 2^{k-1}\frac{1}{2^k} = 1/2$. It follows that $s_{2^n-1} = \sum_{i=1}^{2^n-1} 1/i = \sum_{k=1}^{n} \sum_{i=2^{k-1}}^{2^k-1} 1/i > \sum_{k=1}^{n} 1/2 = n/2$, so that the sequence s_n diverges to infinity. □

Exercises.

i. Show that the series $\sum_{i=1}^{\infty} 1/i^i$ converges. Find an upper bound for the sum.

ii. Show that the series $\sum_{i=1}^{\infty} 1/2i$ and $\sum_{i=0}^{\infty} 1/(2i+1)$ diverge.

iii. Show that the sum of the reciprocals of natural numbers whose decimal expansion contains at least a zero $1/10 + \ldots + 1/90 + 1/100 + 1/101 + \ldots + 1/109 + 1/110 + 1/120 + \ldots$ diverges.

iv. Let $k \in \mathbb{N}$ be ≥ 2. Show that the series $\sum_{i=1}^{\infty} 1/n^i$ converges. (Hint: See Exercise iv, page 72).

v. Suppose $\sum_{i=0}^{\infty} a_i$ converges. Let $(n_k)_k$ be a strictly increasing sequence of natural numbers with $n_0 = 0$. Set $b_k = a_{n_k} + \ldots + a_{n_{k+1}-1}$. Show that $\sum_{k=0}^{\infty} b_k$ converges to $\sum_{i=0}^{\infty} a_i$.

7.2 Easy Consequences of the Definition

Proposition 7.2.1 *If $\sum_{i=0}^{\infty} a_i$ converges then $\lim_{i\to\infty} a_i = 0$.*

Proof: Let $\epsilon > 0$. Since $\sum_{i=0}^{\infty} a_i$ converges, the partial sums $s_n = \sum_{i=0}^{n} a_i$ form a Cauchy sequence. Thus there exists an N such that for all n, $m > N$, $|s_n - s_m| < \epsilon$. Taking $m = n+1$, we see that for all $n > N$, $|a_{n+1}| = |s_n - s_{n+1}| < \epsilon$. Thus for all $n > N+1$, $|a_n| < \epsilon$. This shows that $\lim_{n\to\infty} a_n = 0$. □

Corollary 7.2.2 *If $\lim_{i\to\infty} a_i \neq 0$ then the series $\sum_{i=0}^{\infty} a_i$ cannot converge.*

Since the convergence of a series is nothing else but the convergence of the sequence of partial sums, we can use the completeness of $\mathbb{R}$:

Proposition 7.2.3 (Cauchy's Criterion) *A sequence $\sum_{i=0}^{\infty} a_i$ converges if and only if for all $\epsilon > 0$, there exists an N such that for all $n > m > N$, $|\sum_{i=m}^{n} a_i| < \epsilon$.*

Proof: The condition just expresses the fact that the sequence of partial sums $(\sum_{i=0}^{n} a_i)_n$ is Cauchy. □

Theorem 7.2.4 *If $a_i \geq 0$ and if the partial sums $\sum_{i=0}^{n} a_i$ are bounded then $\sum_{i=0}^{\infty} a_i$ converges.*

Proof: Follows from Theorem 6.7.2. □

Corollary 7.2.5 (Comparison Test) *Let $0 \leq a_i \leq b_i$.*
a. If $\sum_{i=0}^{\infty} b_i$ converges, then $\sum_{i=0}^{\infty} a_i$ converges as well. Furthermore $\sum_{i=0}^{\infty} a_i \leq \sum_{i=0}^{\infty} b_i$.
b. If $\sum_{i=0}^{\infty} a_i = \infty$ then $\sum_{i=0}^{\infty} b_i = \infty$.

Corollary 7.2.6 *Suppose that u_n, $v_n > 0$ and $u_{n+1}/u_n \leq v_{n+1}/v_n$ eventually. If $\sum_n v_n$ converges then $\sum_n u_n$ converges. Hence if $\sum_n u_n$ diverges then $\sum_n v_n$ diverges.*

Proof: Let n_0 be a witness for the "eventuality". Let $n > n_0$. Multiply the inequalities from n_0 to $n-1$ to get $u_n/u_{n_0} \leq v_n/v_{n_0}$, i.e. $u_n \leq r v_n$ where $r = \frac{u_{n_0}}{v_{n_0}}$. Clearly $\sum_n v_n$ converges if and only if $\sum_n r v_n$ converges. Now apply the comparison test (Corollary 7.2.5). □

Exercises.

i. **Cauchy Condensation Test.** Let $(x_i)_i$ be a nonincreasing, nonnegative real sequence. Show that $\sum_{i=0}^{\infty} x_i$ converges if and only if $\sum_{i=0}^{\infty} 2^i x_{2^i}$ converges. [**A**, page 53]

ii. Let $a_n > 0$, $b_n > 0$. Show that if $\lim_{i\to\infty} b_i/a_i = 0$ and $\sum_{i=0}^{\infty} a_i$ converges then $\sum_{i=0}^{\infty} b_i$ converges as well.

iii. Let $a_i > 0$, $b_i > 0$. Show that if $\lim_{i\to\infty} b_i/a_i = \infty$ and $\sum_{i=0}^{\infty} a_i = \infty$ then $\sum_{i=0}^{\infty} b_i = \infty$.

7.3 Absolute Convergence

If $\sum_{i=0}^{\infty} |a_i|$ converges, then we say that the series $\sum_{i=0}^{\infty} a_i$ **converges absolutely**.

Theorem 7.3.1 (Cauchy) *An absolutely convergent series is convergent.*

Proof: Let $\sum_{i=0}^{\infty} a_i$ be an absolutely convergent series. We will show that $\sum_{i=0}^{\infty} a_i$ converges by using Cauchy's Criterion (Proposition 7.2.3). Let $\epsilon > 0$. Let N such that for all $n > m > N$, $\sum_{i=m}^{n} |a_i| < \epsilon$. Then for all $n > m > N$, $|\sum_{i=m}^{n} a_i| \leq \sum_{i=m}^{n} |a_i| < \epsilon$. □

But the converse of this theorem is false. Indeed, we will see that the series $\sum_{i=1}^{\infty}(-1)^i/i$ converges (Corollary 7.4.2), but we know that it does not converge absolutely (Theorem 7.1.3).

We will later show that permuting the terms of a series may change its value. However, this is not the case with the absolutely convergent series.

Theorem 7.3.2 (Rearrangement of the Terms) *Let $\sum_{i=0}^{\infty} a_i$ be an absolutely convergent series. Let $f : \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection. Let $b_i = a_{f(i)}$. Then $\sum_{i=0}^{\infty} b_i$ is also absolutely convergent and its sum is equal to $\sum_{i=0}^{\infty} a_i$*

Proof: Let s_n and t_n be the partial sums of the series $\sum_{i=0}^{\infty} a_i$ and $\sum_{i=0}^{\infty} b_i$ respectively. Let $a = \sum_{i=0}^{\infty} a_i$. We will show that $\lim_{n\to\infty} t_n = a$ as well.

Let $\epsilon > 0$. Since $\lim_{m\to\infty} s_m = a$, there is an N_1 such that $|s_m - a| < \epsilon/2$ for all $m > N_1$.

Since $\sum_{i=0}^{\infty} a_i$ converges absolutely, there is an N_2 such that for all $m > N_2$, $\sum_{k=i}^{\infty} |a_i| < \epsilon/2$.

Choose an $m > \max(N_1, N_2)$.

Let N be such that $\{0, \ldots, m\} \subseteq \{f(0), \ldots, f(N)\}$. Now for $n > N$, we have, $|t_n - a| \leq |t_n - s_m| + |s_m - a| < |t_n - s_m| + \epsilon/2 = |(b_0 + \ldots + b_n) - (a_0 + \ldots + a_m)| + \epsilon/2 = |(a_{f(0)} + \ldots + a_{f(n)}) - (a_0 + \ldots + a_m)| + \epsilon/2 \leq \sum_{k=m+1}^{\infty} |a_k| + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$. □

We can also partition an absolutely convergent series in two (or more) pieces:

Theorem 7.3.3 *Let us partition the terms* $(a_i)_i$ *of an absolutely convergent series* $\sum_{i=0}^{\infty} a_i$ *in two disjoint and infinite subsets* $(b_i)_i$ *and* $(c_i)_i$*. Then* $\sum_{i=0}^{\infty} b_i$ *and* $\sum_{i=0}^{\infty} c_i$ *are absolutely convergent series and* $\sum_{i=0}^{\infty} a_i = \sum_{i=0}^{\infty} b_i + \sum_{i=0}^{\infty} c_i$*.*

Proof: Let α_n, β_n and γ_n be the partial sums of the three series in this order. Given n, we can find m_n and p_n such that every term of the partial sum α_n appears in one of the partial sums β_{m_n} and γ_{p_n}. Thus $0 \leq \beta_{m_n} + \gamma_{p_n} - \alpha_n \leq \sum_{i=n+1}^{\infty} a_i$, so that $\lim_{n\to\infty}(\beta_{m_n} + \gamma_{p_n} - \alpha_n) = 0$. □

Corollary 7.3.4 *Let* $\sum_i a_i$ *be absolutely convergent. Let* $(b_i)_i$ *be the positive terms and* $(c_i)_i$ *the negative terms of* $(a_i)_i$*. Then* $\sum_i a_i = \sum_i b_i + \sum_i c_i$*.*

Exercises.

i. Suppose $\sum_i a_i$ converge absolutely. Show that $|\sum_i a_i| \leq \sum_i |a_i|$. (**First Solution:** Let $R = \sum_i |a_i|$. Then $\sum_i^n a_i \in \overline{B}(0, R)$. Since $\overline{B}(0, R)$ is closed (why?), the limit $\sum_i^{\infty} a_i$ is still in $\overline{B}(0, R)$. **Second Solution:** Let $x_n = \sum_i^n a_i$. By Exercise v, page 63, $|\sum_i^{\infty} a_i| = |\lim_{n\to\infty} x_n| = \lim_{n\to\infty} |x_n| \leq \lim_{n\to\infty} \sum_i^n |a_i| = \sum_i^{\infty} |a_i|$.)

ii. Show that $\sum_{i=0}^{\infty} a_i$ converges absolutely, then $\sum_{i=0}^{\infty} a_i^2$ converges (absolutely) as well. (See also Exercise vi, page 89).

7.4 Alternating Series

A series of the form $\sum_{i=0}^{\infty}(-1)^i a_i$ where $a_i \geq 0$ is called **alternating series**.

Theorem 7.4.1 (Alternating Series Test) *Let* $(a_i)_i$ *be a decreasing sequence such that* $\lim_{i\to\infty} a_i = 0$*. Then* $\sum_{i=0}^{\infty}(-1)^i a_i$ *converges. Furthermore,*

$$\sum_{i=0}^{2n+1}(-1)^i a_i \leq \sum_{i=0}^{\infty}(-1)^i a_i \leq \sum_{i=0}^{2n}(-1)^i a_i.$$

for all n*.*

Proof: Let $s_n = a_0 - a_1 + \ldots + (-1)^n a_n$. We consider the sequences $(s_{2n})_n$ and $(s_{2n+1})_n$ of even and odd partial sums.

Since $s_{2n+1} = (a_0 - a_1) + \ldots + (a_{2n} - a_{2n+1})$, the sequence $(s_{2n+1})_n$ is increasing.

Since $s_{2n} = a_0 - (a_1 - a_2) - \ldots - (a_{2n-1} - a_{2n})$, the sequence $(s_{2n})_n$ is decreasing.

Since $s_{2n+1} - s_{2n} = -a_{2n+1} \leq 0$, $s_{2n+1} \leq s_{2n}$. By the Monotone Convergence Theorem (6.7.2), both the series $(s_{2n})_n$ and $(s_{2n+1})_n$ converge, say to a and b respectively. Then $a - b = \lim_{n\to\infty} s_{2n} - \lim_{n\to\infty} s_{2n+1} = \lim_{n\to\infty}(s_{2n} - s_{2n+1}) = \lim_{n\to\infty} a_{2n+1} = 0$, so that $a = b$. Hence the sequence $(s_n)_n$ converges to a as well (see Exercise iii, page 67). □

Corollary 7.4.2 $\sum_{i=1}^{\infty}(-1)^i/i$ *converges to a number in* $(0, 1)$*.*

Example. [**A**, page 36] We now give an example (necessarily non-absolutely convergent, see Theorem 7.3.2) of a convergent series that, when the terms are permuted, converges to a different number. We consider series $\sum_{i=1}^{\infty}(-1)^i/i$. Let S be its sum. Multiply the series by 1/2 and compute $S + S/2$:

$$3S/2 = S + S/2 = \sum_{i=1}^{\infty}(-1)^i/i + \sum_{i=1}^{\infty}(-1)^i/2i$$

and it is easy to see that the partial sums of the right hand side are exactly the partial sums of the sequence

$$1 + 1/3 - 1/2 + 1/5 + 1/7 - 1/4 + 1/9 + \ldots = 1 + \sum_{i=1}^{\infty}\left(\frac{1}{4i-1} - \frac{1}{2i} + \frac{1}{4i+1}\right),$$

which is a rearrangement of the alternating series $\sum_{i=1}^{\infty}(-1)^i/i$. Since this sum is nonzero, we see that a different rearrangement of the terms of a series may change the sum.

Exercises.

i. Show that $\sum_{i=1}^{\infty}(-1)^i/\sqrt{i}$ converges.

ii. Prove the Alternating Series Test by showing that the sequence of partial sums is a Cauchy sequence.

iii. Show that $\sum_n \frac{1}{(2n+1)(2n+3)}$ converges. (Hint: Note that $\frac{1}{2n+1} - \frac{1}{2n+2} = \frac{1}{(2n+1)(2n+3)}$

iv. Prove the Alternating Series Test by using the Nested Intervals Property (Theorem 3.1.12).

v. Find diverging series $\sum_{i=0}^{\infty} x_i$ and $\sum_{i=0}^{\infty} y_i$ such that $\sum_{i=0}^{\infty} x_i y_i$ converges.

vi. Find a converging series $\sum_{i=0}^{\infty} x_n$ such that $\sum_{i=0}^{\infty} x_i^2$ is divergent.

7.5 Criteria for Convergence

We first generalize Theorem 7.1.2:

Proposition 7.5.1 *Let $(x_i)_i$ be a sequence such that for some $r \in (0,1)$, $|x_{i+1}| \leq r|x_i|$ eventually. Then $\sum_{i=0}^{\infty} x_i$ converges absolutely.*

Proof: We may assume that $x_i \in \mathbb{R}^{\geq 0}$ and that the condition $|x_{i+1}| \leq r|x_i|$ holds for all i. By induction $x_i \leq r^i x_0$. Thus $0 \leq \sum_{i=0}^{n} x_i \leq x_0 \sum_{i=0}^{n} r^i$. Since the latter sequence converges, $\sum_{i=0}^{\infty} x_i$ converges as well (Corollary 7.2.5). $\square$

Corollary 7.5.2 (Ratio Test, d'Alembert) *Let* $(x_i)_i$ *be a nonzero sequence in a Banach space such that*

$$\lim_{n\to\infty} |x_{i+1}/x_i|$$

exists and is < 1. *Then* $\sum_{i=0}^{\infty} x_i$ *converges absolutely.*

Proof: We may assume that $x_i > 0$ for all i. Let s be the limit of the sequence $(x_{i+1}/x_i)_i$. By hypothesis $0 \leq s < 1$. Let $\epsilon = \frac{1-s}{2}$. Then $\epsilon > 0$ and there is an N such that for all $i > N$, $|s - \frac{x_{i+1}}{x_i}| < \epsilon$. Let $r = s + \epsilon$. Then $\frac{x_{i+1}}{x_i} < r$ and the corollary follows from Proposition 7.5.1. □

Examples.

i. The series $\sum_{i=0}^{\infty} z^n/n!$ converges for all z in all Banach spaces.

ii. Consider the series

$$1 + \alpha z + \frac{\alpha(\alpha-1)}{2!}z^2 + \ldots \frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}z^n + \ldots$$

where $\alpha \in \mathbb{C}$. The series converges for $|z| < 1$ according to d'Alembert (Corollary 7.5.2).

Corollary 7.5.3 (d'Alembert) *If* $\overline{\lim}_{i\to\infty}|a_{i+1}/a_i| < 1$, *then* $\sum_{i=0}^{\infty} a_i$ *converges absolutely. If* $\underline{\lim}_{i\to\infty}|a_{i+1}/a_i| > 1$, *then* $\sum_{i=0}^{\infty} a_i$ *diverges.*

Proof: As the proof of Corollary 7.5.2, using Lemma 6.10.1. □

Example. Let

$$a_i = \begin{cases} 1/i^2 & i \text{ is even} \\ 2/(i+1)^2 & i \text{ is odd} \end{cases}$$

Then $\overline{\lim}_{i\to\infty}|a_{i+1}/a_i| = 2 > 1$ and $\underline{\lim}_{i\to\infty}|a_{i+1}/a_i| = 1/2 < 1$. But $\sum_{i=0}^{\infty} a_i$ converges.

Corollary 7.5.4 *For any* $z \in \mathbb{C}$, *the series*

$$\begin{array}{rcl} \exp(z) & = & \sum_{i=0}^{\infty} z^i/i! \\ \sin(z) & = & \sum_{i=0}^{\infty}(-1)^i z^{2i+1}/(2i+1)! \\ \cos(z) & = & \sum_{i=0}^{\infty}(-1)^i z^{2i}/(2i)! \end{array}$$

converge absolutely.

These series then give rise to functions from $\mathbb{C}$ into $\mathbb{C}$. They are called **exponentiation**, **sine** and **cosine** respectively.

Theorem 7.5.5 (Root Test, Cauchy) *If* $\overline{\lim}_{i\to\infty}|a_i|^{1/i} < 1$, *then* $\sum_{i=0}^{\infty} a_i$ *converges absolutely. If* $\overline{\lim}_{i\to\infty}|a_i|^{1/i} > 1$, *then* $\sum_{i=0}^{\infty} a_i$ *diverges.*

Proof: Let $r = \overline{\lim}_{i\to\infty}|a_i|^{1/i} < 1$. Let $\epsilon = \frac{1-r}{2}$. Note that $r + \epsilon < 1$. By Lemma 6.10.1 there is an N such that for all $i > N$, $|a_i|^{1/i} < r + \epsilon$, i.e. $|a_i| < (r + \epsilon)^i$. Now our theorem follows from theorems 7.2.5 and 7.1.2. $\square$

Corollary 7.5.6 *If $R = 1/\overline{\lim}_{i\to\infty}|a_i|^{1/i}$, then $\sum_{i=0}^{\infty} a_i z^i$ converges absolutely if $|z| < R$ and diverges if $|z| > R$.*

$R = 1/\overline{\lim}_{i\to\infty}|a_i|^{1/i}$ is called the **radius of convergence** of the series $\sum_{i=0}^{\infty} a_i z^i$.

A series of the form $\sum_{i=0}^{\infty} a_i z^i$ is called a **power series**. If R is its radius of convergence, such a series gives rise to a function from the ball of center 0 and radius R into $\mathbb{C}$.

Theorem 7.5.7 *Let $f(z) = \sum_{i=0}^{\infty} a_i z^i$ be a power series with radius of convergence R. Then $\sum_{i=1}^{\infty} i a_i z^{i-1}$ has the same radius of convergence.*

Theorem 7.5.8 (Abel, form due to Lejeune-Dirichlet) *The (complex or real) series $\sum_{i=0}^{\infty} v_n \epsilon_n$ converges if*

i. The sum $|v_m + \ldots + v_n|$ is bounded for all $m \leq n$.

ii. The series $\sum_{n=0}^{\infty} |\epsilon_n - \epsilon_{n+1}|$ converges.

iii. $\lim_{n\to\infty} \epsilon_n = 0$.

Proof: We will use Cauchy criterion of convergence (7.2.3). Let $\epsilon > 0$. For $m \leq n$, let $V_{m,n} = v_m + \ldots + v_n$. By hypothesis, there is an A such that $|V_{m,n}| < A$ for all $m \leq n$. Let N_1 be such that for all $n \geq m > N_1$, $|\epsilon_m - \epsilon_{m+1}| + \ldots + |\epsilon_{n-1} - \epsilon_n| < \epsilon/2A$ and let N_2 be such that for all $n > N_2$, $|\epsilon_n| > \epsilon/2A$. Let $N = \max\{N_1, N_2\}$. For $n \geq m > N$ we compute as follows:

Since $V_{k,k+1} - V_{k,k} = v_{k+1}$, we have, $v_m\epsilon_m + \ldots + v_n\epsilon_n = V_{m,m}\epsilon_m + (V_{m,m+1} - V_{m,m})\epsilon_{m+1} + \ldots + (V_{m,n} - V_{m,n-1})\epsilon_n = V_{m,m}(\epsilon_m - \epsilon_{m+1}) + V_{m,m+1}(\epsilon_{m+1} - \epsilon_{m+2}) + \ldots + V_{m,n-1}(\epsilon_{n-1} - \epsilon_n) + V_{m,n}\epsilon_n$. Thus $v_m\epsilon_m + \ldots + v_n\epsilon_n \leq A(|\epsilon_m - \epsilon_{m+1}| + |\epsilon_{m+1} - \epsilon_{m+2}| + \ldots + |\epsilon_{n-1} - \epsilon_n| + |\epsilon_n|) < A(\epsilon/2A) + A(\epsilon/2A) = \epsilon$. $\square$

Corollary 7.5.9 *The (complex or real) series $\sum_{i=0}^{\infty} v_n \epsilon_n$ converges if*

i. The sum $|v_m + \ldots + v_n|$ is bounded for all $m \leq n$.

ii. $(\epsilon_n)_n$ is positive, nondecreasing and $\lim_{n\to\infty} \epsilon_n = 0$.

Proof: We check that the above conditions are met. We only have to be concerned with condition (ii): $\sum_{n=0}^{\infty} |\epsilon_n - \epsilon_{n+1}| = \sum_{n=0}^{\infty} (\epsilon_n - \epsilon_{n+1}) = \epsilon_0$. $\square$

We can obtain the alternating series test (Theorem 7.4.1) as a consequence:

Corollary 7.5.10 (Alternating Series Test) *Let $(v_i)_i$ be a decreasing sequence such that* $\lim_{i\to\infty} v_i = 0$. *Then $\sum_{i=0}^{\infty} (-1)^i v_i$ converges.*

Proof: Take $v_n = (-1)^n$ in the corollary above. $\square$

Exercises.

i. Discuss the convergence and absolute convergence of the alternating series $1 - 1/2^\alpha + 1/3^\alpha - 1/4^\alpha + \dots$ for various values of $\alpha \in \mathbb{Q}$.

ii. **Decimal Expansion.** Let $r \in \mathbb{R}^{\geq 0}$. Then there are $k \in \mathbb{Z}$ and $a_i \in \{0, 1, \dots, 9\}$ such that $r = \sum_{i=k}^{\infty} a_i 10^{-i}$.

iii. Let $n > 0$ be a natural number. Let $r \in \mathbb{R}^{\geq 0}$. Then there are $k \in \mathbb{Z}$ and $a_i \in \{0, 1, \dots, n-1\}$ such that $r = \sum_{i=k}^{\infty} a_i n^{-i}$.

iv. Show that $1/3 = \sum_{i=1}^{\infty} 2^{-2i}$.

v. Let $(a_i)_i$ be a positive decreasing sequence, k an integer greater than 1. Show that the series $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} k^i a_{k^i}$ either both converge or both diverge.

vi. Show that if the ratio test says that a sequence converges, so does the root test.

vii. Show that if $a_i > 0$ and $\lim_{i \to \infty} i a_i$ exists and is nonzero, then $\sum_{i=0}^{\infty} a_i$ diverges.

viii. Assume $a_i > 0$ and $\lim_{i \to \infty} i^2 a_i$ exists. Show that $\sum_{i=0}^{\infty} a_i$ converges.

7.6 Supplementary Problems

i. Show that the sum of the reciprocals of natural numbers whose decimal expansion does not contain a 0, i.e. the series $1/1 + \dots + 1/9 + 1/11 + \dots + 1/19 + 1/21 + \dots + 1/29 + 1/31 + \dots + 1/99 + 1/111 + \dots + 1/119 + 1/121 + \dots + 1/129 + \dots$ is convergent.

ii. Let $\sum_{i=0}^{\infty} a_i$ is given. For each $i \in \mathbb{N}$, let $p_i = a_i$ if $a_i \geq 0$ and assign $p_i = 0$ if $a_i < 0$. In a similar manner, let $q_i = a_i$ if $a_i \leq 0$ and $q_i = 0$ otherwise.

   a. Show that if $\sum_{i=0}^{\infty} a_i$ diverges, then at least one of $\sum_{i=0}^{\infty} p_i$ or $\sum_{i=0}^{\infty} q_i$ diverges (to ∞ and $-\infty$ resp.)

   b. Show that if $\sum_{i=0}^{\infty} a_i$ converges conditionally (i.e. not absolutely), then $\sum_{i=0}^{\infty} p_i = \infty$ and $\sum_{i=0}^{\infty} q_i = -\infty$. [**A**, Ex. 2.7.3, page 68]

   c. Show that if $\sum_{i=0}^{\infty} a_i$ converges conditionally and $r \in \mathbb{R}$, then there is a bijection $f : \mathbb{N} \longrightarrow \mathbb{N}$ such that $\sum_{i=0}^{\infty} a_{f(i)} = r$. [**A**, Section 2.9]

iii. Let $x_i = 1/1^2 + 1/2^2 + 1/3^2 + \dots + 1/i^2$.

   a. Show that for all integers $i \geq 1$, $x_i \leq 2 - 1/i < 2$.

   b. Conclude that the series $\sum_{i=1}^{\infty} 1/i^2$ converges.

   c. Show that for an integer i large enough, $i^2 \leq 2^i$.

   d. Conclude that $\sum_{i=1}^{\infty} 1/i^2 \leq 1 + 1/4 + 1/9 + 1/8 = 107/72$.

iv. Show that the series $\sum_{i=1}^{\infty} \frac{1}{i^2+i}$ converges. Find its value.

v. Show that the series $\sum_{i=1}^{\infty} \frac{1}{i^2+2i}$ converges. Find its value.

vi. Which of the following series converge? Can you estimate their sum?

$$\sum_{i=1}^{\infty} \frac{1}{i^2-i}$$
$$\sum_{i=3}^{\infty} \frac{1}{i^2-3i+2}$$
$$\sum_{i=0}^{\infty} \frac{1}{3i-2}$$
$$\sum_{i=0}^{\infty} \frac{i}{3i^2-2}$$
$$\sum_{i=0}^{\infty} \frac{i}{3i^3-2}$$
$$\sum_{i=0}^{\infty} \frac{3^i+4^i}{5^i}$$
$$\sum_{i=0}^{\infty} \frac{2^i}{i!}$$
$$\sum_{i=0}^{\infty}(-1)^i \frac{i}{i+1}$$
$$\sum_{i=0}^{\infty}(-1)^i \frac{i}{i^2-i+1}$$
$$\sum_{i=1}^{\infty} \frac{(-1)^i}{\sqrt{i}}$$
$$\sum_{i=0}^{\infty}(-1)^i \frac{2^i i^3}{i!}$$

7.6.1 Midterm of Math 152

i. Decide the convergence of the series
$$\sum_n \frac{1}{\sqrt{|n^2-2|}}.$$

ii. Decide the convergence of the series
$$\sum_n \frac{1}{\sqrt{n^2+1}}.$$

iii. Decide the convergence of the series
$$\sum_n \frac{1}{\sqrt{|n^4-6|}}.$$

iv. Suppose that the series $\sum_n a_n$ is convergent. Show that $\lim_{n\to\infty} a_n = 0$.

v. Suppose that $(a_n)_n$ is a positive and decreasing sequence and that the series $\sum_n a_n$ is convergent. Show that $\lim_{n\to\infty} na_n = 0$.

vi. Find a positive sequence $(a_n)_n$ such that the series $\sum_n a_n$ is convergent but that $\lim_{n\to\infty} na_n \neq 0$.

vii. Suppose that series $\sum_n a_n$ is absolutely convergent and that the sequence $(b_n)_n$ is Cauchy. Show that the series $\sum_n a_n b_n$ is absolutely convergent.

viii. Let $(a_n)_n$ be a sequence. Suppose that $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges. Such a sequence is called of **bounded variation**. Show that a sequence of bounded variation converges.

7.6.2 Solutions of the Midterm of Math 152

i. Decide the convergence of the series

$$\sum_n \frac{1}{\sqrt{|n^2-2|}}.$$

Answer: Since for all $n > 1$,

$$\frac{1}{\sqrt{|n^2-2|}} = \frac{1}{\sqrt{n^2-2}} \geq \frac{1}{\sqrt{n^2}} = \frac{1}{n}$$

and since $\sum_n \frac{1}{n}$ diverges, the series $\sum_n \frac{1}{\sqrt{|n^2-2|}}$ diverges as well.

ii. Decide the convergence of the series

$$\sum_n \frac{1}{\sqrt{n^2+1}}.$$

Answer: Since for $n > 0$,

$$\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+n^2}} = \frac{1}{n\sqrt{2}}$$

and since $\sum_n \frac{1}{n}$ diverges, the series $\sum_n \frac{1}{\sqrt{n^2+1}}$ diverges as well.

iii. Decide the convergence of the series

$$\sum_n \frac{1}{\sqrt{|n^4-6|}}.$$

Answer: Since for $n > 1$,

$$\frac{1}{\sqrt{|n^4-6|}} = \frac{1}{\sqrt{n^4-6}} \leq \frac{1}{\sqrt{n^4-n^4/2}} = \frac{\sqrt{2}}{n^2}$$

and since $\sum_n \frac{1}{n^2}$ converges, the series $\sum_n \frac{1}{\sqrt{n^4-6}}$ converges as well.

iv. Suppose that the series $\sum_n a_n$ is convergent. Show that $\lim_{n\to\infty} a_n = 0$.

Proof: Let $s_n = a_0 + \ldots + a_n$ and $s = \sum_n a_n$. Thus $\lim_{n\to\infty} s_n = s$. We have $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(s_n - s_{n-1}) = s - s = 0$.

v. Suppose that $(a_n)_n$ is a positive and decreasing sequence and that the series $\sum_n a_n$ is convergent. Show that $\lim_{n\to\infty} na_n = 0$.

Proof: We know that $\lim_{n\to\infty} a_n = 0$. Let $s_n = a_0 + \ldots + a_n$ and $s = \sum_n a_n$. Thus $\lim_{n\to\infty} s_n = s$.

Since $(a_n)_n$ is decreasing,

$$na_{2n} \le a_{n+1} + \ldots + a_{2n} = s_{2n} - s_n.$$

Thus $\lim_{n\to\infty} na_{2n} = \lim_{n\to\infty}(s_{2n} - s_n) = s - s = 0$. Hence $\lim_{n\to\infty} 2na_{2n} = 0$.

Also $0 < (2n+1)a_{2n+1} \le (2n+1)a_{2n} = 2na_{2n} + a_{2n}$. By the above and the fact that $\lim_{n\to\infty} a_{2n} = 0$, the right hand side converges to 0. Hence by the squeezing lemma $\lim_{n\to\infty}(2n+1)a_{2n+1} = 0$.

From the above two paragraphs it follows that $\lim_{n\to\infty} na_n = 0$.

(Remark: As we will see later, the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges. This example shows that the conditions that $(a_n)_n$ is positive and decreasing and that $\lim_{n\to\infty} na_n = 0$ are not enough for the series $\sum_n a_n$ to be convergent.)

vi. Find a positive sequence $(a_n)_n$ such that the series $\sum_n a_n$ is convergent but that $\lim_{n\to\infty} na_n \ne 0$.

Solution: Take $a_n = 1/n^2$ if n is not a square and $a_n = 1/n$ if n is a square. Then $(na_n)_n$ does not converge as $\lim_{n\to\infty}(n^2+1)a_{n^2+1} = 0$ and $\lim_{n\to\infty} n^2 a_{n^2} = 1$. On the other hand $\sum_n a_n = \sum_{n \text{ non square}} a_n + \sum_{n \text{ a square}} a_n = \sum_{n \text{ non square}} 1/n^2 + \sum_n 1/n^2 < 2\sum_n 1/n^2 < 4$.

vii. Suppose that series $\sum_n a_n$ is absolutely convergent and that the sequence $(b_n)_n$ is Cauchy. Show that the series $\sum_n a_n b_n$ is absolutely convergent.

Proof: Since $(b_n)_n$ is Cauchy the sequence $(|b_n|)_n$ is bounded. In fact that is all we need to conclude. Indeed, let B be an upper bound of the sequence $(|b_n|)_n$. Then $|a_n b_n| \le B|a_n|$ and since $\sum_n |a_n|$ converges, $\sum_n |a_n b_n|$ converges as well.

viii. Let $(a_n)_n$ be a sequence. Suppose that $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges. Such a sequence is called of **bounded variation**. Show that a sequence of bounded variation converges.

Proof: Since $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges, $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ converges as well. Thus the sequence of partial sums whose terms are

$$\sum_{i=1}^{n-1}(a_i - a_{i+1}) = a_1 - a_n$$

converges, say to a. Thus $(a_n)_n$ converges to $a_1 - a$.

Chapter 8

Supplementary Topics

8.1 Liouville Numbers

A real number r is called **algebraic** (over $\mathbb{Q}$) if there is a nonzero polynomial $p(x) \in \mathbb{Q}[x]$ (equivalently $p(x) \in \mathbb{Z}[x]$) such that $p(r) = 0$. Otherwise r is called **nonalgebraic** or **transcendental** over $\mathbb{Q}$.

Consider the root $\alpha \in \mathbb{R}$ of a polynomial $f(x) \in \mathbb{Z}[x]$. Set $f(x) = \sum_{i=0}^{n} a_i x^i$ where $a_n \neq 0$.

Lemma 8.1.1 *If* $M = \max\{|a_i/a_n| : i = 0, \ldots, n-1\}$ *then* $|\alpha| < 1 + M$.

Proof: We may assume $\alpha \neq 0$. Since $0 = f(\alpha) = a_0 + a_1\alpha^i + \ldots + a_n\alpha^n$ and $a_n\alpha^n \neq 0$, we have

$$-1 = \frac{a_{n-1}\alpha^{n-1} + \ldots + a_1\alpha + a_0}{a_n\alpha^n} = \frac{a_{n-1}}{a_n\alpha} + \ldots + \frac{a_0}{a_n\alpha^n}.$$

Now if $|\alpha| \geq 1 + M$, then the absolute value of the right hand side is less than

$$M(\frac{1}{1+M} + \frac{1}{(1+M)^2} + \ldots + \frac{1}{(1+M)^n}) <$$

$$M(\frac{1}{1+M} + \frac{1}{(1+M)^2} + \ldots + \frac{1}{(1+M)^n} + \ldots) = \frac{M}{1+M}\frac{1}{1-\frac{1}{1+M}} = 1,$$

a contradiction. $\square$

Assume now $0 < \alpha \notin \mathbb{Q}$ is a root of $f(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{Q}[x]$ with $a_n \neq 0$. Let M be as above. Let $q \in \mathbb{N}$ be any natural number large enough so that,

1) $1/q < \alpha$,
2) f has no root in $(\alpha - 1/q, \alpha + 1/q)$,
3) $\alpha + 1/q < 1 + M$, and
4) $q > n\sum_{i=1}^{n} |a_i|(1+M)^{i-1}$.

Note that if q_0 satisfies these conditions then any $q > q_0$ also satisfies these conditions.

Note also that there are exactly two natural numbers such that $|\alpha - p/q| < 1/q$: if p is the smallest such natural number then $\alpha - 1/q < p/q < \alpha < (p+1)/q < \alpha + 1/q < 1 + M$. Let $p \in \mathbb{N}$ be such that $|\alpha - p/q| < 1/q$. We will show that $|\alpha - p/q| > 1/q^{n+1}$.

Clearly $f(p/q) = A/q^n$ for some $A \in \mathbb{Z}$. By (2), $A \neq 0$. We also have,

$$-A/q^n = -f(p/q) = f(\alpha) - f(p/q) = (\alpha - p/q)f'(\beta)$$

for some β between p/q and α. Note that $\beta < \max\{p/q, \alpha\} < 1 + M$. Since $f'(x) = \sum_{i=1}^{n} i a_i x^{i-1}$,

$$|f'(\beta)| = |\sum_{i=1}^{n} i a_i \beta^{i-1}| \leq n \sum_{i=1}^{n} |a_i| \beta^{i-1} \leq n \sum_{i=1}^{n} |a_i|(1+M)^{i-1} < q.$$

Thus, since $A \neq 0$,

$$1/q^n \leq |-A/q^n| = |\alpha - p/q||f'(\beta)| < |\alpha - p/q|q.$$

Hence $|\alpha - p/q| > 1/q^{n+1}$.

Thus if α is an irrational root of a polynomial of degree n, for large enough q, if $|\alpha - p/q| < 1/q$ then $|\alpha - p/q| > 1/q^{n+1}$.

It follows that if α is irrational and if there are infinitely many natural numbers q for which $|\alpha - p/q| < 1/q^{n+1}$ $(< 1/q)$ for some natural number p (that may depend on q), then α is not the root of a polynomial of degree $\leq n$. If this happens for infinitely many integers n, then α is not algebraic. This is what happens for numbers of the form

$$\alpha = \sum_{i=1}^{\infty} \frac{a_i}{10^{i!}}$$

where $a_i = 0, 1, \ldots, 9$, but the values of a_i are not all 0 after a while.

Because the decimal expansion is not periodic, $a \notin \mathbb{Q}$. We check the second condition.

Let n be any natural number and $m > n + 1$. Let $q = q_m = 10^{m!}$ and $p/q = \sum_{i=1}^{m} \frac{a_i}{10^{i!}}$. Then

$$0 < \alpha - p/q = \sum_{i=m+1}^{\infty} \frac{a_i}{10^{i!}} \leq \sum_{i=m+1}^{\infty} \frac{9}{10^{i!}} = \frac{9}{10^{(m+1)!}} \sum_{i=0}^{\infty} \frac{1}{10^i} =$$

$$= \frac{10}{10^{(m+1)!}} < \frac{1}{(10^{m!})^m} = 1/q^m.$$

Thus α cannot be an algebraic number. Such a number is called a **Liouville number**.

Theorem 8.1.2 (Liouville) *If*

$$\alpha = \sum_{i=1}^{\infty} \frac{a_i}{10^{i!}}$$

where $a_i = 0, 1, \ldots, 9$, *but the values of* a_i *are not all 0 after a while, then* α *is not an algebraic number.*

Chapter 9

Convergence of Functions

9.1 Pointwise Convergence of a Sequence of Functions

Let X be a set and (Y, d) be a metric space. Let $(f_n)_n$ be a sequence of functions from X into Y. Assume that for all $x \in X$, the sequence $(f_n(x))_n$ of the metric space Y converges. Let $f(x)$ be this limit:

$$f(x) := \lim_{n\to\infty} f_n(x).$$

Then the rule $x \mapsto f(x)$ defines a function f from X to Y. we say that the function f is the **pointwise limit** of the sequence $(f_n)_n$. In this case we write

$$f = \lim_{n\to\infty} f_n$$

or sometimes

$$f \stackrel{p}{=} \lim_{n\to\infty} f_n$$

to precise that the convergence is pointwise (soon we will define other kinds of convergence).

Clearly, if $f_n = g$ for all n, then $\lim_{n\to\infty} f_n \stackrel{p}{=} g$.

Exercises and Examples.

i. Let $X = \mathbb{R}$, $Y = \mathbb{R}$ with the usual metric and $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f_n(x) = x/n$. Then the zero function is the pointwise limit of the sequence $(f_n)_n$.

ii. Let $X = Y = (0, 1)$ and let $f_n(x) = 1/n$ for all $n \in \mathbb{N} \setminus \{0\}$. Unfortunately, the pointwise limit of the sequence $(f_n)_n$ does not exist because $0 \notin Y$. This is of course only a minor problem. It is enough to change Y with $[0, 1)$, or even with $\mathbb{R}$, in which case the pointwise limit is the constant zero function.

iii. Let $X = Y = [0,1]$ and $f_n(x) = x^n$. Then the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1 \end{cases}$$

is the limit of the sequence $(f_n)_n$.

iv. Let $X = Y = \mathbb{R}$ and let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f_n(x) = x^n/n!$. Then the zero function is the pointwise limit of the sequence $(f_n)_n$.

v. Let $X = Y = [0,1]$. Let

$$f_n(x) = \begin{cases} 2(x - \frac{k}{2^n}) & \text{if } \frac{k}{2^n} \leq x \leq \frac{k+1}{2^n} \text{ and } k \in 2\mathbb{N} \\ 2(\frac{k+1}{2^n} - x) & \text{if } \frac{k}{2^n} \leq x \leq \frac{k+1}{2^n} \text{ and } k \in \mathbb{N} \setminus 2\mathbb{N} \end{cases}$$

Draw the graph of f_n. What is its length? Show that the zero function is the pointwise limit of the sequence $(f_n)_n$. What is the length of the limit function?

vi. Let $X = Y = \mathbb{C}$ (or $\mathbb{R}$). define

$$\begin{array}{rcl} f_n(z) & = & \sum_{i=0}^n z^i/i! \\ g_n(z) & = & \sum_{i=0}^n (-1)^i z^{2i+1}/(2i+1)! \\ h_n(z) & = & \sum_{i=0}^n (-1)^i z^{2i}/(2i)! \end{array}$$

Then

$$\begin{array}{rcl} \lim_{n\to\infty} f_n & = & \exp \\ \lim_{n\to\infty} g_n & = & \sin \\ \lim_{n\to\infty} f_n & = & \cos\,. \end{array}$$

Theorem 9.1.1 *Let $Y = \mathbb{R}^k$ with the usual metric. Let $\lim_{n\to\infty} f_n \stackrel{p}{=} f$ and $\lim_{n\to\infty} g_n \stackrel{p}{=} g$. Then*

i. $\lim_{n\to\infty}(f_n + g_n) \stackrel{p}{=} f + g$

ii. For all $r \in \mathbb{R}$, $\lim_{n\to\infty} rf_n \stackrel{p}{=} rf$.

iii. If $Y = \mathbb{R}$, then $\lim_{n\to\infty}(f_n g_n) \stackrel{p}{=} fg$.

iv. If $Y = \mathbb{R}$ and $f(x) \neq 0$ for all $x \in \mathbb{R}$, then $\lim_{n\to\infty} 1/f_n \stackrel{p}{=} 1/f$.

Proof: Trivial □

Note that the first two parts of the theorem above says that the set of pointwise convergent sequences of functions from X into the Euclidean space $\mathbb{R}^n$ is a real vector space.

9.2 Uniform Convergence of a Sequence of Functions

As we defined the convergence of a series of numbers $\sum_{i=0}^{\infty} a_i$, we can also define the convergence of a series of functions $\sum_{i=0}^{\infty} f_i$ as the pointwise limit of the sequence of functions $\sum_{i=0}^{n} f_i$. Here f and each f_i are functions from a set X into $\mathbb{R}^k$ (fixed k) or even into any Banach space (a complete normed vector space). Thus, we say that the series $\sum_{i=0}^{\infty} f_i$ **converges** (pointwise) to f if for all $x \in X$, the series $\sum_{i=0}^{\infty} f_i(x)$ converges to $f(x)$.

Let us rewrite the definition of pointwise convergence: Let X be a set and (Y, d) be a metric space. Let $(f_n)_n$ be a sequence of functions from X into Y and f a function from X into Y. Assume that $f \stackrel{p}{=} \lim_{n\to\infty} f_n$. This means that for all $x \in X$, $f(x) = \lim_{n\to\infty} f_n(x)$; in other words, for all $x \in X$ and all $\epsilon > 0$, there is an N such that $d(f_n(x), f(x)) < \epsilon$ whenever $n > N$. Here, N depends on ϵ, but also on x. That is why, one sometimes writes $N_{\epsilon,x}$ instead of N.

When N is independent of x, we say that the convergence is **uniform**. More precisely, the sequence $(f_n)_n$ **converges uniformly** to f if for all $\epsilon > 0$ there is an N (that depends only on ϵ) such that for all $x \in X$ and $n > N$, $d(f_n(x), f(x)) < \epsilon$. When that is the case, one writes $f \stackrel{u}{=} \lim_{n\to\infty} f_n$.

Clearly, if $f_n = g$ for all n, then $\lim_{n\to\infty} f_n \stackrel{u}{=} g$.

It is also clear that the uniform convergence is stronger than the pointwise convergence, in other words, if $f \stackrel{u}{=} \lim_{n\to\infty} f_n$ then $f \stackrel{p}{=} \lim_{n\to\infty} f_n$. But the converse fails as the following example shows.

Examples.

i. Let $X = (-1, 1)$ and $Y = \mathbb{R}$ with the usual metric. Consider the sequence $(f_n)_n$ of functions from X into Y defined by

$$f_n(x) = \frac{1 - x^n}{1 - x}.$$

It is clear that the pointwise limit of the sequence $(f_n)_n$ is the function f defined by

$$f(x) = \frac{1}{1 - x}.$$

Therefore the uniform limit of this sequence - if it exists at all - should also be the same function. Assume $\lim_{n\to\infty} f_n \stackrel{u}{=} f$. Then for all $\epsilon > 0$ there is an N such that

$$\frac{|x|^n}{1 - x} = |f_n(x) - f(x)| < \epsilon$$

for all $n > N$ and $x \in X$. But whatever n is, the function $\frac{|x|^n}{1-x}$ is unbounded on X, so that there cannot be such an N (independent of all x).

ii. Let $\alpha \in (0,1)$. Let $X = (-1, \alpha)$ and $Y = \mathbb{R}$ with the usual metric. Consider the sequence $(f_n)_n$ of functions from X into Y defined by
$$f_n(x) = \frac{1-x^n}{1-x}$$
(as above). It is clear that the pointwise limit of the sequence $(f_n)_n$ is the function f defined by
$$f(x) = \frac{1}{1-x}.$$
Therefore the uniform limit of this sequence – if it exists at all – should also be the same function. Indeed it is. Let us prove that. Let $\epsilon > 0$. Since $0 < \alpha < 1$, $\lim_{n\to\infty} \alpha^n = 0$, so that there is an N such that $\alpha^n < \epsilon(1-\alpha)$ for all $n > N$. (Note that N depends only on ϵ). Now, for all $n > N$ and **for all** $x \in X$, we have,
$$|f_n(x) - f(x)| = \frac{|x|^n}{1-x} < \frac{|x|^n}{1-\alpha} < \frac{\alpha^n}{1-\alpha} < \epsilon.$$
Thus $\lim_{n\to\infty} f_n \stackrel{u}{=} f$.

iii. Let $f_n : [-1,1] \longrightarrow \mathbb{R}$ be defined by $f_n(x) = |x|^{1+\frac{1}{2n+1}}$. Show that $\lim_{n\to\infty} f_n(x) \stackrel{u}{=} |x|$.

The second example above shows that the domain X is important for the uniform convergence. For this reason, one says that the sequence $(f_n)_n$ "converges uniformly on X".

Theorem 9.2.1 *Let X be any set and $Y = \mathbb{R}^n$ with the usual metric. Let f_n, g_n, f, g ($n \in \mathbb{N}$) be functions from X into Y. Let $\lim_{n\to\infty} f_n \stackrel{u}{=} f$ and $\lim_{n\to\infty} g_n \stackrel{u}{=} g$. Then*
i. $\lim_{n\to\infty}(f_n + g_n) \stackrel{u}{=} f + g$
ii. For all $r \in \mathbb{R}$, $\lim_{n\to\infty} rf_n \stackrel{u}{=} rf$.

Proof: Trivial. □

It is false that if $(f_n)_n$ and $(g_n)_n$ are sequences functions that converge uniformly, then the sequence $(f_n g_n)_n$ converges uniformly.

Suppose the sequence $(f_n)_n$ of functions from a set into a metric space converges uniformly (hence pointwise) to f. Let $T_n = \sup_x\{d(f_n(x), f(x))\} \in \mathbb{R}^{\geq 0} \cup \{infty\}$. Take $\epsilon = 1$ in the definition to find an N such that for all $n > N$ and x, $d(f_n(x), f(x)) < 1$, i.e. for all $n > N$, $T_n < 1$. It follows that if the sequence $(f_n)_n$ of functions from a set into a metric space converges uniformly (hence pointwise) to f, then the sequence $T_n = \sup_x\{d(f_n(x), f(x))\} \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ is eventually bounded.

Lemma 9.2.2 *Suppose the sequence $(f_n)_n$ of functions from a set into a metric space converges pointwise to f. Let $T_n = \sup_x\{d(f_n(x), f(x))\} \in \mathbb{R}^{\geq 0} \cup \{infty\}$. Then $\lim_{n\to\infty} f_n = f$ if and only if $\lim_{n\to\infty} T_n = 0$.*

Proof: Trivial. $\square$

Consider the set of functions $\mathcal{F}$ from a set X into a metric space (Y, d). For $f,\, g \in \mathcal{F}$, let $d_\infty(f, g) = \min\{\sup_{x\in X}\{d(f(x), g(x))\}, 1\}$.

Lemma 9.2.3 *$(\mathcal{F}, d_\infty)$ is a metric space and a sequence $(f_n)_n$ from $\mathcal{F}$ converges to a function in $f \in \mathcal{F}$ if and only if $\lim_{n\to\infty} d_\infty(f_n, g) = 0$.*

Proof: Easy. $\square$

If we consider the set $\mathcal{B}$ of bounded functions from X into the metric space (Y, d), then in the lemma above we can take $d_\infty(f, g) = \sup_{x\in X}\{d(f(x), g(x))\}$ as the next lemma will show.

If (X, d) is a normed vector space $(V, |\ |)$, then $\|f\| = \sup\{|f(x)| : x \in X\}$ is a norm on the vector space of bounded functions from the set X into V, as it can easily be checked.

Lemma 9.2.4 *Suppose the sequence $(f_n)_n$ of functions from a set X into a metric space (Y, d) converges uniformly to f. Assume that the functions f_n are eventually bounded. Then f is bounded.*

Proof: Take $\epsilon = 1$ in the definition of the uniform convergence. Let N be such that for all $n > N$ and all x, $d(f_n(x), f(x)) < 1$. Let $n_\circ > N$ be such that $f_{n_\circ}$ is bounded. Let $b \in Y$ and $R \in \mathbb{R}$ be such that $f_{n_\circ}(x) \in B(b, R)$ for all $x \in R$. Then for all $x \in \mathbb{R}$, $d(f(x), b) \leq d(f(x), f_{n_\circ}(x)) + d(f_{n_\circ}(x), b) < 1 + R$. $\square$

Let $\mathcal{F}$ be the set functions from X into the metric space (Y, d). Set $d_\infty(f, g) = \sup_{x\in X}\{d(f(x), g(x)), 1\}$. Then d_∞ is a distance on $\mathcal{F}$ and a sequence $(f_n)_n$ of functions converge uniformly if and only if the sequence converges in this metric space.

The following criterion à la Cauchy shows the uniform convergence of a sequence without explicitly calculating the limit:

Theorem 9.2.5 (Cauchy Criterion for Uniform Convergence) *A sequence $(f_n)_n$ of functions from a set X into a complete metric space (Y, d) converges uniformly if and only if for any $\epsilon > 0$ there is an N_ϵ such that for all n, $m > N$ and for all $x \in X$, $d(f_n(x), f_m(x)) < \epsilon$.*

Proof: By hypothesis, for all $x \in X$, $(f_n(x))_n$ is a Cauchy sequence, hence it has a limit. Let $f(x)$ denote this limit. We will prove that $\lim_{n\to\infty} f_n \stackrel{u}{=} f$. Let $\epsilon > 0$.

By hypothesis there is an N, independent of x, such that for all n, $m > N$, $d(f_n(x), f_m(x)) < \epsilon/2$.

Let $x \in X$ be given. Since the sequence $(f_n(x))_n$ converges to $f(x)$, there is an N_x (that may depend on x) such that for all $m > N_x$, $d(f_m(x), f(x)) < \epsilon/2$.

Let $n > N$ be any natural number. Let $x \in X$ be any. We will show that $d(f_n(x), f(x)) < \epsilon$, proving that $(f_n)_n$ converges to f uniformly. Let $m_x > \max\{N, N_x\}$ be fixed. Then $d(f_n(x), f(x)) \leq d(f_n(x), f_{m_x}(x)) + d(f_{m_x}(x), f(x)) < \epsilon/2 + \epsilon/2 = \epsilon$. $\square$

Theorem 9.2.6 *Let $(f_n)_n$ be a sequence of functions defined on a subset A of $\mathbb{R}$ that converges uniformly to a function f. If each f_n is continuous at a point $a \in A$, then f is continuous at a.*

Proof: Let $\epsilon > 0$. We want to find a $\delta > 0$ such that for $x \in A$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. For all n and $x \in A$ we have

$$|f(x) - f(a)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(a)| + |f_n(a) - f(a)|.$$

We will make each term of the right hand side less than $\epsilon/3$, for large enough n and x close enough to a.

Since $\lim_{n\to\infty} f_n \overset{=}{u} f$, there is an N such that for $n > N$ and $x \in A$, $|f(x) - f_n(x)| < \epsilon/3$. Thus we also have $|f_n(a) - f(a)| < \epsilon/3$. (Here we use the uniform continuity).

Pick such an $n > N$. Since f_n is continuous at a, there is a $\delta > 0$ such that for $x \in A$ and $|x - a| < \delta$, $f_n(x) - f_n(a)| < \epsilon/3$. □

Example. The function $f_n(x) = (1 - |x|)^n$ defined on $[-1, 1]$ converges pointwise (but not uniformly) and its limit is not continuous at $a = 0$.

This example shows also that pointwise convergence on a compact set does not imply the uniform convergence.

Exercises.

i. Show that the functions $f_n(x) = \frac{x}{nx+1}$ converge uniformly on $[0, 1]$.

ii. Determine the pointwise or the uniform convergence of the following functions:

   i. $f_n(x) = \sum_{k=0}^{n} x^k$ on $X = (-1, 1)$.

   ii. $f_n(x) = \sum_{k=0}^{n} x^k$ on $X = (-1/2, 1/2)$.

   iii. $f_n(x) = x^n$ on $X = (0, 1)$.

   iv. $f_n(x) = x^n$ on $X = [-\alpha, \alpha]$ where $\alpha \in (0, 1)$.

   v. $f_n(x) = \frac{x^n}{1-x}$ on $X = (0, 1)$.

   vi. $f_n(x) = \frac{x^n}{1-x}$ on $X = (0, 1 - \alpha)$ for any $\alpha \in (0, 1)$.

iii. Let $f_n(x) = \frac{1}{1+x^n}$

   a. Find the set $A = \{x \in R : (f_n(x))_n \text{ converges}\}$.

   For $x \in A$, let $f(x) = \lim_{n\to\infty} f_n(x)$.

   b. What is f?

   c. Is the convergence uniform? Justify your answer.

   d. Discuss the uniform convergence of $(f_n)_n$ in the (open or closed) intervals contained in A.

iv. Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as follows:

$$f_n(x) = \begin{cases} x^2 & \text{if } |x| \le n \\ n^2 & \text{if } |x| \ge n \end{cases}$$

Does $(f_n)_n$ converge uniformly?

v. Let $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as follows:

$$f_n(x) = \begin{cases} x^2 & \text{if } |x| \ge n \\ n^2 & \text{if } |x| \le n \end{cases}$$

Does $(f_n)_n$ converge uniformly?

vi. Is part iv of Theorem 9.1.1 true for uniform convergence?

vii. Let $(f_n)_n$ be a sequence of functions from X into a metric space Y. Let A, $B \subseteq X$. Assume that the sequence $(f_n)_n$ converges uniformly on A and also on B. Show that $(f_n)_n$ converges uniformly on $A \cup B$.

viii. Study the convergence of $f_n(x) = \frac{nx}{1+nx^2}$ on intervals of $\mathbb{R}$.

ix. Study the convergence of $f_n(x) = \sin(nx)/n$.

x. Study the convergence of $f_n(x) = \frac{x}{1+x^n}$.

xi. Assume $(f_n)_n$ converges uniformly to a function f on a compact set $K \subseteq \mathbb{R}$. Let $g : K \longrightarrow \mathbb{R}$ be a continuous function satisfying $g(x) \neq 0$. Show that $(f_n/g)_n$ converges uniformly to f/g on K.

xii. Give an example of functions $(f_n)_n$ and $(g_n)_n$ that converge uniformly, but whose product sequence $(f_n g_n)_n$ does not converge uniformly.

9.3 Uniform Convergence of a Series of Functions

The pointwise or the uniform convergence of a series $\sum_{n\in\mathbb{N}} f_n(x)$ of functions is defined in the expected way as the pointwise or the uniform convergence of the partial sums $\sum_{i=0}^{n} f_i$. We define it more precisely

Let f_n $(n \in \mathbb{N})$ and f be functions defined on a subset A of $\mathbb{R}$. The series of $\sum_{i=0}^{\infty} f_n$ is said to **converge uniformly** to f if the sequence of partial sums $(s_n)_n$ where $s_n = f_1 + \ldots + f_n$ converges uniformly to f.

Theorem 9.3.1 *Assume each f_i is continuous and $\sum_{i=0}^{\infty} f_i$ converges uniformly to f on A. Then f is continuous on A.*

Proof: Follows directly from Theorem 9.2.6. □

Theorem 9.3.2 (Cauchy's Criterion for Uniform Convergence of Series) *The series $\sum_{i=0}^{\infty} f_n$ converges uniformly on $A \subseteq \mathbb{R}$ if and only if for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $N < m < n$, $|\sum_{i=m+1}^{n} f_n| < \epsilon$.*

Proof: Exercise. □

Theorem 9.3.3 (Weierstrass M-Test) *Let $(f_i)_i$ be a sequence of functions from a set X into $\mathbb{C}$. Suppose that $|f_i(x)| \leq M_i$ for all $x \in X$ and for all $i \in \mathbb{N}$. If the series $\sum_{i=0}^{\infty} M_i$ converges, then the sequence $\sum_{i=0}^{\infty} f_i$ converges uniformly on X.*

Proof: We will use Theorem 9.3.2. Let $\epsilon > 0$. Since the series $\sum_{i=0}^{\infty} M_i$ converges, by Cauchy Criterion for series, there is an $N \in \mathbb{N}$ such that for all $N < m < n$, $\sum_{i=m+1}^{n} M_i < \epsilon$. Thus for all $N < m < n$ and all $x \in X$, $|\sum_{i=m+1}^{n} f_i(x)| \leq \sum_{i=m+1}^{n} M_i < \epsilon$. □

Theorem 9.3.4 *If a power series $\sum_{i=0}^{\infty} a_n x^n$ converges absolutely at a point x_0 then it converges uniformly on the closed ball $\overline{B}(0, |x_0|)$.*

Proof: Follows directly from Weierstrass M-Test (Theorem 9.3.3). □

What happens at the end points? Here is the answer:

Theorem 9.3.5 (Abel's Theorem) *Let the power series $\sum_{n=0}^{\infty} a_n x^n$ converge at the point $R > 0$ (resp. at the point $-R < 0$) . Then the series converges uniformly on the interval $[0, R]$ (resp. on the interval $[-R, 0]$).*

To prove this theorem, we need the following lemma:

Lemma 9.3.6 (Abel's Lemma) *Let $(b_n)_n$ satisfy $b_1 \geq b_2 \geq \ldots \geq 0$. Let $\sum_{i=0}^{\infty} a_i$ be a series whose partial sums are bounded, by A say. Then for all $n \in \mathbb{N}$, $|a_1 b_1 + \ldots a_n b_n| \leq A b_1$.*

Proof: To be proven. □

Proof of Abel's Theorem. To be proven. □

Corollary 9.3.7 *If a power series converges pointwise on the set $A \subseteq \mathbb{R}$, then it converges uniformly on any compact set $K \subseteq A$.*

Proof: By Theorem 9.3.4 and Abel's Theorem (9.3.5). □

Corollary 9.3.8 *A power series is continuous at every point at which it is convergent.*

Exercises.

i. Show that if $\sum_{i=0}^{\infty} f_i$ converges uniformly, then $(f_n)_n$ converges uniformly to the zero function.

ii. Show that the series $\sum_{i=1}^{\infty} \cos(2^i x)/2^i$ converges uniformly. Therefore the limit function is continuous.

iii. Show that $\sum_{n=1}^{\infty} x^n/n^2$ is continuous on $[-1, 1]$.

9.4 Uniform Convergence and Metric

Let X be a set and $(V, |\ |)$ a normed vector space. Consider the normed vector space $B(X,V)$ of function from X into Y which are bounded. (See Exercise iii, page 53). As any normed vector space, $B(X,V)$ is a metric space (Lemma 5.3.1) and so we may speak about the convergence of a sequence of $B(X,V)$.

Lemma 9.4.1 *Let $(f_n)_n$ be a sequence from $B(X,V)$ and $f \in B(X,V)$. If $\lim_{n\to\infty} f_n = f$ in the metric of $B(X,V)$, then $\lim_{n\to\infty} f_n \stackrel{u}{=} f$.*

Conversely if $(f_n)_n$ is a sequence from $B(X,V)$ that converges uniformly to a function $f : X \longrightarrow V$, then $f \in B(X,V)$ and $\lim_{n\to\infty} f_n = f$ in the metric of $B(X,V)$.

Proof: We first show that if $(f_n)_n$ is a sequence from $B(X,V)$ that converges uniformly to a function $f : X \longrightarrow V$, then $f \in B(X,V)$

Let $\epsilon > 0$. Since $\lim_{n\to\infty} f_n = f$ in the metric of $B(X,V)$, there is an N such that for all $n > N$, $|f_n(x) - f(x)| < \epsilon$, i.e., by the very definition of the metric on $B(X,V)$, $\sup\{|f_n(x) - f(x)| : x \in X\} < \epsilon$. This means that $|f_n(x) - f(x)| < \epsilon$ for all $x \in X$, proving the uniform convergence.

Let us now show the converse. TO BE COMPLETED $\square$

9.5 Limits of Functions

Let (X, d) be a metric space. Let $A \subseteq X$ be a subset. A point $x \in X$ is called a **limit point** of A if any open subset of X containing x contains an element of A different from x.

Let (X, d) and (Y, d') be two metric spaces, A a subset of X and $f : A \longrightarrow Y$ a function. Let $a \in X$ be a limit point of A and $b \in Y$. We say that the **limit** of $f(x)$ when x goes to a is b if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$, if $x \neq a$ and $d(x, a) < \delta$, then $d'(f(x), b) < \epsilon$. We then write

$$\lim_{x\to a} f(x) = b.$$

If $X = \mathbb{R}$ or $\mathbb{C}$ and A is an unbounded subset of X, we would like to be able to speak about the limit at infinity. The definitions in $\mathbb{R}$ or in $\mathbb{C}$ are different.

Let A be a subset of $\mathbb{R}$ without upper bound. Let (Y, d') be a metric space. Let $f : A \longrightarrow Y$ be a function. Let $b \in Y$. We say that the **limit** of $f(x)$ when x goes to ∞ is b if for all $\epsilon > 0$ there is a $N > 0$ such that for all $x \in A$, if $x > N$ then $d'(f(x), b) < \epsilon$. We then write

$$\lim_{x\to\infty} f(x) = b.$$

The limit when x goes to $-\infty$ is defined similarly.

Note that when $A = \mathbb{N}$, we obtain the definition of the limit of a sequence in the metric space Y.

Now let A be an unbounded subset of $\mathbb{C}$. Let (Y, d') be a metric space. Let $f : A \longrightarrow Y$ be a function. Let $b \in Y$. We say that the **limit** of $f(z)$ when z goes to ∞ is b if for all $\epsilon > 0$ there is a $N > 0$ such that for all $z \in A$, if $|z| > N$ then $d'(f(z), b) < \epsilon$. We then write

$$\lim_{z \to \infty} f(z) = b.$$

Exercises.

i. Show that $\lim_{x \to 5}(x^2 - 3x + 5) = 15$ by using the definition of limits.

ii. Let $X = \mathbb{R}$, $A = \mathbb{R}^{>0}$, $Y = \mathbb{R}$ and $f : A \longrightarrow Y$ be defined by $f(x) = 1/x$. Show that $\lim_{x \to 0} f(x)$ does not exist.

iii. Let $X = A = Y = \mathbb{R}$ with the usual metric. Let $f : A \longrightarrow Y$ be defined as follows:
$$f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$
Show that $\lim_{x \to 0} f(x)$ does not exist.

iv. Let $A = (0, 1)$, $X = [0, 1]$, $Y = \mathbb{R}$ with the usual metric. Let $f : A \longrightarrow Y$ be defined as follows:
$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \notin \mathbb{Q} \end{cases}$$
Show that $\lim_{x \to 1} f(x)$ does not exist.

Theorem 9.5.1 *The limit* $\lim_{x \to a} f(x)$ *of a function* f *is unique whenever it exists.*

Theorem 9.5.2 *The relationship of the limits with the functional operations.*

9.6 Convergence of a Family of Functions

If $f(x, y)$ is a function that depends on two parameters, sometimes we may want to view the parameter x as an index set. In this case we may write $f_x(y)$ instead of $f(x, y)$. Suppose f is such a function going from $A \times Y$ into Z where $A \subseteq X$ and X and Z are metric spaces. Suppose that $x_0 \in X$ is a limit point of A. Let $f : Y \longrightarrow Z$ be a function. Then we say that f is the **pointwise limit** of the family $(f_x(y))_{x \in A}$ when x approaches x_0 and we write $\lim_{x \to x_0} f_x \overset{p}{=} f$ if for all $y \in Y$, $\lim_{x \to x_0} f_x(y) = f(y)$. This makes sense because for fixed $y \in Y$, $x \mapsto f_x(y)$ is map from the metric space X into the metric space Z. More formally this means the following: For all $y \in Y$ and $\epsilon > 0$, there is a $\delta = \delta_{\epsilon, y} > 0$ such that if $x \in B(x_0, \delta) \setminus \{x_0\}$ then $f_x(y) \in B(f(y), \epsilon)$. We emphasize the fact that δ depends on y. If we can choose δ independent of y, then we say that the convergence is uniform.

If x varies over $\mathbb{R}$ or $\mathbb{C}$, we would like to be able to make x to ∞. This is easy to do. Assume first $A \subseteq \mathbb{R}$ has no upper bound. Define $\lim_{x\to\infty} f_x \overset{p}{=} f$ if $\lim_{x\to\infty} f_x(y) = f(y)$ for all $y \in Y$, i.e. if for all $y \in Y$ and $\epsilon > 0$ there is an N such that for all $x \in A$, if $x > N = N_{\epsilon,y}$ then $d'(f_x(y), f(y)) < \epsilon$. Limit when x goes to $-\infty$ is defined similarly. If $A \subseteq \mathbb{C}$ is an unbounded subset, then we define $\lim_{x\to\infty} f_x \overset{p}{=} f$ if $\lim_{x\to\infty} f_x(y) = f(y)$ for all $y \in Y$, i.e. if for all $y \in Y$ and $\epsilon > 0$ there is an $N = N_{\epsilon,y}$ such that for all $x \in A$, if $|x| > N$ then $d'(f_x(y), f(y)) < \epsilon$.

The convergence is said to be **uniform** if N can be chosen independently of y.

Theorem 9.6.1 *The uniform limit of continuous functions is continuous.*

NOTE: I should think about defining these convergence definitions once for all, without specific cases.

9.7 Supplementary Topics

Theorem 9.7.1 (Weierstrass) *Let $(f_i)_i$ be a sequence of functions from an unbounded subset X of $\mathbb{R}$ into $\mathbb{C}$. Suppose that $|f_i(x)| \leq M_i$ for all $x \in X$ and for all $i \in \mathbb{N}$. Assume that the series $\sum_{i=0}^{\infty} M_i$ converges. Then the sequence $\sum_{i=0}^{\infty} f_i(x)$ converges uniformly on X. Assume also that $\lim_{x\to\infty} f_i(x) = \ell_i$ for all i. Then $\sum_{i=0}^{\infty} \ell_i$ converges absolutely and $\lim_{x\to\infty} \sum_{i=0}^{\infty} f_i(x) = \sum_{i=0}^{\infty} \ell_i$.*

Proof: The first part is just Weierstrass' M-Test. We prove the last part. Clearly $|\ell_i| \leq M_i$, so that the absolute convergence occurs. Let $\epsilon > 0$. Let n be such that $\sum_{i=n+1}^{\infty} M_i < \epsilon/4$. Choose x_0 such that for $x > x_0$ and $i = 0, 1, \ldots, n$, $|f_i(x) - \ell_i| < \epsilon/2(n+1)$. We have

$$\begin{array}{rcl} |\sum_{i=0}^{\infty} f_i(x) - \sum_{i=0}^{\infty} \ell_i| & = & |\sum_{i=0}^{\infty}(f_i(x) - \ell_i)| \leq \sum_{i=0}^{\infty} |f_i(x) - \ell_i| \\ & = & \sum_{i=0}^{n} |f_i(x) - \ell_i| + \sum_{i=n+1}^{\infty} |f_i(x) - \ell_i| \\ & \leq & \sum_{i=0}^{n} |f_i(x) - \ell_i| + 2\sum_{i=n+1}^{\infty} M_i \\ & < & \sum_{i=0}^{n} \epsilon/2(n+1) + 2\epsilon/4 = \epsilon. \end{array}$$

This proves the theorem. $\square$

Example.

i. Let $n \in \mathbb{N}^{>0}$ and $z \in \mathbb{C}$. We have

$$(1 + z/n)^n = \sum_{i=0}^{n} \binom{n}{i} (z/n)^i.$$

Set

$$f_1(n) = 1 + z,$$

and for $i > 1$,

$$f_i(n) = \begin{cases} \begin{pmatrix} n \\ i \end{pmatrix} (z/n)^i = \frac{n(n-1)\dots(n-i+1)}{i!}(z/n)^i & \text{if } i \le n \\ 0 & \text{if } i > n \end{cases}$$

Since, for $i \le n$,

$$f_i(n) = \frac{z^i}{i!}\left(1 - \frac{1}{n}\right)\dots\left(1 - \frac{i-1}{n}\right),$$

we have

$$\lim_{n\to\infty} f_i(n) = z^i/i! =: M_i$$

and $|f_i(n)| \le M_i$ and $\sum_i M_i$ converges by Example i, page 90. Thus the conditions are realized and so $\lim_{n\to\infty}(1 + z/n)^n$ exists and is equal to $\sum_{i=0}^{\infty} z^i/i!$.

Chapter 10

Topological Spaces

10.1 Definition and Examples

A set X together with a set τ of subsets of X is called a **topological space** if τ satisfies the three conditions of Proposition 5.4.3:

i. $\emptyset$, $X \in \tau$.
ii. If $U_1, \ldots, U_n \in \tau$ then $U_1 \cap \ldots \cap U_n \in \tau$.
iii. If $U_i \in \tau$ for all i then $\bigcup_i U_i \in \tau$.

The elements of τ are called the **open subsets** of the topological space X.

Every metric space induces a topological space as stated in Proposition 5.4.3. But the converse is false as the following example shows: Let X be any set. Let $\emptyset$ and X be the only open subsets, i.e. let $\tau = \{\emptyset, X\}$. Then X becomes a topological space. If $|X| \geq 2$ then there is no metric on X that gives this topology. The topology on X is called the **coarsest topology** or the **weakest** topology on X.

A topological space whose topology is induced by a metric is called a **metrisable** topology.

Sometimes two different metrics on the same set give the same topologies, i.e. the same open sets. In this case we say that the two metrics are equivalent.

Exercises.

i. Show that the following metrics on $\mathbb{R}^2$ are equivalent.

ii. Show that every metric d on a set X is equivalent to the metric d' defined by $d'(x, y) = \max\{d(x, y), 1)$.

iii. Show that for equivalent metrics, Cauchy sequences, convergent sequences and limits do not change.

Discrete Topology. In the topology induced by the discrete metric, every subset is open. This is called the **discrete** or the **finest topology** on X.

Induced Topology. Let X be a topological space and $Y \subseteq X$. By letting the open subsets of Y to be the traces of open subsets of X on Y, we define a topology on Y. Thus an open subset of Y is of the form $U \cap Y$ where U is an open subset of X. This topology on Y is called the **restricted** or the **induced** topology.

Note that the open subsets of Y and X are different. So, we must be careful when speaking about the open subsets of Y. For this reason, we may say that a subset of Y is X-open or Y-open.

But if Y is open in X, then subsets of Y which Y-open are exactly X-open subsets of Y.

If the topology on X is induced by a metric and $Y \subseteq X$, then the restricted topology on Y is induced by the restricted metric on Y.

Order Topology. TO BE DEFINED.

Topology Generated. Let X be a set. Let $(\tau_i)_i$ be a family of topologies on X, i.e. each τ_i is the set of open subsets of a topology on X. Then $\cap_i \tau_i$ defines a topology on X. An open subset of $\cap_i \tau_i$ is a subset of X that is open in all these topologies.

Let $\wp$ be a set of subsets of a set X. Consider the intersection of all the topologies that contain $\wp$. By the previous item this is a topology. It is the smallest topology that contains the subsets of $\wp$ as open sets. We will denote this topology by $\langle \wp \rangle$ and call it the **topology generated by** $\wp$.

We can describe the topology above more precisely. Consider the subsets which are arbitrary unions of finite intersections of elements from $\wp$. These subsets form the open subsets of a topology and this topology is obviously the smallest topology that contains all the elements of $\wp$.

Examples. Let X be a set. The topology generated by $\emptyset$ consists of $\emptyset$ and X. If $A \subseteq X$, the topology generated by $\{A\}$ is $\{\emptyset, A, X\}$. If A, $B \subseteq X$, the topology generated by $\{A, B\}$ is $\{\emptyset, A \cap B, A, B, A \cup B, X\}$. The topology generated by the singleton sets is the set of all subsets of X.

Extended Real Line

Exercises.

i. Show that in a metric space a finite subset is closed.

ii. Let X be any set. Show that the cofinite subsets of X define a topology on X. Show that this topology is not metrisable if X is infinite.

iii. Show that a pseudometric on a set defines a topology in a natural way. ($d : X \times X \longrightarrow \mathbb{R}^{\geq 0}$ is a pseudometric if $d(x, x) = 0$, $d(x, y) = d(y, x)$ and $d(x, y) \leq d(x, z) + d(z, y)$.

iv. The topologies defined on $\mathbb{R}^n$ by the metrics d_p and d_∞ are all equal, i.e. any open subset with respect to any of them is also open in the other one.

v. Let X be a set and A, B, C be three subsets of X. Show that the topology generated by $\{A, B, C\}$ has at most ??? open subsets.

vi. Let $X = \mathbb{R}$ and $\wp$ the set of open and bounded intervals of $\mathbb{R}$. Show that the topology generated by $\wp$ is the topology induced by the usual metric.

vii. Let $X = \mathbb{R}$ and $\wp$ the set of intervals of the form $[a, b)$ where a, $b \in \mathbb{R}$. Is the topology generated by $\wp$ metrisable?

viii. Let $X = \mathbb{R}$ and $\wp$ the set of intervals of the form $[a, b)$ where a, $b \in \mathbb{Q}$. Is the topology generated by $\wp$ equal to the topology above?

ix. Let X be a set. Let d and d' be two distance functions on X. Show that the two metrics generate the same topology if and only if any sequence of X that converges for one of the metrics, converges to the same element for the other metric.

10.2 Closed Subsets

The complement of an open subset of a topological space is called a **closed subset**. The properties of open subsets reflect upon closed subsets naturally:

Proposition 10.2.1 *Let X be a topological space. Then*
i. $\emptyset$ and X are closed subsets.
ii. A finite union of closed subsets is a closed subset.
iii. An arbitrary intersection of closed subsets is closed.

10.3 Interior

Let X be a topological space. Let $A \subseteq X$. Then the union A° of all the open subsets of X which are in A is also an open subset of X which is in A. Thus A° is the largest open subset of X which is in A, in other words it contains all the open subsets (in X) of A. The set A° is called the **interior** of A.

Lemma 10.3.1 *Let A, $B \subseteq X$. Then the following hold:*
i. A is open if and only if $A^\circ = A$. In particular $A^{\circ\circ} = A^\circ$.
ii. If $A \subseteq B$ then $A^\circ \subseteq B^\circ$.
iii. $A^\circ \cap B^\circ = (A \cap B)^\circ$.
iv. $A^\circ \cup B^\circ \subseteq (A \cup B)^\circ$.

Proof: i. Clearly, if A is open then A is the largest open subset of itself, so $A = A^\circ$.
ii. Since $A^\circ \subseteq A \subseteq B$, A° is an open subset of B. Thus $A^\circ \subseteq B^\circ$.

10.4 Closure

Let X be a topological space. Let $A \subseteq X$. Then the intersection $\overline{A}$ of all the closed subsets of X that contain A is also an closed subset of X that contains A. Thus $\overline{A}$ is the smallest closed subset of X which contains A. The set $\overline{A}$ is called the **closure** of A.

Lemma 10.4.1 *Let A, $B \subseteq X$. Then the following hold:*
i. A is closed if and only if $\overline{A} = A$. In particular $\overline{\overline{A}} = \overline{A}$.
ii. If $A \subseteq B$ then $\overline{A} \subseteq \overline{B}$.
iii. $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.
iv. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Lemma 10.4.2 *$x \in \overline{A}$ if and only if every open subset containing x intersects A nontrivially.*

Exercises.

i. Find an example where the inclusion of Lemma 10.3.1.iii is strict.

ii. Find an example where the inclusion of Lemma 10.4.1.iii is strict.

iii. Show that $(\overline{A})^c = (A^c)^\circ$ and that $(A^\circ)^c = \overline{A^c}$.

iv. Find the interior and the closure of an arbitrary subset in the finest topology.

v. Find the interior and the closure of an arbitrary subset in the coarsest topology.

vi. Find the closure and the interior of an arbitrary interval in the usual topology of $\mathbb{R}$.

10.5 Base of a Topology

Let X be a topological space. Let $\mathcal{B}$ be a set of open subsets of X such that every open subset of X is a union of open sets from $\mathcal{B}$. Then $\mathcal{B}$ is called a **base** of the topology. Obviously, the set of open subsets of X form a base of the topological space X.

Examples.
1. Assume X is a metric space. Then the open balls of X form a base of the topology.
2. Assume $X = \mathbb{R}^n$ with the Euclidean topology. Then the balls of the form $B(a, q)$ such that $a \in \mathbb{Q}^n$ and $q \in \mathbb{Q}$ form a base of the topology. Note that this base is countable.
3. Let $\wp$ be a set of subsets of a set X. The topology $\langle \wp \rangle$ generated by $\wp$ has a base consisting of sets of the form $U_1 \cup \ldots \cup U_n$ where $n \in \mathbb{N}$ and $U_i \in \wp$.

Let X be a topological space and $x \in X$. Let $\mathcal{B}_x$ be a set of open subsets of X that contain x such that every open subset of X that contains x contains one of the of open sets from $\mathcal{B}_x$. Then $\mathcal{B}_x$ is called a **base** of the topology at the point x.

Assume X is a metric space and $x \in X$. Then the open balls of the form $B(x, q)$ where $q \in \mathbb{Q}$ form a base of the topology at x. Note that this base at x is countable.

10.6 Compact Subsets

A topological space X is called **compact** if any open covering of X has a finite subcovering. A subspace A of X is compact if A is compact with the induced topology.

If X has finitely many open sets then any subset of X is compact.

A subset A of a topological space is called compact if A is compact as a topological space, i.e. if any family of open subsets of X that covers A has a finite subcover that covers A.

Singleton subsets of a topological set are certainly compact. Clearly the union of finitely many compact subsets of a topological space is compact.

Lemma 10.6.1 *Let K be a closed subset of a compact topological space X. Then K is compact.*

Proof: Let $(U_i)_{i \in I}$ be an open covering of K. Then $(U_i)_{i \in I} \cup \{K^c\}$ is an open covering of X. Hence a finite number of $(U_i)_{i \in I} \cup \{K^c\}$ cover X. Hence a finite number of $(U_i)_{i \in I}$ cover K. $\square$

Proposition 10.6.2 *A compact subset of a metric space is closed and bounded.*

Proof: Let K be a compact subset of the metric space (X, d). Let $a \in X$ be any subset. Then the concentric open balls $(B(a, n))_{n \in \mathbb{N}}$ cover K. Therefore a finite number of them, hence only one of them, covers K. This shows that K is bounded.

We now show that K is closed, by showing that its complement K^c is open. Let $a \in K^c$. The the concentric open subsets $(\overline{B}(a, 1/n)^c)_{n \in \mathbb{N}}$ cover K. Hence only one of them cover K. If $K \subseteq \overline{B}(a, 1/n)^c$, then $B(a, 1/n) \subseteq K^c$. Thus K^c is open and K is closed. $\square$

Exercises.

i. The converse of the above proposition is false. Find a counterexample.

Theorem 10.6.3 (Heine-Borel Theorem) *A closed and bounded subset of $\mathbb{R}^n$ (with the usual metric) is compact.*

Proof: Assume first that $n = 1$. Let K be a closed and bounded subset of $\mathbb{R}$. By Lemma 10.6.1, we may assume that $K = [a, b]$ is a closed interval. Assume $[a, b]$ is not compact. Let $(U_i)_{i\in I}$ be an open covering of $[a, b]$ without a finite subcover. We want to choose sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ so that for all n,

i) $x_n \leq x_{n+1} < y_{n+1} \leq y_n$,

ii) $0 < y_n - x_n < (b - a)/2^n$,

iii) No finite subcover of $(U_i)_{i\in I}$ covers $[x_n, y_n]$.

Let $x_0 = a$ and $y_0 = b$. Assume $x_0, x_1, \ldots, x_n$ and $y_0, y_1, \ldots, y_n$ are chosen so that (i, ii, iii) hold, except of course (i) for n (since x_{n+1} is not defined yet). We choose x_{n+1} and y_{n+1} as follows. One of the two intervals $[x_n, \frac{x_n+y_n}{2}]$ and $[\frac{x_n+y_n}{2}, y_n]$ is not covered by a finite subcover of $(U_i)_{i\in I}$. If $[x_n, \frac{x_n+y_n}{2}]$ is not covered by a finite subcover of $(U_i)_{i\in I}$, let $x_{n+1} = x_n$ and $y_{n+1} = \frac{x_n+y_n}{2}$. Otherwise, $[\frac{x_n+y_n}{2}, y_n]$ is not covered by a finite subcover of $(U_i)_{i\in I}$ and in this case we let $x_{n+1} = \frac{x_n+y_n}{2}$ and $y_{n+1} = y_n$. Clearly (i, ii, iii) hold.

By Theorem 3.1.12 (see also Exercise x, page 58), $\cap_{n\in\mathbb{N}}[x_n, y_n]$ is a singleton set, say $\{c\}$. Since $c \in [a, b]$, there is an $i \in I$ such that $c \in U_i$. Let $\epsilon > 0$ be such that $[c - \epsilon, c + \epsilon] \subseteq U_i$. By (ii) there is an $n \in \mathbb{N}$ such that $y_n - x_n < \epsilon$. Then $[x_n, y_n] \subseteq [c - \epsilon, c + \epsilon] \subseteq U_i$ as one can show easily. But this contradicts (iii), because only one of the U_i's suffices to cover $[x_n, y_n]$.

The proof for $n > 1$ is similar, instead of the intervals $[x_n, y_n]$ be find closed cubes $(C_n)_n$ by dividing the previous cube C_n to 2^n equal parts and choosing C_{n+1} the one which is not covered by a finite subcover of $(U_i)_{i\in I}$. □

Theorem 10.6.4 *If $(K_n)_{n\in\mathbb{N}}$ is a descending sequence of nonempty compact subsets of $\mathbb{R}^n$, then $\cap_n K_n \neq \emptyset$.*

Proof: Assume otherwise. Then $\cup_n K_n^c = \mathbb{R}$, so that $(K_n^c)_{n\in\mathbb{N}}$ is an open covering of K_0. Thus a finite subsequence of $(K_n^c)_{n\in\mathbb{N}}$ covers K_0, hence $K_0 \subseteq K_n^c$. Since $K_n \subseteq K_0$, this implies that $K_n = \emptyset$, a contradiction. □

Exercises.

i. Let $(K_i)_i$ be a family of nonempty compact subsets of a topological space. Show that if a finite intersection of the K_i's is nonempty, then $\cap_i K_i \neq \emptyset$.

ii. X be a complete metric space. Let p be an element not in X. Show that the metric of X cannot be extended to a metric on $X \cup \{p\}$ without p being isolated.

   Conclude that if $f : \mathbb{R} \longrightarrow \mathbb{R}^{>0}$ is a function satisfying $|f(x) - f(y)| \leq |x - y| \leq f(x) + f(y)$ for all $x, y \in \mathbb{R}$ then f is bounded away from 0, i.e. there is an $\epsilon > 0$ such that $f(x) > \epsilon$ for all $x \in \mathbb{R}$. (**Hint**: Set $d(x, p) = f(x)$ for $x \in \mathbb{R}$).

iii. Let $F \subseteq \mathbb{R}^n$ be a nonempty closed subset. Let $A \in \mathbb{R}^n$. Show that there is a $B \in F$ such that $d(A, B) = \inf\{d(A, P) : P \in F\}$. (**Proof:** Let $d = \inf\{d(A, P) : P \in F\}$. Let $\epsilon > 0$. Then $\overline{B}(A, d+\epsilon) \cap F$ is a closed and bounded subset of $\mathbb{R}^n$, hence it is compact. The distance function f from the compact subset $\overline{B}(A, d+\epsilon) \cap F$ into $\mathbb{R}$ defined by $f(P) = d(A, P)$ is continuous, so it attains its infimum d.)

10.7 Convergence and Limit Points

Exercises. For each of the topological spaces (X, τ), describe the convergent sequences and discuss the uniqueness of their limits.

i. $\tau = \wp(X)$. ($\wp(X)$ is the set of all subsets of X, 2 pts.).

Answer: Only the eventually constant sequences converging to that constant.

ii. $\tau = \{\emptyset, X\}$ (2 pts.).

Answer: All sequences converge to all elements.

iii. $a \in X$ is a fixed element and τ is the set of all subsets of X that do not contain a, together with X of course. (5 pts.)

Answer: First of all, all sequences converge to a. Second: If a sequence converges to $b \neq a$, then the sequence must be eventually the constant b.

iv. $a \in X$ is a fixed element and τ is the set of all subsets of X that contain a, together with $\emptyset$ of course. (5 pts.)

Answer: Only the eventually constant sequences converge to a. A sequence converge to $b \neq a$ if and only if the sequence eventually takes only the two values a and b.

v. τ is the set of all cofinite subsets of X, together with the $\emptyset$ of course. (6 pts.)

Answer: All the sequences without more than one infinitely repeating terms converge to all elements. Eventually constant sequences converge to the constant. There are no others.

vi. Let τ be the topology on $\mathbb{R}$ generated by $\{[a, b) : a,\, b \in \mathbb{R}\}$. Compare this topology with the Euclidean topology. (3 pts.) Is this topology generated by a metric? (20 pts.)

Answer: Any open subset of the Euclidean topology is open in this topology because $(a, b) = \cup_{n=1}^{\infty}[a + 1/n, b)$. But of course $[0, 1)$ is not open in the usual topology.

Assume a metric generates the topology. Note that $[0, \infty)$ is open as it is the union of open sets of the form $[0, n)$ for $n \in \mathbb{N}$. Thus the sequence $(-1/n)_n$ cannot converge to 0. In fact for any $b \in \mathbb{R}$, no sequence can converge to b from the left. Thus for any $b \in \mathbb{R}$ there is an $\epsilon_b > 0$ such that $B(b, \epsilon_b) \subseteq [b, \infty)$. Let b_0 be any point of $\mathbb{R}$. Let $\epsilon_0 > 0$ be such that $B(b_0, \epsilon_0) \subseteq [b_0, \infty)$. Since $\{b_0\}$ is not open, there is $b_1 \in B(b_0, \epsilon_0) \setminus \{b_0\}$.

Let $0 < \epsilon_1 < \epsilon_0/2$ be such that $B(b_1, \epsilon_1) \subseteq [b_1, \infty) \cap B(b_0, \epsilon_0)$. Inductively we can find $(b_n)_n$ and $(\epsilon_n)_n$ such that $B(b_n, \epsilon_n) \subseteq [b_n, \infty) \cap B(b_{n-1}, \epsilon_{n-1}) \setminus \{b_{n-1}\}$ and $\epsilon_n < \epsilon_0/2^n$. Then $(b_n)_n$ is a strictly increasing convergent sequence, a contradiction.

10.8 Connected Sets

A topological space is called **connected** if it is not the union of two nonempty disjoint subsets. TConclude hus a topological space X is connected if whenever U and V are disjoint open subsets of X whose union is X, one of the open subsets must be the emptyset. Otherwise X is called **disconnected**. Note that if $X = U \cup V$ with $U \cap V = \emptyset$, then U and V are also closed. Note also that a singleton set is connected.

Lemma 10.8.1 *The union of two connected subspace with a nonempty intersection is connected. The union of a chain of connected subsets is connected.*

Lemma 10.8.2 *An open and connected subset of a topological space is contained in a maximal open and connected subset, which is necessarily closed.*

Theorem 10.8.3 *A subset X of $\mathbb{R}$ is connected if and only if X is an interval (all possible kinds).*

Define **totaly disconnected**. Show $\mathbb{Q}$ is totally disconnected. Show $\mathbb{Z}$ and $\mathbb{Z}_p$ are totally disconnected with the p-adic metric.

Chapter 11

Continuity

11.1 Continuity on Metric Spaces

Let (X, d) and (Y, d') be two metric spaces, $f : X \longrightarrow Y$ a function and $a \in X$. We say that f is **continuous at** a if for any $\epsilon > 0$ there is a $\delta > 0$ such that for all $x \in X$ if $d(x, a) < \delta$ then $d'(f(x), f(a)) < \epsilon$. The last condition says that if $x \in B(a, \delta)$ then $f(x) \in B(f(a), \epsilon)$, i.e. that $f(B(a, \delta)) \subseteq B(f(a), \epsilon)$, i.e. that $B(a, \delta) \subseteq f^{-1}(B(f(a), \epsilon))$. Thus $f : X \longrightarrow Y$ is continuous at $a \in X$ if and only if a is in the interior of $f^{-1}(B(f(a), \epsilon))$ for any $\epsilon > 0$. It follows that $f : X \longrightarrow Y$ is continuous at $a \in X$ if and only if a is in the interior of $f^{-1}(V)$ for any open subset V of Y containing $f(a)$. We note this for future reference:

Lemma 11.1.1 *Let (X, d) and (Y, d') be two metric spaces, $f : X \longrightarrow Y$ a function and $a \in X$. Then $f : X \longrightarrow Y$ is continuous at $a \in X$ if and only if a is in the interior of $f^{-1}(V)$ for any open subset V of Y containing $f(a)$.* □

Note that if $a \in X$ is an **isolated point** of X, i.e. if $B(a, \delta) = \{a\}$ for some $\delta > 0$ then f is always continuous at a. A point of X which is not an isolated point is called a **limit point** of X.

Lemma 11.1.2 *Let (X, d) and (Y, d') be two metric spaces, $f : X \longrightarrow Y$ a function and $a \in X$ a limit point of X. Then f is* **continuous at** *a if and only if $\lim_{x \to a} f(x) = f(a)$.*

Proof: This is immediate. □

Lemma 11.1.3 *Let (X, d) and (Y, d') be two metric spaces, $f : X \longrightarrow Y$ a function and $a \in X$. Then f is* **continuous at** *a if and only if for any sequence $(x_n)_n$ of X converging to a, $(f(x_n))_n$ is a sequence of Y converging to $f(y)$.*

Proof: Suppose f is continuous at a. Let $(x_n)_n$ be a sequence of X converging to a. Let $\epsilon > 0$. Let $\delta > 0$ be such that $f(B(a, \delta)) \subseteq B(f(a), \epsilon)$. Let N be such that for all $n > N$, $d(x_n, a) < \delta$, i.e. $x_n \in B(a, \delta)$. Hence $f(x_n) \in$

$f(B(a,\delta)) \subseteq B(f(a),\epsilon)$, i.e. $d'(f(x_n),f(a)) < \epsilon$. This shows that the series $(f(x_n))_n$ converges to $f(a)$.

Conversely, suppose that for any sequence $(x_n)_n$ of X converging to a, $(f(x_n)_n$ is a sequence of Y converging to $f(y)$. Assume that f is not continuous at ϵ. Let $\epsilon > 0$ be such that for every $\delta > 0$, there is an $x_\delta \in B(a,\delta)$ for which $f(x_\delta) \notin B(f(a),\epsilon)$. Choose $\delta = 1/n$ for $n \in \mathbb{N}^{>0}$ and let $y_n = x_{1/n}$. Then $(y_n)_n$ converges to a because $d(y_n,a) = 1/n$. But $d(f(y_n),f(a)) > \epsilon$ and so the sequence $(f(y_n))_n$ does not converge to $f(a)$. This is a contradiction. □

Let (X,d) and (Y,d') be two metric spaces and $f : X \longrightarrow Y$ a function. We say that f is **continuous** if f is continuous at every point of X.

Lemma 11.1.4 *Let (X,d) and (Y,d') be two metric spaces and $f : X \longrightarrow Y$ a function. The following conditions are equivalent:*
i. f is continuous
ii. The preimage of every open subset of Y is an open subset of X.
iii. The preimage of any open ball in Y is an open subset of X.
iv. The preimage of every closed subset of Y is a closed subset of X.
v. For any convergent sequence $(x_n)_n$ of X, $f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$.

Proof: (i $\to$ ii) Suppose f is continuous. It is enough to show that the preimage of an open ball of Y is an open subset of X. Let $b \in Y$ and $r > 0$. Let $a \in f^{-1}(B(b,r))$. Then $f(a) \in B(b,r)$. Let $\epsilon = r - d(f(a),b) > 0$. Then $B(f(a),\epsilon) \subseteq B(b,r)$. Since f is continuous at a, there is a $\delta > 0$ such that $B(a,\delta) \subseteq f^{-1}(B(f(a),\epsilon)) \subseteq f^{-1}(B(b,r))$. It follows that $f^{-1}(B(b,r))$ is open.

(ii $\to$ i) Suppose now that the preimage of every open subset of Y is an open subset of X. Let $a \in X$. Let $\epsilon > 0$. Since $f^{-1}(B(f(a),\epsilon))$ is open and a is in this set, there is a $\delta > 0$ such that $B(a,\delta) \subseteq f^{-1}(B(f(a),\epsilon))$. It follows that f is continuous at a.

The rest of the equivalences are now immediate. □

Note that in the lemmas 11.1.1 and 11.1.4 above the existence of a metric on X and Y disappeared, only topological concepts are left. This will be the basis for extending the concept of continuity of a function between metric spaces to the concept of continuity of a function between topological spaces. We will do this in the next subsection.

Corollary 11.1.5 *Continuity of functions between metric spaces depends on the topologies generated by the metrics rather than on the metrics themselves. In other words continuity of functions is preserved under equivalent metrics.*

Lemma 11.1.6 *Let (X,d) be a metric space. Then the map $d : X \times X \longrightarrow \mathbb{R}$ is continuous.*

Proof: It is enough to show that the inverse image of any open bounded interval (r,s) is open. Let us take the sup distance on $X \times X$. Let $(a,b) \in X \times X$ be such that $d(a,b) \in (r,s)$. Let $x \in B(a,s) \setminus \overline{B}(a,r)$ and $y \in B(b,s) \setminus \overline{B}(b,r)$ Then $r < \sup(d(a,x),d(b,y)) = d((a,b),(x,y)) = \sup(d(a,x),d(b,y)) < s$. Since $B(a,s) \setminus \overline{B}(a,r)$ and $B(b,s) \setminus \overline{B}(b,r)$ are open in X, this proves that d is continuous. □

Exercises.

i. Let $f : [a,b] \longrightarrow [a,b]$ be a function with the following property: There exists a real number $c \in (0,1)$ such that for any $x, y \in [a,b]$ one has

$$|f(x) - f(y)| < c|x - y|.$$

a) Show that f is continuous, and

b) Show that there exists an $x \in [a,b]$ such that $f(x) = x$.

ii. Assume that X is a complete metric space which is also connected. Show that for any $a \in X$ and any $r \geq 0$ there is a $b \in X$ such that $d(a,b) = r$. Conclude that if x has more than one point, show that X must be uncountable.

11.2 Continuity on Topological Spaces

Let X and Y be two topological spaces. A function $f : X \to Y$ is called **continuous** if $f^{-1}(V)$ is open for all open subsets V of Y. The function f is said to be continuous at a if a is in the interior of $f^{-1}(V)$ for any open subset V of Y containing $f(a)$. By lemmas 11.1.1 and 11.1.4, this generalizes continuity in metric spaces.

Lemma 11.2.1 *Let X and Y be two topological spaces. A function $f : X \to Y$ is continuous if and only if f is continuous at a for all $a \in X$.*

Proof: Left as an exercise. $\square$

Examples.

i. Let X be the discrete topological space (i.e. every subset is open). Then any function from X into any topological space is continuous.

ii. Let X be any topological space. The identity function Id_X from X into X is continuous.

iii. One should note that the identity map $\mathrm{Id}_X : X \longrightarrow X$ from any topological space into itself is continuous if one considers the domain X and the arrival set X with the same topologies (i.e. the same open subsets). Otherwise this may be false. For example if X_1 denotes the topological space on the set X where only X and $\emptyset$ are open and X_1 denotes the discrete topological space on X, then the identity map $\mathrm{Id}_X : X_1 \longrightarrow X_2$ is not continuous unless $|X| = 1$.

iv. Let X be a set. Let X_1 and X_2 be two topologies on the set X. Then the map $\mathrm{Id}_X : X_1 :\longrightarrow X_2$ is continuous if and only if any subset of X which is open for the topological space X_2 is open in the topological space X_1. In this case we say that X_1 is a **refinement** of X_2.

v. Let X and Y be any two topological spaces. Then any constant map from X into Y is continuous.

Proposition 11.2.2 *i. Composition of continuous functions is continuous.*

ii. A function is continuous if and only if the inverse image of a closed set is closed.

iii. Let X and Y be topological spaces. Let $\mathcal{B}$ be a base of Y. Then a function $f : X \longrightarrow Y$ is continuous if and only if the inverse image of a set in $\mathcal{B}$ is open in X.

iv. A function $f : X \longrightarrow Y$ is continuous if and only if the inverse image of a closed subset of Y is closed in X.

Let X be a set, X_i be topological spaces and $f_i : X \longrightarrow X_i$ be functions. Then there is a smallest/weakest topology on X that makes all the functions f_i continuous. This topology is generated by the set

$$\{f_i^{-1}(U) : i \in I \text{ and } U \subseteq X_i \text{ is open}\}.$$

Exercises.

i. By using the definition of continuity, show that the function $f(x) = \frac{x}{x-1}$ is continuous in its domain of definition.

ii. Let X be a topological space. Given two continuous numerical functions f and g on X, show that $\max\{f(x), g(x)\}$ is also continuous.

iii. Show that if $+ : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ and $\times : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous.

iv. Show that if $f,\, g : X \longrightarrow \mathbb{R}^n$ are continuous, then so is $f + g$.

v. Show that if $f,\, g : X \longrightarrow \mathbb{C}$ are continuous, then so are $f + g$ and fg.

vi. Show that any polynomial map from $\mathbb{C}$ into $\mathbb{C}$ is continuous.

vii. Show that the map $x \mapsto 1/x$ from $(0, 1)$ into $\mathbb{R}$ is continuous.

viii. Show that the map $x \mapsto \frac{(x-1)^2}{x(x-2)}$ is continuous from $(0, 2)$ into $\mathbb{R}$.

ix. Show that the map $x \mapsto \frac{1}{1+x^2}$ is a homeomorphism from $(0, \infty)$ onto $(0, 1)$.

x. Show that there is no homeomorphism from $(0, 1)$ onto $[0, 1)$.

xi. Show that the map that sends rational numbers to 0 and irrational numbers to 1 is not continuous.

xii. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function. Let $Z = \{r \in \mathbb{R} : f(r) = 0\}$. Show that $1/f : \mathbb{R} \setminus Z \longrightarrow \mathbb{R}$ is continuous.

xiii. Let X be a topological space and $Y \subseteq X$. Show that the smallest topology on Y that makes the inclusion $i : Y \hookrightarrow X$ is the induced topology.

xiv. Let X, Y be topological spaces, $f : X \longrightarrow Y$ a continuous function, $A \subseteq X$ and $f(A) \subseteq B \subseteq Y$. Show that $f_{|A} : A \longrightarrow B$ is continuous.

xv. Let X, Y be topological spaces, $f : X \longrightarrow Y$ a function and $f(X) \subseteq B \subseteq Y$. Show that $f : X \longrightarrow Y$ is continuous if and only if $f : X \longrightarrow B$ is continuous.

xvi. Consider the spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ with their usual topology. Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. Let $f : X \longrightarrow Y$ be a function. Show that f is continuous if and only if each $f_i = \pi_i \circ f : X \longrightarrow \mathbb{R}$ is continuous.

xvii. Let $f : X \longrightarrow Y$ be continuous and $B \subseteq Y$. Show that $f^{-1}(B^\circ) = f^{-1}(B)^\circ$ and $\overline{f^{-1}(B)} = f^{-1}(\overline{B})$.

xviii. **II. Subgroup Topology on $\mathbb{Z}$.** Let $\tau = \{n\mathbb{Z} + m : n, m \in \mathbb{Z}, n \neq 0\} \cup \{\emptyset\}$. We know that $(\mathbb{Z}, \tau)$ is a topological space.

a. Let $a \in \mathbb{Z}$. Is $\mathbb{Z} \setminus \{a\}$ open in τ?

Answer: Yes. $\cup_{n\neq 0,\pm 1} n\mathbb{Z} \cup (3\mathbb{Z} + 2) = \mathbb{Z} \setminus \{1\}$.Translating this set by $a - 1$, we see that $\mathbb{Z} \setminus \{a\}$ is open.

b. Find an infinite non open subset of $\mathbb{Z}$.

Answer: The set of primes is not an open subset. Because otherwise, for some $a \neq 0$ and $b \in \mathbb{Z}$, the elements of $a\mathbb{Z} + b$ would all be primes. So b, $ab + b$ and $2ab + b$ would be primes, a contradiction.

c. Let a, $b \in \mathbb{Z}$. Is the map $f_{a,b} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f_{a,b}(z) = az + b$ continuous? (Prove or disprove).

Answer: Translation by b is easily shown to be continuous. Let us consider the map $f(z) = az$. If $a = 0, 1, -1$ then clearly f is continuous. Assume $a \neq 0, \pm 1$ and that f is continuous. We may assume that $a > 1$ (why?) Choose a b which is not divisible by a. Then $f^{-1}(b\mathbb{Z})$ is open, hence contains a subset of the form $c\mathbb{Z} + d$. Therefore $a(c\mathbb{Z} + d) \subseteq b\mathbb{Z}$. Therefore $ac = \pm b$ and so a divides b, a contradiction. Hence f is not continuous unless $a = 0, \pm 1$.

d. Is the map $f_{a,b} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(z) = z^2$ continuous? (Prove or disprove).

e. Is the topological space $(\mathbb{Z}, \tau)$ compact? (Prove or disprove).

Answer: First Proof: Note first the complement of open subsets of the form $a\mathbb{Z} + b$ are also open as they are unions of the form $a\mathbb{Z} + c$ for $c = 0, 1, \ldots, a - 1$ and $c \not\equiv b \bmod a$. Now consider sets of the form $U_p = p\mathbb{Z} + (p - 1)/2$ for p an odd prime. Then $\cap_p U_p = \emptyset$ because if $a \in \cap_p U_p$ then for some $x \in \mathbb{Z} \setminus \{-1\}$, $2a + 1 = px + p$, so that a is divisible by all primes p and $a = 0$. But if $a = 0$ then $(p-1)/2$ is divisible by p, a contradiction. On the other hand no finite intersection of the U_p's can be emptyset as $(a\mathbb{Z} + b) \cap (c\mathbb{Z} + b) \neq \emptyset$ if a and b are prime to each other (why?) Hence $(U_p^c)_p$ is an open cover of $\mathbb{Z}$ that does not have a finite cover. Therefore $\mathbb{Z}$ is not compact.

First Proof: Let p be a prime and $a = a_0 + a_1 p + a_2 p^2 + \ldots$ be a p-adic integer which is not in $\mathbb{Z}$. Let

$$b_n = a_0 + a_1 p + a_2 p^2 + \ldots + a_{n-1} p^{n-1}.$$

Then $\cap_n p^n \mathbb{Z} + b_n = \emptyset$ but no finite intersection is empty. We conclude as above.

xix. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the squaring map. Suppose that the arrival set is endowed with the usual Euclidean topology. Find the smallest topology on the domain that makes f continuous. (5 pts.)

Answer: The smallest such topology is the set

$$\{U \cap -U : U \text{ open in the usual topology of } \mathbb{R}\}.$$

xx. Let $(V, |\ |)$ be a normed real metric space. Show that the maps $V \times V \longrightarrow V$ and $R \times V \longrightarrow V$ defined by $(v, w) \mapsto v + w$ and $(r, v) \longrightarrow rv$ respectively are continuous.

xxi. Show that an isometry between metric spaces is a homeomorphism between the topological spaces that they induce.

We end this subsection with easy but important results.

Theorem 11.2.3 *i. The image of a compact set under a continuous map is compact.*
ii. The image of a connected set under a continuous map is compact.

Proof: Let $f : X \longrightarrow Y$ be a continuous map.
i. Let $K \subseteq X$ be a compact subset of X. Let $(V_i)_{i \in I}$ be an open covering of $f(K)$. Then $(f^{-1}(V_i))_{i \in I}$ is an open covering of $f^{-1}(f(K))$. Since $K \subseteq f^{-1}(f(K))$, $(f^{-1}(V_i))_{i \in I}$ is also an open covering of K. Hence K is covered by $f^{-1}(V_{i_1}), \ldots, f^{-1}(V_{i_k})$ for some $i_1, \ldots, i_k \in I$, i.e. $K \subseteq f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_k})$. By applying f to both sides, we get $f(K) \subseteq f(f^{-1}(V_{i_1}) \cup \ldots \cup f^{-1}(V_{i_k})) = f(f^{-1}(V_{i_1})) \cup \ldots \cup f(f^{-1}(V_{i_k})) \subseteq V_{i_1} \cup \ldots \cup V_{i_k}$. This shows that $f(K)$ is compact. □
ii. Let $f(X) \subseteq U \cap V$ where U and V are open in Y and $U \cap V \cap f(X) = \emptyset$. Then $X = f^{-1}(f(X)) \subseteq f^{-1}(U) \cap f^{-1}(V)$ and $\emptyset = f^{-1}(U \cap V \cap f(X)) = f^{-1}(U) \cap f^{-1}(V) \cap f^{-1}(f(X)) = f^{-1}(U) \cap f^{-1}(V) \cap X = f^{-1}(U) \cap f^{-1}(V)$, so either $f^{-1}(U) = \emptyset$ of $f^{-1}(V) = \emptyset$, i.e. either $U \cap f(X) = \emptyset$ or $V \cap f(X) = \emptyset$. □

Corollary 11.2.4 *If K is a compact set and $f : K \longrightarrow Y$ a continuous bijection into a topological space Y then f is a homeomorphism from K onto $f(K)$.*

Proof: Let $F \subseteq K$ be closed. Then F is compact. So $f(F)$ is compact in the compact subset $f(K)$. So $f(F)$ is closed in $f(K)$. Thus f^{-1} is continuous. □

11.3 Continuous Functions and $\mathbb{R}$

Continuous functions assume their extreme values on compact subsets:

Theorem 11.3.1 *Let X be a compact topological space and $f : X \longrightarrow \mathbb{R}$ be continuous. Then there are $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.*

Proof: By Theorem 11.2.3, $f(X)$ is a compact subset of $\mathbb{R}$. By Proposition 10.6.2, $f(X)$ is closed and bounded. Clearly, a closed and bounded subset of $\mathbb{R}$ contains its upper bound. Thus $f(X)$ has a maximal element, say $f(b)$. Similarly for the existence of a. $\square$

A continuous real valued function which is positive once, remains positive for a while, i.e. if a continuous function is positive at a point then it is positive in a neighborhood of this point:

Lemma 11.3.2 *Let $f : (a, b) \longrightarrow \mathbb{R}$ and $c \in (a, b)$. Assume that f is continuous at c. If $f(c) > 0$ then $f > 0$ in an open neighborhood of c. If $f(c) < 0$ then $f < 0$ in an open neighborhood of c.*

Proof: Let $\epsilon = f(c)$. Since f is continuous at c, there is a $\delta > 0$ such that if $x \in (c - \delta, c + \delta)$ then $f(x) \in (f(c) - \epsilon, f(c) + \epsilon) = (0, 2\epsilon)$. $\square$

A continuous real valued function which assumes two values assumes all the values between them:

Theorem 11.3.3 (Intermediate Value Theorem) *If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous and d is between $f(a)$ and $f(b)$, then $d = f(c)$ for some $c \in [a, b]$.*

Proof: Replacing f by $f - d$, we may assume that $d = 0$. Replacing f by $-f$ if necessary, we may assume that $f(a) \leq 0 \leq f(b)$. We may further assume that $f(a) < 0 < f(b)$. Consider the set $A := \{x \in [a, b] : f(x) < 0\}$. Since $a \in A$, $A \neq \emptyset$. Let $c = \sup(A) \in [a, b]$. By Lemma 11.3.2 $a < c < b$. We will show that $f(c) = 0$.

If $f(c) < 0$ then we get a contradiction by Lemma 11.3.2.

If $f(c) > 0$ then we again get a contradiction by Lemma 11.3.2.

Thus $f(c) = 0$. $\square$

The image of a compact and connected subset under a continuous real valued function is a closed interval. **Prove this under this generality.**

Corollary 11.3.4 *If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then $f([a, b]) = [m, M]$ where $m = \inf(f([a, b]))$ and $M = \sup(f([a, b]))$.* $\square$

Theorem 11.3.5 *Let $f : [a, b] \longrightarrow [m, M]$ be one to one, onto and continuous. Then $f^{-1} : [m, M] \longrightarrow [a, b]$ is continuous also.*

Exercises.

i. **[Cauchy]** Show that for all real numbers $r_1, \ldots, r_n$, $(\sum_i r_i)^2 \le n \sum_i r_i^2$. (**Solution:** Replacing r_i by $|r_i|$, we may assume that all $r_i \ge 0$. If $\sum_i r_i = 0$, the result is obvious. Replacing r_i by $r_i/\sum_i r_i$, we may assume that $\sum_i r_i = 1$. So we have to show that if $\sum_i r_i = 1$, then $\sum_i r_i^2 \le 1/n$. Let $A = \{(r_1, \ldots, r_n) \in \mathbb{R}^{\ge 0} : \sum_i r_i = 1\}$ and let $f : A \longrightarrow \mathbb{R}$ be defined by $f(r_1, \ldots, r_n) = \sum_i r_i^2$. Then A is a compact subset of $\mathbb{R}^n$ because A is a closed subset of the compact set $[0,1]^n$ (Heine-Borel Theorem, Theorem 10.6.3), being the inverse image of 1 under the continuous map $(r_1, \ldots, r_n) \mapsto \sum_i r_i$. Thus f assumes its minimal value. We will show that $f(r_1, \ldots, r_n) \ge f(1/n, \ldots, 1/n) = 1/n$. Let $R = (r_1, \ldots, r_n)$ be the point where the minimal value is assumed. Assume $r_i \ne r_j$ for some i and j. Assume $r_i < r_j$. Let $0 < \epsilon < r_j - r_i$. Now look at the point S where all the coordinates are the same as the coordinates of R except that the i-th coordinate is $r_i + \epsilon$ and the j-th coordinate is $r_j - \epsilon$. Then $S \in A$ and an easy calculation shows that $f(S) < f(R)$, a contradiction. Thus $r_i = r_j$ for all i, j. Since $\sum_i r_i = 1$, this implies that $r_i = 1/n$ for all n.)

ii. Let $f : [a,b] \longrightarrow [a,b]$ be a continuous function. Show that f has a fixed point.

iii. Show that if f is one to one and continuous on $[a,b]$, then f is strictly monotone on $[a,b]$.

iv. Let f be a continuous numerical function on the closed interval $[a,b]$. Let $x_1, \ldots, x_n$ be arbitrary points in $[a,b]$. Show that $f(x_0) = (f(x_1) + ... + f(x_n))$ for some $x_0 \in [a,b]$.

v. Show that there does not exist any continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which assumes every $x \in \mathbb{R}$ twice.

vi. Does there exist a continuous function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which assumes every real number three times? (Yes!)

vii. Assume $f, g : [0,1] \longrightarrow [0,1]$ be continuous functions and assume that $f(0) \ge g(0)$ and $f(1) \le g(1)$. Show that there exists an $x \in [0,1]$ such that $f(x) = g(x)$.

viii. Assume that f is a continuous real-valued function on $[0,1]$ and that $f(0) = f(1)$. If $n \in \mathbb{N}$ is positive show that there is a point x in $[0,1]$ such that $f(x) = f(x + \frac{1}{n})$.

11.4 Uniform Continuity

Exercises.

i. Assume $(f_n)_n$ converges uniformly to f on a subset A of $\mathbb{R}$ and that each f_n is uniformly continuous on A. Show that f is uniformly continuous on A.

11.5 Uniform Convergence and Continuity

Theorem 11.5.1 *Uniform limit of a family of continuous functions is continuous.*

11.6 Supplementary Topics

11.6.1 A Continuous Curve Covering $[0,1]^2$

Proposition 11.6.1 $[0,1]$ *and* $[0,1]^2$ *are not homeomorphic.*

Proof: Take away one point from $[0,1]$. □

Corollary 11.6.2 *There is no continuous bijection from* $[0,1]$ *onto* $[0,1]^2$.

Proof: From Corollary 11.2.4 and Proposition 11.6.1. □

Theorem 11.6.3 (Peano, 1890) *There is a continuous map (curve) from* $[0,1]$ *onto* $[0,1]^2$.

Proof: Let $I=[0,1]$. Subdivide I into four closed intervals of equal length $[0,1/4]$, $[1/4,1/2]$, $[1/2,3/4]$ and $[3/4,1]$. Subdivide I^2 into four smaller squares of length $1/2\times 1/2$: $[0,1/2]\times[0,1/2]$, $[0,1/2]\times[1/2,1]$, $[1/2,1]\times[1/2,1]$ and $[1/2,1]\times[0,1/2]$. Make a correspondence between the four smaller intervals and the four smaller squares in the order they appear above. Call this "stage 1". At "stage 2", subdivide each segment and each square into four equal parts again. Make the correspondence in such a way that the intersecting intervals correspond to neighboring squares. Continue in this fashion. Now for any point x of $[0,1]$, consider the set of intervals that x belongs to. The point x will belong to either one or two of the four intervals of stage n, of total length $\leq 1/2^{n-1}$. To these intervals correspond closed squares of stage n, one inside the other and of total area $\leq /2^{2n-1}$. Because of the size of the squares, these corresponding squares intersect at a unique point, say $f(x)$. The map f so constructed is a map from I onto I^2, but is not a bijection.

This map is continuous, because for x, $y\in I$ close to each other, the images $f(x)$ and $f(y)$ are also close to each other, as can be easily checked. In fact if $|x-y|<1/4^n$ then $d(f(x),f(y))\leq 2\sqrt{2}/2^n$. □

Note that I and I^2 are not homeomorphic, because taking one point away from I disconnects I, a phenomenon that does not occur in I^2. Indeed the map f^{-1} defined above is not continuous, as the images under $f^{-1/2}$ of two points close to the center of I^2 which belong to the centers of opposite squares $[0,1/2]\times[0,1/2]$ and $[1/2,1]\times[1/2,1]$ are never at a distance $<1/4$.

Question. Is there a continuous bijection from I onto I^2?

Exercises.

i. Show that a subgroup of $\mathbb{R}$ is either discrete, in which case it is one-generated, or is dense.

ii. Let $\mathbb{A} \in \mathbb{R}$. Show that $\mathbb{Z} + \alpha\mathbb{Z}$ is discrete if and only if $\alpha \in \mathbb{Q}$.

Chapter 12

Differentiable Functions

12.1 Definition and Examples

DO ALSO DIFFERENTIATION IN $\mathbb{C}$, $\mathbb{R}^n$ and in normed vector spaces more generally, all at once

Let $X \subseteq \mathbb{R}$, $f : X \longrightarrow \mathbb{R}$ a function and $a \in X^\circ$. We say that f is **differentiable at** a if

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

exists. We then write

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}.$$

If X is open, the function f is called **differentiable on** X if f is differentiable at all the points of X. Such a function defines a function $f' : X \longrightarrow \mathbb{R}$ called the **derivative** of f.

Examples.

i. Let $n \in \mathbb{N}$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^n$. Then

$$\begin{aligned}
\lim_{h\to 0} \tfrac{f(x+h)-f(x)}{h} &= \lim_{h\to 0} \tfrac{(x+h)^n - x^n}{h} \\
&= \lim_{h\to 0} \frac{\sum_{i=0}^{n} \binom{n}{i} x^i h^{n-i} - x^n}{h} \\
&= \lim_{h\to 0} \frac{\sum_{i=0}^{n-1} \binom{n}{i} x^i h^{n-i}}{h} \\
&= \lim_{h\to 0} \sum_{i=0}^{n-1} \binom{n}{i} x^i h^{n-i-1} \\
&= \lim_{h\to 0} \binom{n}{n-1} x^{n-1} h^0 = n x^{n-1}.
\end{aligned}$$

ii. Let $f_n : [0,1] \longrightarrow [0,1]$ be defined by $f_n(x) = x^n$. Then the sequence of differentiable functions $(f_n)_n$ converges to the function f defined by

$$f(x) = \begin{cases} 0 & \text{for } 0 \leq x < 1 \\ 1 & \text{for } x = 1 \end{cases}$$

and f is not differentiable.

Lemma 12.1.1 *If f is differentiable at a then f is continuous at a. Thus a differentiable function is continuous.*

Exercises.

i. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the squaring function $f(x) = x^2$. Show that f is differentiable and $f'(x) = 2x$.

ii. Let $n \in \mathbb{N} \setminus \{0\}$ and $f_n : [-1,1] \longrightarrow \mathbb{R}$ be defined by $f_n(x) = |x|^{1+1/n}$. Show that f_n is differentiable on $[-1,1]$.

iii. Show that the function

$$f(x) = \begin{cases} x^2 sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is differentiable but f' is not continuous.

12.2 Differentiation of Complex Functions

12.3 Basic Properties of Differentiable Functions

From now on we let f and a be as above.

Lemma 12.3.1 *If f is differentiable at a and if $f(a)$ is maximal then $f'(a) = 0$.*

Exercises.

i. Let $f : U \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) be differentiable at $a \in U$, where U is open. Assume that there is a sequence $(x_n)_n$ of U converging to a such that $f(x_n) = f(x_m)$ all n, m. Then $f(a) = f(x_n)$ and $f'(a) = 0$. (Hint: Follows almost directly from the definition of derivatives).

ii. Let $f : [a,b] \longrightarrow \mathbb{R}$ be differentiable. Assume that $f'(x) \neq 0$ for $x \in [a,b]$. Then f has finitely many zeroes in $[a,b]$. (Hint: See Exercise i, page 132).

12.4 Rules of Differentiation

12.5 Relationship Between a Function and Its Derivative

Let$f : X \longrightarrow \mathbb{R}$ be a function from a topological space into $\mathbb{R}$. We say that a point $x_\circ$ is a **local maximum** (resp. local minimum) of f if there is an open subset U of X containing $x_\circ$ such that for all $x \in U$, $f(x) \leq f(x_\circ)$ (resp. $f(x) \geq f(x_\circ)$). An **absolute maximum** (resp. absolute minimum) of f is a point $x_\circ$ such that $x \in U$, $f(x) \leq f(x_\circ)$ (resp. $f(x) \geq f(x_\circ)$). An extremum (local or absolute) point is a maximum or a minimum point.

Theorem 12.5.1 *Let $x_\circ \in X \subseteq \mathbb{R}$ and $f : X \longrightarrow \mathbb{R}$ be a function differentiable at $x_\circ$. If $x_\circ$ is a local extremum then $f'(x_\circ) = 0$.*

Proof: Suppose $x_\circ$ is a local maximum. Then there is a $\delta > 0$ such that $(x_\circ - \delta, x_\circ + \delta) \subseteq X$ and

$$\begin{array}{ll} f(x) \leq f(x_\circ) & \text{if } x \in (x_\circ - \delta, x_\circ) \\ f(x) \geq f(x_\circ) & \text{if } x \in (x_\circ, x_\circ + \delta). \end{array}$$

Now

$$\begin{array}{ll} \frac{f(x_\circ+h)-f(x_\circ)}{h} \leq 0 & \text{for } h \in (0, \delta) \\ \frac{f(x_\circ+h)-f(x_\circ)}{h} \geq 0 & \text{for } h \in (-\delta, 0). \end{array}$$

Thus $f'(x_\circ) = \lim_{n\to\infty} = \frac{f(x_\circ+h)-f(x_\circ)}{h} \geq 0$. If $x_\circ$ is a local minimum the proof is similar. □

Theorem 12.5.2 (Rolle's Theorem) *For $a < b$, let $f : [a, b] \longrightarrow \mathbb{R}$ be differentiable on (a, b). If $f(a) = f(b)$ then $f'(c) = 0$ for some $c \in (a, b)$.*

Proof: If f is constant then $f' = 0$ and there is nothing to prove. Assume f is not a constant. By Theorem 11.3.1, f has an absolute maximum and an absolute minimum. Since f is not constant, these are two distinct points. Since $f(a) = f(b)$, one of these points, say c, must be in the open interval (a, b). By Theorem 12.5.1, $f'(c) = 0$. □

The next theorem is one of the most important result in this series. We will have several opportunities to use it.

Theorem 12.5.3 (Mean Value Theorem of Differential Calculus) *For $a < b$, let $f : [a, b] \longrightarrow \mathbb{R}$ be differentiable on (a, b). Then*

$$\frac{f(a) - f(b)}{a - b} = f'(c)$$

for some $c \in (a, b)$.

Proof: For $t \in [0,1]$, let $g(t) = f(ta + (1-t)b) - tf(a) + (1-t)f(b)$. Then $g(0) = g(1) = 0$. By Rolle's Theorem (12.5.2), $g'(t_\circ) = 0$ for some $t_\circ \in (0,1)$. But $g'(t) = (a-b)f'(ta + (1-t)b) - f(a) + f(b)$. Thus $0 = g'(t_\circ) = (a-b)f'(t_\circ a + (1-t_\circ)b) - f(a) + f(b)$. If $c = t_\circ a + (1-t_\circ)b$, then $f'(c) = \frac{f(a)-f(b)}{a-b}$. □

Theorem 12.5.4 *If f is differentiable on (a,b) and $f'(x) \geq 0$ for every $x \in (a,b)$ then f is monotone decreasing on (a,b).*

Proof: Let $a < x_1 < x_2 < b$. By the Mean Value Theorem of Differential Calculus (12.5.3), there is an $x_\circ \in (x_1, x_2)$ such that $f'(x_\circ) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$. Since $f'(x_\circ) \geq 0$ and $x_2 - x_1 > 0$, $f(x_2) - f(x_1) \geq 0$. □

The fact that $f'(c) > 0$ at some c does not imply that f is increasing around c. It is easy to find a counterexample, find one.

12.6 Uniform Convergence and Differentiation

The fact that a sequence of differentiable function $(f_n)_n$ converges uniformly to a function f, does not imply that f is differentiable.

Example. Let $f_n : [-1,1] \longrightarrow \mathbb{R}$ be defined by $f_n(x) = |x|^{1+\frac{1}{2n+1}}$. Then each f_n is differentiable on $[0,1]$ (see Exercise ii, page 132). Also $\lim_{n\to\infty} f_n(x) \stackrel{u}{=} |x|$. (See Exercise iii, page 132). But the function $f(x) = |x|$ is not differentiable at $x = 0$.

Question. Suppose $(f_n)_n$ is a sequence of differentiable functions that converge uniformly to a differentiable function f. Is it true that $f' \stackrel{p}{=} \lim_{n\to\infty} f_n'$?

Theorem 12.6.1 *Let the sequence of differentiable functions $(f_n)_n$ converge pointwise to f on the closed interval $[a,b]$. Assume that the sequence $(f_n')_n$ converges uniformly. Then f is differentiable and $f' = \lim_{n\to\infty} f_n'$.*

Proof: Let $\lim_{n\to\infty} f_n' \stackrel{u}{=} g$. Let $c \in [a,b]$. We want to show that

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = g(c).$$

Thus given an $\epsilon > 0$, we want to find a $\delta > 0$ such that

$$\left| \frac{f(x) - f(c)}{x - c} - g(c) \right| < \epsilon$$

for all $x \in [a,b]$ that satisfies $0 < |x - c| < \delta$. Let $\epsilon > 0$ be given. By the triangular inequality, for any n, we have,

$$\left|\frac{f(x)-f(c)}{x-c} - g(c)\right| \leq \left|\frac{f(x)-f(c)}{x-c} - \frac{f_n(x)-f_n(c)}{x-c}\right| + \left|\frac{f_n(x)-f_n(c)}{x-c} - f_n'(c)\right| + |f_n'(c) - g(c)|.$$

We will make each term of the right hand side less than $\epsilon/3$. It is easy to make the second term small by choosing δ small enough. It is also easy to make the last term small by choosing n large enough. The main problem is with the first term, that we deal first.

Assume $c < x$. The proof is the same if $x < c$. Let n and m be any natural numbers. Applying the Mean Value Theorem to the function $f_n - f_m$ on the interval $[c, x]$ we can find an $\alpha = \alpha_{m,n} \in (c, x)$ such that

$$f'_m(\alpha) - f'_n(\alpha) = \frac{f_m(x) - f_n(x)}{x - c} - \frac{f_m(c) - f_n(c)}{x - c}.$$

On the other hand, by the Cauchy Criterion for Uniform Convergence (Theorem 9.2.5), there exists an N_1 such that for all n, $m > N_1$ and $x \in [a, b]$,

$$|f'_m(x) - f'_n(x)| < \epsilon/3.$$

In particular,

$$|f'_m(\alpha) - f'_n(\alpha)| < \epsilon/3.$$

(It is worth while to notice that, since α depends on n and m, we really need the uniform convergence of $(f'_n)_n$ for this part of the argument). Thus

$$\left| \frac{f_m(x) - f_n(x)}{x - c} - \frac{f_m(c) - f_n(c)}{x - c} \right| < \epsilon/3$$

for all n, $m > N_1$ and all $x \in [a, b]$. Now let m go to infinity, to get

$$\left| \frac{f(x) - f(c)}{x - c} - \frac{f_n(x) - f_n(c)}{x - c} \right| < \epsilon/3 \tag{1}$$

for all $n > N_1$ and any $x \in [a, b]$.

Since $\lim_{n\to\infty} f'_n = g$, there is an N_2 large enough so that,

$$|f'_n(c) - g(c)| < \epsilon/3 \tag{2}$$

for all $n > N_2$.

Let $n = \max\{N_1, N_2\} + 1$. Note that (1) and (2) hold for this n. Choose $\delta > 0$ such that if $x \in [a, b]$ and $|x - c| < \delta$, then

$$\left| \frac{f_n(x) - f_n(c)}{x - c} - f'_n(c) \right| < \epsilon/3. \tag{3}$$

Now (1), (2) and (3) give us the result. □

In the theorem above we do not really need the pointwise convergence of $(f_n)_n$ on $[a, b]$, it is enough to know that $(f_n(x_0))_n$ converges for some n to get the pointwise, in fact the uniform convergence on $[a, b]$.

Theorem 12.6.2 *Let $(f_n)_n$ be a sequence of differentiable functions defined and assume that the sequence $(f'_n)_n$ converges uniformly on $[a, b]$. If there exists a point $x_0 \in [a, b]$ such that $(f_n(x_0))_n$ is convergent. Then $(f_n)_n$ converges uniformly on $[a, b]$ and $\lim_{n\to\infty} f_n$ is differentiable and $(\lim_{n\to\infty} f_n)' = \lim_{n\to\infty} f'_n$.*

Proof: In view of the previous theorem, it is enough to prove the uniform convergence of $(f_n)_n$ on $[a, b]$.

COMPLETE THE PROOF.

Exercises.

12.7 Second and Further Derivatives

Chapter 13

Analytic Functions

13.1 Power Series

We collect the main results and a few more about power series. This will show that the power series – whenever and wherever they converge – behave like polynomials, as much as they can.

Theorem 13.1.1 *Let $\sum_{i=0}^{\infty} a_i z^i$ be a power series with $R = \overline{\lim}\{1/|a_i|^{1/i}\} \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ its "**radius of convergence**". Let $0 < r < R$. Then the following hold.*

i. If $|z| < R$, then $\sum_{i=0}^{\infty} a_i z^i$ converges absolutely. If $|z| > R$, then $\sum_{i=0}^{\infty} a_i z^i$ diverges.

ii. $\sum_{i=0}^{\infty} a_i z^i$ converges uniformly on $\overline{B}(0, r)$.

iii. The radius of convergence of $\sum_{i=1}^{\infty} i a_i z^{i-1}$ converges uniformly on $\overline{B}(0, r)$.

iv. $\sum_{i=0}^{\infty} a_i z^i$ is infinitely differentiable in $B(0, R)$ and

$$(\sum_{i=0}^{\infty} a_i z^i)' = \sum_{i=1}^{\infty} i a_i z^{i-1}.$$

v. If $f(z) = \sum_{i=0}^{\infty} a_i z^i$ in $B(0, R)$, then f is infinitely differentiable in $B(0, R)$ and $a_i = f^{(i)}(0)/i!$. In other words, the coefficients of a power series are unique and if the function $f(z)$ is a power series, then

$$f(z) = \sum_{i=0}^{\infty} \frac{f^{(i)} i!}{z}^i.$$

Proof: Part (i) is Corollary 7.5.6.

Part (ii) is by Proposition 6.8.3 and part (i).

iii. Since $\sum_{i=0}^{\infty} a_i z^i$ converges absolutely for $|z| = r$, the series $\sum_{i=0}^{\infty} a_i r^i$ converges. By Weierstrass M-Test (Theorem 9.3.3), the series $f = \sum_{i=0}^{\infty} a_i z^i$ converges uniformly on $\overline{B}(0, r)$.

iv. By parts (i, iii), the series $\sum_{i=1}^{\infty} ia_i z^{i-1}$ converges uniformly on $\overline{B}(0,r)$. Let $f_n = \sum_{i=0}^{n} a_i z^i$. By Theorem 12.6.1, $f = \sum_{i=0}^{\infty} a_i z^i$ is differentiable and $f' = \lim_{n\to\infty} f_n' = \lim_{n\to\infty} \sum_{i=1}^{n} ia_i z^{i-1} = \sum_{i=1}^{\infty} ia_i z^{i-1}$ for $z \in \overline{B}(0,r)$, so also for all $z \in B(0,R)$.

v. Direct application of the previous facts. □

Corollary 13.1.2 *Consider the infinite series $\sum_{i=0}^{\infty} a_i(z-a)^i$ with $0 < R = \overline{lim}\{1/|a_i|^{1/i}\} \in \mathbb{R}^{\geq 0} \cup \{\infty\}$ its "***radius of convergence***". Let $0 < r < R$. Then the following hold.*

i. *If $z \in B(a,R)$, then $\sum_{i=0}^{\infty} a_i(z-a)^i$ converges absolutely. If $z \notin B(a,R)$, then $\sum_{i=0}^{\infty} a_i(z-a)^i$ diverges.*

ii. *The radius of convergence of $\sum_{i=0}^{\infty} a_i(z-a)^i$ is also R.*

iii. *$\sum_{i=0}^{\infty} a_i(z-a)^i$ converges uniformly on $\overline{B}(a,r)$.*

iv. *$\sum_{i=0}^{\infty} a_i(z-a)^i$ is infinitely differentiable in $B(a,R)$ and*

$$(\sum_{i=0}^{\infty} a_i(z-a)^i)' = \sum_{i=1}^{\infty} ia_i(z-a)^{i-1},$$

the convergence being uniform in $\overline{B}(a,r)$.

v. *If $f(z) = \sum_{i=0}^{\infty} a_i(z-a)^i$ for $z \in B(a,R)$, then f is infinitely differentiable in $B(a,R)$ and $a_i = f^{(i)}(a)/i!$.*

Exercises.

i. Let $\sum_{i=1}^{\infty} a_i/n^i$ be convergent for all $n > 1$. Show that $\lim_{n\to\infty} \sum_{i=0}^{\infty} a_i/n^i = a_0$. (**Solution:** Consider the power series $f(x) := \sum_{i=1}^{\infty} a_i x^i$ which is convergent at $x = 1/2$, thus its radius of convergence is at least $1/2$. Thus $f(x)$ is continuous at 0. It follows that for all $\epsilon > 0$ there is a $\delta > 0$ such that if $|x| < \delta$ and $x \neq 0$ then $|f(x) - f(0)| < \epsilon$. Let $1/\delta - 1 < N = [1/\delta] \leq 1/\delta$. Then for $n > N$, $1/n \leq 1/(N+1) < \delta$ and so $|f(1/n) - f(0)| < \epsilon$, i.e. $|\sum_{i=1}^{\infty} a_i/n^i - a_0| < \epsilon$. Thus $\lim_{n\to\infty} \sum_{i=0}^{\infty} a_i/n^i = a_0$.)

13.2 Taylor Series

As seen above, the power series $\sum_{i=0}^{\infty} a_i z^i$ and their cousin $\sum_{i=0}^{\infty} a_i(z-a)^i$ are easy to handle, they behave almost like polynomials. According to these results, it would be helpful if every function could be expressed as such an infinite series around any a. But since such a series is infinitely differentiable if it converges around a, a function which is not infinitely differentiable around a cannot be expressed as such a series. As it happens, even this condition is not sufficient for a function to be expressed as such a series. We will see an example of such a function later.

A function f defined on an open subset U (of $\mathbb{R}$ or of $\mathbb{C}$) is said to be **analytic** at a point $a \in U$ if there is a sequence $(a_i)_i$ such that $f(z) = \sum_{i=0}^{\infty} a_i(z-a)^i$ for $z \in B(a,R)$ for some $R > 0$. The coefficients a_i are unique and equal to

$\phi^{(i)}(a)/i!$ as Theorem 13.1.1 shows. Also we can find a maximal R where the equality $f(z) = \sum_{i=0}^{\infty} a_i(z-a)^i$ holds.

If f is a complex function, then the fact that f is differentiable on U once is enough to insure that for any $a \in U$, $f = \sum_{i=0}^{\infty} a_i(z-a)^i$ for $z \in B(a,R)$ for some $R > 0$. This result is part of a subject called complex analysis and will be seen somewhere else. The situation in $\mathbb{R}$ is much more mysterious and is the subject of this section.

Lemma 13.2.1 *Let U be an open subset of $\mathbb{R}$ and $f : U \longrightarrow \mathbb{R}$ an $(n+1)$-times differentiable function. Let $[a,b] \subseteq U$. Then*

$$f(b) = f(a) + \frac{f'(a)}{1!}(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}$$

for some $c \in (a,b)$.

Proof: Let

$$g(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2 f'(x)}{2!} - \ldots - \frac{(b-x)^n f^{(n)}(x)}{n!} - \frac{(b-x)^{n+1}\gamma}{(n+1)!}$$

where γ is chosen so that $g(a) = 0$. Since $g(b) = 0$ also, by Rolle's Theorem (12.5.2), there is a $c \in (a,b)$ such that $g'(c) = 0$. Note that, after simplification,

$$\begin{aligned} g'(x) &= [f(b) - \textstyle\sum_{i=0}^{n} \frac{(b-x)^i f^{(i)}(x)}{i!} - \frac{(b-x)^{n+1}\gamma}{(n+1)!}]' \\ &= \textstyle\sum_{i=1}^{n} \frac{(b-x)^{i-1} f^{(i)}(x)}{(i-1)!} - \sum_{i=0}^{n} \frac{(b-x)^i f^{(i+1)}(x)}{i!} + \frac{(b-x)^n\gamma}{n!} \\ &= -\frac{(b-x)^n f^{(n+1)}(x)}{n!} + \frac{(b-x)^n\gamma}{n!}. \end{aligned}$$

Thus $\gamma = f^{(n+1)}(c)$. Now the equation $g(a) = 0$ gives the desired result. $\square$

Corollary 13.2.2 *Let U be an open subset of $\mathbb{R}$ and $f : U \longrightarrow \mathbb{R}$ an $(n+1)$-times differentiable function. Let $[a,b] \subseteq U$. Then for all $x \in [a,b)$ there is a $\xi \in (x,b)$ (that depends on a and x) such that*

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}.$$

It follows that the infinitely differentiable real function f is equal to its **Taylor series**

$$\sum_{i=0}^{\infty} \frac{f^{(i)}}{i!}(z-a)^i$$

around a (so is analytic) if and only if $|\frac{f^{(n+1)}(\xi)}{(n+1)!}(z-a)^{n+1}|$ converges to 0 as n goes to infinity. The term $(n+1)!$ is helpful to make the quantity small. If z is close to a, $(z-a)^{n+1}$ can also be made small. Only the term $f^{(n+1)}(\xi)$ is bothersome. Note that here ξ depends on n, a and x.

13.2.1 Calculating Taylor Polynomials

Theorem 13.2.3 *Suppose that on an open subset we have*

$$f(x) = \sum_{i=0}^{n} a_i x^i + x^n \epsilon(x)$$

where $\lim_{x\to\infty} \epsilon(x) = 0$ *and* f *is* $(n+1)$*-times differentiable. Then* $Tf_n(x) = \sum_{i=0}^{n} a_i x^i$. *????*

Taylor Polynomials of $1/\cos x$

We start noting that since $1/\cos x$ is an even function, its derivatives of odd degree must be 0 at $x = 0$.

First Method. Let $f(x) = 1/\cos x$ and start computing its derived series. This may take a long time and is not advised.

Second Method. Let us compute the first 6 (or 7) terms of the Taylor series of $1/\cos x$ around 0. For this we take the inverse of the Taylor series for $\cos x$ around 0.

$$\begin{aligned}
\frac{1}{\cos x} &= \frac{1}{1-x^2/2!+x^4/4!-x^6/6!+x^7f(x)} \\
&= \frac{1}{1-(x^2/2!-x^4/4!+x^6/6!-x^7f(x))} \\
&= 1+(x^2/2!-x^4/4!+x^6/6!-x^7f(x))+ \\
&\quad +(x^2/2!-x^4/4!+x^6/6!-x^7f(x))^2+ \\
&\quad +(x^2/2!-x^4/4!+x^6/6!-x^7f(x))^3 \\
&= 1+(x^2/2!-x^4/4!+x^6/6!)+(x^4/4-x^6/4!)+x^6/8+x^7g(x) \\
&= 1+x^2/2-x^4(1/4!-1/4)+x^6(1/6!-1/4!+1/8)+x^7g(x) \\
&= 1+x^2/2+5x^4/24+61x^6/720+x^7g(x),
\end{aligned}$$

where f and g are functions that converge to 0 when x goes to 0.

Taylor Polynomial of $\tan x$

Note first that since $\tan x$ is an odd function, its derivatives of even degree must be 0.

First Method. Let $f(x) = \tan x$ and start computing its derived series. This is long and tiring and is not advised.

Second Method. We use the Taylor polynomial of $1/\cos x$ that has been computed above:

$$\begin{aligned}
\tan x &= \sin x \times \frac{1}{\cos x} \\
&= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + x^6 f(x)\right)\left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + x^6 g(x)\right) \\
&= x + \left(\frac{-1}{3!} + \frac{1}{2}\right)x^3 + \left(\frac{1}{5!} - \frac{1}{12} + \frac{5}{24}\right)x^5 + x^6 h(x) \\
&= x + \frac{1}{3}x^3 + \frac{2}{15}x^5,
\end{aligned}$$

where f, g and h are functions that converge to 0 when x goes to 0.

Third Method for those who know how to integrate. Since $\tan' x = 1/\cos^2 x$, if we know the Taylor polynomial of $1/\cos^2 x$, we can integrate to find the Taylor polynomial of $\tan x$, except may be the first term. But since $\tan 0 = 0$, the first term must be 0.

Fourth Method. Since $\tan x$ is $(-\ln(\cos x))' = \tan x$, if we know the Taylor polynomial of $-\ln(\cos x)$, then by differentiating, we can find the Taylor polynomial of $\tan x$:

$$\begin{array}{rcl}
\ln(\cos x) & = & \ln\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} - x^7 f(x)\right) \\
& = & \ln\left(1 - \left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + x^7 f(x)\right)\right) \\
& = & -\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + x^7 f(x)\right) - \frac{1}{2}\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + x^7 f(x)\right)^2 \\
& & -\frac{1}{3}\left(\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + x^7 f(x)\right)^3 \\
& = & -\frac{1}{2}x^2 - \frac{1}{12}x^4 - \frac{2}{90}x^6 + x^7 g(x)
\end{array}$$

Hence

$$\tan x = (-\ln(\cos x))' = \left(\frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{2}{90}x^6 + x^7 g(x)\right)' = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + x^6 h(x)$$

where the limits of g and h when x goes to 0 are 0.

Taylor Polynomial of $\sin^3 x$

We compute up to ninth degree:

$$\begin{array}{rcl}
\sin^3(x) & = & \left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + x^8 f(x)\right)^3 = x^3\left(\frac{1}{1!} - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + x^7 f(x)\right)^3 \\
& = & x^3\Big[1 + 3\cdot\frac{1\cdot 1\cdot(-1)}{1!1!3!}x^2 + \left(3\cdot\frac{1\cdot 1\cdot 1}{1!1!5!} + 3\cdot\frac{1\cdot(-1)\cdot(-1)}{1!3!3!}\right)x^4 + \\
& & + \left(\frac{(-1)\cdot(-1)\cdot(-1)}{3!3!3!} + 3!\cdot\frac{1\cdot(-1)\cdot 1}{1!3!5!} + 3\cdot\frac{1\cdot 1\cdot(-1)}{1!1!7!}\right)x^6 + x^7\epsilon'(x)\Big] \\
& = & x^3\left(1 - \frac{1}{2}x^2 + \frac{13}{120}x^4 - \frac{41}{3024}x^6 + x^7\epsilon'(x)\right) \\
& = & x^3 - \frac{1}{2}x^5 + \frac{13}{120}x^7 - \frac{41}{3024}x^9 + x^{10}\epsilon'(x)
\end{array}$$

where the limits of f, ϵ and ϵ' when x goes to 0 are 0.

Exercises.

i. Show that the derivatives of even (resp. odd) degree of an analytic function which is odd (resp. even) must be 0.

ii. Let $f : \mathbb{R}^{>0} \longrightarrow \mathbb{R}$ be twice differentiable. Assume that $f > 0$ and $f' < 0$ then f'' cannot be always negative.

   Proof: Assume $f''(x) < 0$ for all x. Let $a \in \mathbb{R}^{>0}$ be fixed. Then for all $x > a$, $f(x) = f(a) + (x-a)f'(a) + (b-x)^2 f'(c)/2!$ for some c. We can choose x large enough so that $f(a) + (x-a)f'(a) < 0$. Then $f(x) < 0$. $\square$

iii. If $f(x) = \frac{1}{1+x}$ and $U = (-1, \infty)$, solve the first two questions.

iv. a. Estimate the error made in replacing the function exp on the interval $[0, 1]$ by its Taylor polynomial of degree 10.

b. On what interval $[0, h]$ does the function exp differ from its Taylor polynomial of degree 10 by no more than 10^{-7}?

c. For what value of n does the function exp differ from its Taylor polynomial of degree n by no more than 10^{-7} on the interval $[0, 1]$?

13.3 Analytic Functions

If f is infinitely differentiable and a, $x \in U$, we set

$$(Tf)(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

Note that $(Tf)(x)$ may or may not be convergent for a given x. But if Tf is convergent, then Tf and f have the same n^{th}-derivatives at 0 for all n. Does this condition implies that $Tf = f$. Thus we have two questions:

1. For what values of x is $(Tf)(x)$ convergent? Is Tf convergent for all $x \in U$.

2. In case $(Tf)(x)$ is convergent on U, do we have $(Tf)(x) = f(x)$ for all x?

The answer to both questions is negative as the next examples show.

Examples.

i. Let f be the real function $f(x) = \frac{1}{1+x^2}$. Then $Tf(x) = \sum_{i=0}^{\infty}(-1)^i x^{2i}$. It can be easily checked that Tf converges for $x \in (-1, 1)$ and diverges elsewhere, although f is defined everywhere. Thus the answer to the first question is negative.

ii. Let

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Then f is infinitely differentiable and $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. Therefore $Tf = 0$ and $Tf(x) = f(x)$ if and only if $x = 0$. Thus the answer to the second question is also negative.

A function f defined on an open subset U (of $\mathbb{R}$ or of $\mathbb{C}$) is said to be **analytic** at a point $a \in U$ if there is a sequence $(a_i)_i$ such that $f(z) = \sum_{i=0}^{\infty} a_i(z-a)^i$ for $z \in B(a, R)$ for some $R > 0$. The coefficients a_i are unique and equal to $\phi^{(i)}(a)/i!$ as Theorem 13.1.1 shows. Also we can find a maximal R where the equality $f(z) = \sum_{i=0}^{\infty} a_i(z-a)^i$ holds.

Proposition 13.3.1 *If $f : B(0, R) \longrightarrow \mathbb{C}$ is analytic and $f = 0$ on $\mathbb{R} \cap B(0, \mathbb{R})$ then $f = 0$.*

Proof: . By Exercise i, page 132, $f' = 0$ on $\mathbb{R}$. Thus $f^{(n)}(0) = 0$ for all n and so $f = Tf = 0$. $\square$

13.4 Transcendental Functions

We define

$$\begin{array}{lcl} \exp(z) & = & \sum_{n=0}^{\infty} z^n/n! \\ \sin(z) & = & \sum_{n=0}^{\infty} (-1)^n z^{2n+1}/(2n+1)! \\ \cos(z) & = & \sum_{n=0}^{\infty} (-1)^n z^{2n}/(2n)! \\ \cosh(z) & = & \sum_{n=0}^{\infty} z^{2n+1}/(2n+1)! \\ \sinh(z) & = & \sum_{n=0}^{\infty} z^{2n}/(2n)! \end{array}$$

Each of these functions are analytic on $\mathbb{C}$ as it can be checked by using Theorem 13.1.1. Also they all take real values at real numbers, i.e. they can also be considered as real functions.

Exercises.

i. Show that $\lim_{n\to\infty}(1+z/n)^n = \exp(z)$. (See Example i, page 111).

ii. Show that $\sin z = \frac{exp(iz)-\exp(-iz)}{2i}$.

13.4.1 Exponentiation and Trigonometric Functions

We have $\exp(0) = 1$, $\cos(0) = 1$, $\sin(0) = 0$. By Theorem 7.5.7, $\exp' = \exp$, $\sin' = \cos$, $\cos' = -\sin$, $\cosh' = \sinh$, $\sinh' = \cosh'$, $\cos^2(x) + \sin^2(x) = 1$.

Theorem 13.4.1 $\exp(x+y) = \exp(x)\exp(y)$.

Proof: Since $\exp(z) = \sum_{n=0}^{\infty} z^n/n!$, we have to prove that the limit of

$$(\sum_{i=0}^{n} x^i/i!)(\sum_{i=0}^{n} y^i/i!) - \sum_{i=0}^{n}(x+y)^i/i!$$

is 0 as $n \to \infty$. We first compute the left hand side:

$$(\sum_{i=0}^{n} x^i/i!)(\sum_{i=0}^{n} y^i/i!) = \sum_{i=0}^{n}\sum_{j=0}^{n}\frac{x^i y^j}{i!j!}.$$

Now we compute the left hand side:

$$\sum_{i=0}^{n}(x+y)^i/i! = \sum_{i=0}^{n}\sum_{j=0}^{i}\left(\begin{array}{c} i \\ j \end{array}\right) x^j y^{i-j}/i! = \sum_{i=0}^{n}\sum_{j=0}^{i}\frac{x^j y^{i-j}}{(i-j)!j!} = \sum_{i=0}^{n}\sum_{k=0}^{i}\frac{x^j y^k}{k!j!}.$$

$\square$

Another proof of Theorem 13.4.1: Let $a \in \mathbb{R}$. Let $f(x) = \exp(x+a)$ and $g(x) = \exp(x)\exp(a)$. Then $f(0) = g(0)$ and $f = f'$ and $g = g'$. Thus $f^{(n)}(0) = g^{(n)}(0)$ all n. So $f = g$ by Theorem 13.1.1. $\square$

Theorem 13.4.2 $\exp(iz) = \cos z + i\sin z$ *for all* $z \in \mathbb{C}$.

Proof: Easy. $\square$

Corollary 13.4.3 $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$ *and* $\cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$ *for all* $x, y \in \mathbb{C}$.

Proof: If $x, y \in \mathbb{R}$, this follows from Theorems 13.4.2 and 13.4.1. Now apply Proposition 13.3.1 to the analytic functions $\sin(x+a) - \sin(x)\cos(a) + \cos(x)\sin(a)$ and $\cos(x+a) - \cos(x)\cos(a) - \sin(x)\sin(a)$. $\square$

Theorem 13.4.4 *The functions* sin *and* cos *restricted to* $\mathbb{R}$ *have a common period.*

Proof: Assume $\cos(x) > 0$ for all x. Then sin is increasing. Since $\sin(0) = 0$, $\sin(x) > 0$ for all $x > 0$. Thus $\cos' = -\sin < 0$ on $\mathbb{R}^{>0}$. Also, $\cos''(x) = -\cos(x) < 0$ and this contradicts Exercise ii, page 141. Thus $\cos(x) = 0$ for some $x > 0$. Then $\sin(x) = \pm 1$. So $\cos(2x) = \cos^2(x) - \sin^2(x) = 1$ and $\sin(2x) = 0$. Thus $A := \{\alpha \in \mathbb{R}^{>0} : \sin(\alpha) = 0\}$ is nonempty. By the formulas $\sin(x+y) = \sin x \cos y + \sin y \cos x$ and $\sin(-x) = -\sin x$, we see that the set A of zeroes of $\sin x = 0$ is an additive subgroup of $\mathbb{R}$. Since sin is continuous, this subgroup is closed. If it had an accumulation point, the accumulation point itself would be in A, but then 0 would be an accumulation point. Let (a, b) be an open interval. Assume $0 < a < b$. Let $\alpha \in A$ be such that $0 < \alpha < b - a$. Then $n\alpha \in (a, b) \cap A$ for some $n \in \mathbb{N}$. So A is dense in $\mathbb{R}$. Since A is also closed, this implies that $A = \mathbb{R}$ and $\sin = 0$, a contradiction. Hence A is discrete. Let $\pi > 0$ be the least positive element of A. It is easy to show that the period of sin is 2π.

Also follows from Lemma i.

DETAILS $\square$

It follows from the above proof that a subgroup of $\mathbb{R}$ is either dense in $\mathbb{R}$ or is generated by one element.

Find an upper bound for π.

The number e. We let $e = \exp(1) = \sum_{i=0}^{\infty} 1/i!$.

Proposition 13.4.5 *e is irrational.*

Proof: Assume $e = m/n$ for m and $n > 0$ integers. Then the number

$$N := n!\left(e - \sum_{i=0}^{n} 1/i!\right) = \sum_{i=n+1}^{\infty} n!/i! > 0$$

is a positive natural number and we have

$$\begin{array}{rcl} N & = & \sum_{i=n+1}^{\infty} n!/i! = \sum_{i=1}^{\infty} 1/(n+1)\dots(n+i) \\ & < & \sum_{i=1}^{\infty} 1/(n+1)^i = \frac{1}{n+1}\sum_{i=0}^{\infty} 1/(n+1)^i \\ & = & \frac{1}{n+1}\frac{1}{1-\frac{1}{n+1}} = 1/n, \end{array}$$

implying $N = 0$, a contradiction. □

See [**AZ**, page 28] for the fact that e^q is irrational for all $q \in \mathbb{Q} \setminus \{0\}$.
The proposition above is a consequence of a more general result:

Theorem 13.4.6 *If $(a_i)_i$ is a sequence of zeroes and ones which is not eventually zero, then $\sum_{i=0}^{\infty} a_i/i!$ is irrational.*

Proof: The same as above. **Take this proof to its appropriate place.** □

Exercises.

i. Find $\cos(15°)$.

ii. Express $\sin(4x)$ and $\cos(4x)$ in terms of $\sin x$ and $\cos x$. (Prove your formula).

iii. Let $f(x) = x^3 - 3x + 5$. Show that $f(\ln a) = 6$ for some $a > 1$.

iv. Show that $\frac{\exp(2iz)-1}{\exp(2iz)+1} = i \tan z$ for all $z \in \mathbb{C}$.

v. Describe the set $\alpha \in \mathbb{R}$ such that the subgroup $\langle 1, \alpha\rangle = \mathbb{Z} + \alpha\mathbb{Z}$ is discrete. (**Solution:** If $\alpha = p/q \in \mathbb{Q}$, then it is easy to show that $\langle 1, \alpha\rangle = \langle 1/q\rangle$ is discrete. Otherwise, the elements of the sequence $(n\alpha - [n\alpha])_{n\in\mathbb{N}}$ are distinct and are all in $[0, 1) \cap \langle 1, \alpha\rangle$. Thus $\langle 1, \alpha\rangle$ has an accumulation point in $[0, 1)$. Thus 0 is an accumulation point of $\langle 1, \alpha\rangle$. Now for $0 \le \beta < \gamma$ in $\mathbb{R}$, choose $\alpha_n \in \langle 1, \alpha\rangle \cap (0, \gamma - \beta)$. For some $k \in \mathbb{N}$, $\beta < k\alpha_n \le \gamma$ and $k\alpha_n \in (\beta, \gamma)$).

vi. Assuming π is irrational, show that $\sin(\mathbb{Z})$ is dense in $[-1, 1]$. (**Solution:** Let $y \in [-1, 1]$. Let $x \in \mathbb{R}$ be such that $\sin x = y$. Since $\mathbb{Z} + 2\pi\mathbb{Z}$ is dense in $\mathbb{R}$, there is a sequence $a_n + 2\pi b_n$ with a_n, $b_n \in \mathbb{Z}$ such that $\lim_{n\to\infty}(a_n + 2\pi b_n) = x$. Since sin is continuous, $y = \sin x = \sin(\lim_{n\to\infty} a_n + 2\pi b_n) = \lim_{n\to\infty} \sin(a_n + 2\pi b_n) = \lim_{n\to\infty} \sin a_n$.)

vii. Show that $\sum_{n=1}^{\infty} \frac{\sin n}{n}$ is convergent. (**Solution:** We will apply Abel's Theorem (Theorem 7.5.8). We only need to show that the partial sums of $|\sum_{j=n}^{m} \sin n|$ are bounded. Since $\sin n = \mathrm{Im}(\epsilon^{in})$, it is enough to show that the partial sums $|\sum_{j=n}^{m} e^{ij}|$ are bounded. We will prove that the sums $|\sum_{j=0}^{m} e^{ij}|$ are bounded. This will prove the result since $|\sum_{j=n}^{m} e^{ij}| = |\sum_{j=0}^{m} e^{ij} - \sum_{j=0}^{n-1} e^{ij}| \le |\sum_{j=0}^{m} e^{ij}| + |\sum_{j=0}^{n-1} e^{ij}|$. Now we compute: $|\sum_{j=0}^{m} e^{ij}| = |\frac{e^{i(m+1)}-1}{e^i - 1}| \le \frac{2}{|e^i-1|}$.)

viii. Find the radius of convergence of $\sum_{n=1}^{\infty} \frac{\sin n}{n} z^n$. (**Solution:** Since $\left|\frac{\sin n}{n} z^n\right| \le |z|^n$, the radius of convergence ≥ 1. If the radius of convergence is > 1, then the radius of convergence of the derived series $\sum_{n=0}^{\infty} \sin n\, z^{n-1}$ is also > 1. But by Exercise vi, page 145, the general term $\sin n\, z^n$ does not converge to 0 if $|z| \not< 1$).

ix. Show that $\lim_{n\to\infty}(1+1/n)^n = e$.

x. Find the following limits:

$$\begin{array}{l}
\lim_{x\to\infty}(1+1/x)^x \\
\lim_{x\to\infty}(1-1/x)^x \\
\lim_{n\to\infty}(1-1/n)^n \\
\lim_{n\to\infty}(1-1/n^2)^{n^2} \\
\lim_{n\to\infty}(1-1/n^2)^n \\
\lim_{n\to\infty}(1-1/n)^{n^2} \\
\lim_{x\to\infty}(1+c/x)^x
\end{array}$$

13.4.2 Inverse Trigonometric Functions

13.4.3 Logarithm

13.4.4 Hyperbolic Functions

13.5 Supplement

13.5.1 Trigonometric Functions

By Example i, page 111, $\lim_{n\to\infty}(1+z/n)^n = \exp(z)$. Thus

$$\sin z = \frac{\exp(iz)-\exp(-iz)}{2i} = \lim_{n\to\infty}\frac{1}{2i}[(1+iz/n)^n-(1-iz/n)^n].$$

On the other hand, we note that $(1+iz/n)^n-(1-iz/n)^n = 0$ if and only if $\left(\frac{1+iz/n}{1-iz/n}\right)^n = 1$ if and only if $\left(\frac{n+iz}{n-iz}\right)^n = 1$ if and only if $\frac{n+iz}{n-iz} = \exp(2ik\pi/n)$ for some $k = 0, 1, \ldots, n-1$ if and only if (as an easy calculation shows) $z = \frac{n}{i}\frac{\exp(2\pi ik/n)-1}{\exp(2\pi ik/n)+1} = n\tan k\pi n$ by Exercise iv, page 145.

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13.5.2 Series

Proposition 13.5.1 *For $x \neq 2k\pi$ $(k \in \mathbb{Z})$ and $(\epsilon_n)_n$ a decreasing sequence whose limit is 0, the series $\sum_{n=0}^{\infty}\epsilon_n e^{inx}$ is convergent.*

Proof: We will apply Theorem 7.5.8. Let $x \neq 2k\pi$ for any $k \in \mathbb{Z}$ and $v_n = v_n(x) = e^{inx}$. For $m < n$, $v_m + \ldots + v_n = e^{imx} + \ldots + e^{inx} = e^{imx}(1 + e^{ix} + \ldots + e^{i(n-m)x} = e^{imx}\frac{1-e^{i(n-m+1)x}}{1-e^ix}$, so that $|v_m + \ldots + v_n| \leq \frac{2}{|1-e^{ix}|}$. Hence the conditions of Theorem 7.5.8 are met. □

Corollary 13.5.2 *For $x \neq 2k\pi$ $(k \in \mathbb{Z})$ and $(\epsilon_n)_n$ a decreasing sequence whose limit is 0, the series $\sum_{n=0}^{\infty}\epsilon_n\sin(nx)$ and $\sum_{i=0}^{\infty}\epsilon_n\cos(nx)$ are convergent.*

Exercises.

i. Show that $\sum_{k=1}^{\infty} sin(kx)/k^2$ is uniformly convergent on $(-\infty, \infty)$.

ii. Show that $\sum_{k=0}^{\infty}(xe^{-x})^k$ converges uniformly on $[0, 2]$.

iii. Show the following:

$$\begin{aligned}
&\sin^2(z) + \cos^2(z) = 1 \\
&\exp(iz) = \cos(z) + i\sin(z) \\
&\cos \text{ is even, } \sin \text{ is odd} \\
&\sinh^2(z) - \cosh^2(z) = 1 \\
&\exp(z) = \cosh(z) + i\sinh(z).
\end{aligned}$$

iv. Let X be a set with n elements. Let $0 \leq p := p(n) \leq 1$. We select a random subset $A := A(p)$ of Xin such a way that the probability that an element $x \in X$ is in A is p.

i. Calculate the probability that A has any element at all.

ii. What is the expected number of elements of A?

iii. Let $p(n) = 1/n$. What is the probability that $A(p) \neq \emptyset$ when n tends to infinity?

iv. Assume that $p = p(n) \gg 1/n$. Show that the probability that $A(p) = \emptyset$ when n tends to infinity is 1.

v. Assume that $p = p(n) \gg 1/n$. Show that the probability that $A(p) = \emptyset$ when n tends to infinity is 0.

13.6 Notes

Analytic functions are not the only functions whose sequence $(f^{(n)}(x_0))_n$ of derivatives determine the function in a neighborhood of x_0. The functions defined and infinitely differentiable on $[a, b]$ such that if $M_n = \max\{|f^{(n)}(x)| : x \in [a, b]\}$, then $\sum_n 1/M_n^{1/n}$ is divergent (Denjoy 1921 and Carleman). Such functions are called **quasi analytic**.

Chapter 14

Graph Drawing

14.1 Drawing in Cartesian Coordinates

14.1.1 Asymptotes

Exercises.

i. Draw with as much care as possible the graph of $f(x) = \frac{x^2}{(x-1)(x+2)}$.

ii. Let $f(x) = x + 2x^2 \sin(1/x)$ if $x = 0$ and $f(0) = 0$.

   i. Show that $f'(0) = 1$.

   ii. Show that any interval containing 0 also contains points where $f'(x) < 0$, so that f cannot be increasing on any interval around 0 although $f'(0) > 0$.

14.2 Parametric Equations

14.3 Polar Coordinates

14.4 Geometric Loci

Exercise. [**BR**] *Let $A = (1, 0)$. Find the set of points M of the plane $\mathbb{R}^2$ such that, if P and Q design the projections of M onto the axes x and y respectively, the point S of intersection of AQ and PM is at distance 1 from A.*

First Solution. Let $M = (x_0, y_0)$. Then $P = (x_0, 0)$ and $Q = (0, y_0)$. The line AQ has equation $y = \frac{0-y_0}{1-0}x + y_0$. Thus the point S has coordinates $(x_0, -y_0x_0 + y_0)$. The square of its distance from $A(1, 0)$ is $(x_0 - 1)^2 + (-y_0x_0 + y_0)^2$, which we want to be 1. Thus the equation of the curve in Cartesian coordinates is $(x - 1)^2 + (-yx + y)^2 = 1$, or $(x - 1)^2(1 + y^2) = 1$, or $y^2 = \frac{1}{(x-1)^2} - 1$, i.e. $y = \pm\left(\frac{1}{(x-1)^2} - 1\right)^{1/2}$. Note that $x \neq 1$ and $\frac{1}{(x-1)^2} - 1 \geq 0$, i.e. $x \in [0, 2] \setminus \{1\}$.

Note also that the curve is symmetric with respect to the x axis. The line $x = 1$ is an asymptote and the curve is also symmetrical with respect to the line $x = 1$. It is possible to draw this curve, even if some painful work is necessary.

Second Solution. Let θ be the angle PAQ. Then one can see immediately that

$$\begin{aligned} x(\theta) &= 1 - \cos\theta \\ y(\theta) &= \tan(\theta) \end{aligned}$$

where $\theta \in [0, 2\pi] \setminus \{\pi/2, 3\pi/2\}$. It is enough to draw the curve when $\theta \in [0, \pi/2)$ because of the obvious symmetries.

It is easy to see that

i) $x(0) = y(0) = 0$.

ii) x and y both increase when θ increases from 0 to $\pi/2$.

iii) When θ goes to $\pi/2$, x goes to 1 and y goes to ∞. Thus there is an asymptote at $\theta = \pi/2$, which is the line $x = 1$.

Differentiating with respect to θ, we find

$$\begin{aligned} dx/d\theta &= \sin\theta \\ dy/d\theta &= 1/\cos^2\theta. \end{aligned}$$

Thus

$$dy/dx = (dy/d\theta)/(dx/d\theta) = 1/\cos^2\theta \sin\theta.$$

It follows that when $\theta \in [0, \pi/2)$, y is an increasing function of x. It also follows that the y axis is tangent to the curve (case $\theta = 0$).

Now we find d^2y/dx^2:

$$d^2y/dx^2 = d(dy/dx)/dx = (d(dy/dx)/d\theta)/(dx/d\theta) = (1/\cos^2\theta \sin\theta)'/\sin\theta$$

$$= \frac{2\sin^2\theta - \cos^2\theta}{\cos^3\theta \sin^3\theta}.$$

There is an inflection point when $2\sin^2\theta - \cos^2\theta = 0$, i.e. when $y = \tan(\theta) = 1/\sqrt{2}$. We can compute the first coordinate x of the inflection point: $1/\cos' 2\theta - 1 = \tan^2\theta = 1/2$, $\cos\theta = \sqrt{2/3}$ and $x = 1 - \cos\theta = 1 - \sqrt{2/3}$. The graph is concave down before $(1 - \sqrt{2/3}, 1/\sqrt{2})$ and concave up afterwards.

Exercise. [**BR**] *To a point $M = (X, Y)$, $Y \neq 0$, we associate a point m as follows: The line OM intersects the line $y = a$ in H. The lines passing from M and H and parallel to the x and y axes intersect at m. Given m, we can also find M.*

1. Calculate the coordinates of m in terms of the coordinates of M.

2. Find the curve that the point m traces when M moves on the circle $X^2 + Y^2 - 2RX = 0$.

3. Find the curve that the point M traces when m moves on the circle $x^2 + y^2 - 2Ry = 0$.

Solution.

14.5 Supplement

Theorem 14.5.1 (Cauchy's Mean-Value Theorem) *If f and g are both continuous on $[a, b]$ and differentiable on (a, b), then there exists a $c \in (a, b)$ such that*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

Proof: Let $F : [a, b] \longrightarrow \mathbb{R}$ be defined by

$$F(x) = f(x)[g(b) - g(a)] - g(x)[f(b) - f(a)].$$

Then F is continuous on $[a, b]$ and differentiable on (a, b). It is easy to check that $F(a) = F(b)$. By Rolle's Theorem (Theorem 12.5.2), there is $c \in (a, b)$ such that $F'(c) = 0$. □

Note that Theorem 12.5.3 follows from the above one by taking $g(x) = x$.

Lemma 14.5.2 *If f is differentiable then f' satisfies the Mean Value Theorem, i.e. if $a, b \in U$, $f'(a) < \gamma < f'(b)$, then $f'(c) = \gamma$ for some $c \in (a, b)$.*

Proof: Since $f'(a) < \gamma < f'(b)$, there is an $h > 0$ such that

$$\frac{f(a+h) - f(a)}{h} < \gamma < \frac{f(b+h) - f(b)}{h}.$$

Fix such an $h > 0$. The function $x \mapsto \frac{f(x+h)-f(x)}{h}$ is continuous on (a, b). By the Intermediate Value Theorem (Theorem 11.3.3) there is an $x = x(h)$ such that $\frac{f(x+h)-f(x)}{h} = \gamma$. Also, by Lemma 12.5.3, there is a $c \in (x, x+h)$ such that $\frac{f(x+h)-f(x)}{h} = f'(c)$. Thus $\gamma = f'(c)$. □

Theorem 14.5.3 *If f' exists and is bounded on some interval I, then f is uniformly continuous on I.*

14.5.1 Lipschitz Condition

14.5.2 A Metric On $\mathbb{R}^n$

Lemma 14.5.4 *i. For $i = 1, \ldots, n$, let (X_i, δ_i) be a metric space. Let $X = X_1 \times \ldots \times X_n$. Let $p > 1$ be any real number. For $x, y \in X$ let $d_p(x, y) = (\sum_{i=1}^n \delta_i(x_i, y_i)^p)^{1/p}$. Then (X, d_p) is a metric space.*

ii. Let everything be as above. Set $d_\infty(x, y) = \max\{\delta_i(x_i, y_i) : i = 1, \ldots, n\}$. Then (X, d_∞) is a metric space.

Proof: i. Setting $a_i = \delta_i(x, y)$, $b_i = \delta_i(x, z)$, $c_i = \delta_i(y, z)$, it is enough to show that if b_i, $c_i \geq 0$, then

$$(\sum_{i=1}^n (b_i + c_i)^p)^{1/p} \leq (\sum_{i=1}^n b_i^p)^{1/p} + (\sum_{i=1}^n c_i^p)^{1/p}.$$

Taking p-th powers, we need to show that

$$\sum_{i=1}^{n}(b_i + c_i)^p \leq ((\sum_{i=1}^{n} b_i^p)^{1/p} + (\sum_{i=1}^{n} c_i^p)^{1/p})^p.$$

We first compute the left hand side:

$$\sum_{i=1}^{n}(b_i + c_i)^p = \sum_{i=1}^{n}\sum_{j=0}^{p}\binom{p}{j} b_i^j c_i^{p-j} = \sum_{i=1}^{n} b_i^p + \sum_{i=1}^{n} c_i^p + \sum_{j=1}^{p-1}\binom{p}{j}\sum_{i=1}^{n} b_i^j c_i^{p-j}.$$

We now compute the right hand side:

$$((\sum_{i=1}^{n} b_i^p)^{1/p} + (\sum_{i=1}^{n} c_i^p)^{1/p})^p = \sum_{j=0}^{p}\binom{p}{j}(\sum_{i=1}^{n} b_i^p)^{j/p}(\sum_{i=1}^{n} c_i^p)^{(p-j)/p} =$$

$$= \sum_{i=1}^{n} b_i^p + \sum_{i=1}^{n} c_i^p + \sum_{j=1}^{p-1}\binom{p}{j}(\sum_{i=1}^{n} b_i^p)^{j/p}(\sum_{i=1}^{n} c_i^p)^{(p-j)/p}.$$

Simplifying, we see that we have to show that

$$\sum_{i=1}^{n} b_i^j c_i^{p-j} \leq (\sum_{i=1}^{n} b_i^p)^{j/p}(\sum_{i=1}^{n} c_i^p)^{(p-j)/p}$$

for all $j = 1, \ldots, p-1$. Setting $r = b_i^p$, $s = c_i^p$, $\alpha = j/p$ and $\beta = (p-j)/p$, we see that it is enough to show that

$$\sum_{i=1}^{n} r_i^{\alpha} s_i^{\beta} \leq (\sum_{i=1}^{n} r)^{\alpha}(\sum_{i=1}^{n} s)^{\beta}$$

if $0 \leq r_i$, $0 \leq s_i$, $0 < \alpha$, $\beta < 1$ and $\alpha + \beta = 1$.

We may assume that some $r_i \neq 0$ and some $s_j \neq 0$, thus $\sum_{i=1}^{n} r_i$ and $\sum_{i=1}^{n} r_i$ are nonzero. Dividing both sides by $(\sum_{i=1}^{n} r_i)^{\alpha}(\sum_{i=1}^{n} r_i)^{\beta}$, we see that we have to show that

$$\sum_{i=1}^{n}(\frac{r_i}{\sum_{i=1}^{n} r_i})^{\alpha}(\frac{s_i}{\sum_{i=1}^{n} s_i})^{\beta} \leq 1.$$

By setting

$$u_i = \frac{r_i}{\sum_{i=1}^{n} r_i} \text{ and } v_i = \frac{r_i}{\sum_{i=1}^{n} s_i},$$

we see that we have to prove that

$$\sum_{i=1}^{n} u_i^{\alpha} v_i^{\beta} = 1$$

if $0 \leq u_i$, $0 \leq v_i$, $\sum_{i=1}^{n} u_i = \sum_{i=1}^{n} v_i = 1$, $0 < \alpha$, $\beta < 1$ and $\alpha + \beta = 1$.

Set $f(\alpha) = \sum_{i=1}^{n} u_i^\alpha v_i^\beta = 1$. Note that $f(0) = f(1) = 1$. Then an easy calculation shows that

$$f'(\alpha) = \sum i = 1^n (\ln(u_i) u_i^\alpha v_i^{1-\beta} - \ln(v_i) u_i^\alpha v_i^{1-\beta}) = \sum i = 1^n u_i^\alpha v_i^{1-\beta} (\ln(u_i) - \ln(v_i)),$$

and that

$$f''(\alpha) = \sum_{i=1}^{n} (\ln(u_i) - \ln(v_i))^2 u_i^\alpha v_i^{1-\beta}).$$

Thus f'' is concave up, hence $f(\alpha) \leq f(0) = f(1) = 1$ for all $\alpha \in (0, 1)$. $\square$

Chapter 15

Riemann Integral

15.1 Definition and Examples

15.2 Fundamental Theorem of Calculus

15.3 How To Integrate?

15.3.1 Power Series

How to integrate polynomials and power series...

15.3.2 Trigonometric Functions

Exercises.

i. Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$. Show that

$$I_{2n} = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2 \cdot 4 \cdot 6 \ldots (2n)} \frac{\pi}{2}$$

and that

$$I_{2n+1} = \frac{2 \cdot 4 \cdot 6 \ldots (2n)}{1 \cdot 3 \cdot 5 \ldots (2n+1)}.$$

ii. **[Wallis' Formula]** Let I_n be as above. Show that $(I_n)_n$ is a decreasing sequence. (Hint: $(\sin^n x)_n$ is a decreasing sequence of functions). Conclude that

$$\frac{(2 \cdot 4 \cdot 6 \ldots (2n))^2}{(3 \cdot 5 \ldots (2n-1))^2 (2n+1)} < \frac{\pi}{2} < \frac{(2 \cdot 4 \cdot 6 \ldots (2n-2))^2 (2n)}{(3 \cdot 5 \ldots (2n-1))^2}.$$

Conclude that

$$\frac{\pi}{2} = \lim_{n \to \infty} \frac{(2 \cdot 4 \cdot 6 \ldots (2n))^2}{(3 \cdot 5 \ldots (2n-1))^2 2n}$$

and that

$$\pi = \lim_{n\to\infty} \frac{(2\cdot 4\cdot 6\ldots(2n))^4}{n((2n)!)^2} = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{n((2n)!)^2}.$$

The last equality is known as **Wallis' formula**.

15.4 Integration of Complex Function

Define $\int_a^b f(t,z)dt$.

15.4.1 Functions Defined by Integration

Theorem 15.4.1 . *Let $f(z,t)$ be a function where $z \in A \subseteq \mathbb{C}$ and $t \in I$, an interval of $\mathbb{R}$. Suppose $\int_I f(z,t)dt$ exists for all $z \in A$. Then $z \mapsto \int_I f(z,t)dt$ is a continuous function from A into $\mathbb{C}$.*

15.5 Applications

15.5.1 Application to Series

Theorem 15.5.1 (Integral Test, Cauchy) *Let f be a nonincreasing positive real valued function which is defined for $x \geq 1$. Then the series $\sum_{n=1}^\infty f(n)$ converges if and only if $\int_1^\infty f(x)dx$ exists. Also,*

$$\int_1^\infty f(x)dx \leq \sum_{n=1}^\infty f(n) \leq f(1) + \int_1^n f(x)dx$$

and

$$\sum_{n=1}^\infty f(n) = \sum_{i=1}^n f(i) + R_n$$

where $\int_{n+1}^\infty f(x)dx \leq R_n \leq \int_n^\infty f(x)dx$.

Proof: For $n < m$ we have,

$$\int_n^m f(x)dx \leq f(n) + f(n+1) + \ldots + f(m-1) \leq \int_{n-1}^{m-1} f(x)dx.$$

This proves the first part. The second part follows by taking $n = 1$ and 2 and sending m to infinity in the above inequalities. The third part is easy as well. $\square$

Assume now the series diverges but that $f(x)$ converges to 0 when x goes to infinity. Let $m = m(n) \geq n$ be any integer valued function. Then

$$0 \leq f(n)+f(n+1)+\ldots+f(m-1) - \int_n^m f(x)dx \leq \int_{n-1}^n f(x)dx - \int_{m-1}^m f(x)dx,$$

so that when n tends to infinity, we get,

$$\lim_{n\to\infty}\left(f(n)+f(n+1)+\ldots+f(m-1)-\int_n^m f(x)dx\right)=0.$$

For example if we take $m=2n$ and $f(x)=1/x$,

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{1+n}+\ldots+\frac{1}{2n-1}-\int_n^{2n} dx/x\right)=0.$$

Since $\int_n^{2n} dx/x=\ln(2n)-\ln(n)=\ln(2)$, we have,

$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{1+n}+\ldots+\frac{1}{2n}\right)=\ln(2).$$

More generally, let

$$\sigma_n=f(1)+f(2)+\ldots+f(n)-\int_1^n f(x)dx.$$

For $m>n$ we have

$$\sigma_n-\sigma_m=\int_n^m f(x)dx-(f(n+1)+\ldots+f(m)).$$

By the theorem this number is nonnegative (replace n and m by $n+1$ and $m+1$ respectively) and

$$|\sigma_n-\sigma_m|=\sigma_n-\sigma_m=\int_n^m f(x)dx-(f(n)+\ldots+f(m-1))+f(n)\leq f(n).$$

Since $\lim_{n\to\infty} f(n)=0$, the double sequence $(\sigma_m-\sigma_n)_{m,n}$ converges to 0. Therefore the sequence $(\sigma_n)_n$ has a limit, say c_f. Thus

$$\lim_{n\to\infty}\left(\sum_{i=1}^n f(i)-\int_1^n f(x)dx\right)=c_f.$$

In particular if $f(x)=1/x$, we see that

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\ldots+\frac{1}{n}-\ln(n)\right)$$

exists. This constant is called **Euler's constant** or **Euler-Mascheroni constant**. Since the sequence is increasing, the Euler constant is a positive number.

We can now state the following result:

Theorem 15.5.2 *Let $f(x)$ be a positive real valued nonincreasing function defined for $x\geq 1$ and which converges to 0 as x goes to infinity. Then*

$$\lim_{n\to\infty}\left(\sum_{i=1}^n f(i)-\int_1^n f(x)dx\right)$$

exists. In particular

$$\lim_{n\to\infty}\left(1+\frac{1}{2}+\ldots+\ldots+\frac{1}{n}-\ln(n)\right)=0.57721\ldots$$

exists.

Corollary 15.5.3 *The series $\sum_{n=1}^{\infty}\frac{1}{n^{1+\epsilon}}$ converges if $\epsilon > 0$ and diverges if $\epsilon \le 0$.*

Proof: Follows from Theorem 15.5.1. $\square$

Corollary 15.5.4 *If $\overline{\lim}_{n\to\infty}\left(1+\frac{\ln|u_n|}{\ln n}\right) < 0$ then the series $\sum_n u_n$ converges absolutely. If $\underline{\lim}_{n\to\infty}\left(1+\frac{\ln|u_n|}{\ln n}\right) < 0$ then the series $\sum_n u_n$ converges absolutely.*

Proof: Follows from Corollary 15.5.3. $\square$

Theorem 15.5.5 (Raabe and Duhamel) *If $\underline{\lim}\, n\left(\left|\frac{u_n}{u_{n+1}}\right|-1\right) > 1$, then the series $\sum_n u_n$ converges absolutely. If $u_n > 0$ and $\overline{\lim}\, n\left(\left|\frac{u_n}{u_{n+1}}\right|-1\right) < 1$ then the series $\sum_n u_n$ diverges.*

Proof: In the first case, there is an $\epsilon > 0$ such that $n\left(\left|\frac{u_n}{u_{n+1}}\right|-1\right) > 1+\epsilon$ eventually. Thus $\left|\frac{u_n}{u_{n+1}}\right| > 1+\frac{1+\epsilon}{n}$ eventually. Let $0<\delta<\epsilon$. Then $1+\frac{1+\epsilon}{n} > (1+1/n)^{\delta+1}$ eventually. Thus

$$\left|\frac{u_{n+1}}{u_n}\right| < \left(\frac{n}{n+1}\right)^{\delta+1} = \frac{1/(n+1)^{\delta+1}}{1/n^{\delta+1}}$$

eventually. Since $\sum_n 1/n^{1+\delta}$ converges, the result follows from Corollary 7.2.6.

Assume now we are in the second case. Then $n\left(\left|\frac{v_n}{v_{n-1}}\right|-1\right) < 1$ eventually. Thus $\frac{v_{n-1}}{v_n} > \frac{1/(n+1)}{1/n}$ eventually. Since $\sum_n 1/n$ diverges, the second part follows from Theorem 7.2.6 as well. $\square$

Example. Consider the series

$$1+\alpha z+\frac{\alpha(\alpha-1)}{2!}z^2+\ldots\frac{\alpha(\alpha-1)\ldots(\alpha-n+1)}{n!}z^n+\ldots$$

where $\alpha\in\mathbb{C}$. The series converges for $|z|<1$ according to d'Alembert (Corollary 7.5.2). Suppose now $|z|=1$. If u_n is the term of z^n, then

$$\left|\frac{u_n}{u_{n+1}}\right| = \left|\frac{n+1}{\alpha-n}\right|.$$

Thus when $|z|=1$, by Raabe and Duhamel's Convergence Rule, the series converges absolutely for $\Re(\alpha)>0$, because a simple computation shows that $n\left(\left|\frac{u_n}{u_{n-1}}\right|-1\right)$ converges to $1+\Re(\alpha)$.

If $\alpha\in\mathbb{R}^{<0}$ and $x=-1$ the series diverges as it can be checked as above.

Exercises.

i. Show that the Euler constant is < 1.

15.5.2 Stirling Formula

We will try to find an approximation for $n!$.

Our first try:

Lemma 15.5.6 $\lim_{n\to\infty} \frac{\ln(n!)}{n\ln n - n} = \lim_{n\to\infty} \frac{\ln(n!)}{n \ln \frac{n}{e}} = 1.$

Proof: Note that $\ln(n!) = \sum_{i=1}^n \ln i$. By the integration method, we have

$$\begin{array}{rcl} n\ln n - n & < & n\ln n - n + 1 = \int_1^n \ln x\,dx \\ & \leq & \sum_{i=1}^n \ln i \leq \int_1^{n+1} \ln x\,dx = (n+1)\ln(n+1) - n; \end{array}$$

furthermore the quotient of the first term to the last term converges to 1:

$$\begin{array}{rcl} \frac{n\ln n - n}{(n+1]\ln(n+1)-n} & = & \frac{\ln n - 1}{(1+1/n)\ln(n+1)-1} \sim \frac{\ln n - 1}{\ln(n+1)-1} \\ & = & \frac{1 - 1\ln n}{\ln(n+1)/\ln n - 1/\ln n} \sim \frac{\ln n}{\ln(n+1)} \sim \frac{1/n}{1/n+1} \sim 1. \end{array}$$

This gives the result. $\square$

Theorem 15.5.7 $\lim_{n\to\infty} \frac{n!}{(n/e)^n\sqrt{2\pi n}} = 1.$

We can obtain a better approximation:

Theorem 15.5.8 $n! = (n/e)^n\sqrt{2\pi n}(1 + 1/12n + f(n)/n^2)$ *where f is a function that is bounded.*

The proof will show that we can extend the approximation. But this will be done by another method in the next subsection.

Question: Does $\lim_{n\to\infty} f(n)$ exist?

Proof: Consider the sequence

$$s_n = \ln(n!) - n(\ln n - 1) = \ln(n!) - n\ln\frac{n}{e}.$$

For $n > 1$, we have,

$$\begin{array}{rcl} u_n & := & s_n - s_{n-1} = 1 + (n-1)\ln(1 - 1/n) \\ & = & 1 - (n-1)(1/n + 1/2n^2 + 1/3n^3 + \ldots) \\ & = & \sum_{i=1}^\infty \frac{1}{i(i+1)n^i} = 1/2n + 1/6n^2 + g(n)/n^3 \end{array}$$

where $g(n) = \sum_{i=3}^\infty 1/i(i+1)n^{i-3}$ is a positive valued function whose limit when n goes to infinity is finite (see Exercise i, page 138).

We now introduce

$$S_n := s_n - \frac{1}{2}\ln n = \ln(n!) - n(\ln n - 1) - \frac{1}{2}\ln n \tag{1}$$

(See Exercise i, page 161). Then

$$\begin{array}{rcl} v_n & := & S_n - S_{n-1} = u_n + \frac{1}{2}\ln\frac{n-1}{n} \\ & = & 1/2n + 1/6n^2 + g(n)/n^3 + \frac{1}{2}\ln(1-1/n) \\ & = & 1/2n + 1/6n^2 + g(n)/n^3 - \frac{1}{2}(1/n + 1/2n^2 + 1/3n^3 + \ldots) \\ & = & (1/6n^2 - 1/4n^2) + \frac{g(n)-(1/3+1/4n+1/5n^2+\ldots)/2}{n^3} \\ & = & -1/12n^2 + h(n)/n^3 \end{array}$$

where $\lim_{n\to\infty} h(n)$ is a constant. Setting $v_1 = 1$, we have

$$S_n = \sum_{i=1}^{n} v_i = -\frac{1}{12}\sum_{i=1}^{n} 1/i^2 + \sum_{i=2}^{n}\frac{h(i)}{i^3}.$$

Thus the sequence $(S_n)_n$, i.e. the series $\sum_{i=1}^{\infty} v_i$ has a limit say S. Then $S_n = S - R_n$ where

$$R_n = \sum_{i=n+1}^{\infty} v_i = -\frac{1}{12}\sum_{i=n+1}^{\infty}\frac{1}{i^2} + \sum_{i=n+1}^{\infty}\frac{h(i)}{i^3}.$$

Now we have two remarks: First we show that $n^2\sum_{i=n+1}^{\infty}\frac{h(i)}{i^3}$ converges to a finite number when n goes to infinity. Since $(h(i))_i$ is bounded, we may suppose $h(i) = 1$. By the integral test, we have,

$$n^2\sum_{i=n+1}^{\infty}\frac{1}{i^3} = n^2\sum_{i=1}^{\infty}\frac{1}{(n+i)^3} < n^2\int_2^{\infty}\frac{1}{(n+x)^3}dx = \frac{n^2}{2(n+2)^2},$$

which shows what we want. Thus we may replace $\sum_{i=n+1}^{\infty}\frac{h(i)}{i^3}$ in the formula above by $t(n)/n^2$ where $t(n)$ is a function with a finite limit:

$$R_n = \sum_{i=n+1}^{\infty} v_i = -\frac{1}{12}\sum_{i=n+1}^{\infty}\frac{1}{i^2} + \frac{t(n)}{n^2}. \tag{2}$$

Here is our second remark:

$$\frac{1}{i} - \frac{1}{i+1} = \frac{1}{i(i+1)} < \frac{1}{i^2} < \frac{1}{i(i-1)} = \frac{1}{i-1} - \frac{1}{i},$$

so that

$$\frac{1}{n+1} = \sum_{i=n+1}^{\infty}\left(\frac{1}{i} - \frac{1}{i+1}\right) < \sum_{i=n+1}^{\infty}\frac{1}{i^2} < \sum_{i=n+1}^{\infty}\left(\frac{1}{i-1} - \frac{1}{i}\right) = \frac{1}{n}. \tag{3}$$

Let

$$k(n) = \frac{n^2}{12}\left(\frac{1}{n} - \sum_{i=n+1}^{\infty}\frac{1}{i^2}\right).$$

By (3), $0 < k(n) < \frac{n^2}{12}(\frac{1}{n} - \frac{1}{n+1}) = \frac{1}{12(1+1/n)} < 1$. Thus $k(n)$ remains bounded. (**Question:** Does it have a limit?) Note that

$$\sum_{i=n+1}^{\infty} \frac{1}{i^2} = \frac{1}{n} - \frac{12k(n)}{n^2}. \tag{4}$$

Now by (1), (2) and (4),

$$\begin{aligned}
\ln(n!) &\overset{(1)}{=} S_n + n\ln\left(\tfrac{n}{e}\right) + \tfrac{1}{2}\ln n = S - R_n + n\ln\left(\tfrac{n}{e}\right) + \tfrac{1}{2}\ln n \\
&\overset{(2)}{=} S + \tfrac{1}{12}\textstyle\sum_{i=n+1}^{\infty} 1/i^2 - \frac{t(n)}{n^2} + n\ln\left(\tfrac{n}{e}\right) + \tfrac{1}{2}\ln n \\
&\overset{(4)}{=} S + \tfrac{1}{12n} - \tfrac{k(n)}{n^2} - \tfrac{t(n)}{n^2} + n\ln\left(\tfrac{n}{e}\right) + \tfrac{1}{2}\ln n.
\end{aligned}$$

By exponentiating and using $e^x = 1 + x + x^2/2! + \ldots$, we get,

$$n! = e^S \left(\frac{n}{e}\right)^n \sqrt{n}\left(1 + \frac{1}{12n} + \frac{f(n)}{n^2}\right)$$

for some $f(n)$ that remains bounded when n goes to infinity. It remains to evaluate $A := e^S$ and we use Wallis formula for this. We replace $n!$ by the formula found above in Wallis formula, which we recall from page 155:

$$\pi = \lim_{n\to\infty} \frac{2^{4n}(n!)^4}{n((2n)!)^2}.$$

We easily find $A = \sqrt{2\pi}$. Thus

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{f(n)}{n^2}\right).$$

Exercises.

i. Show that $s_n - \frac{1}{2}\ln n$ can be made as close to a constant. (**Solution:** We continue with the notation of the proof of the theorem above. Note that $s_n - 1 = s_n - s_1 = \sum_{i=2}^n u_i = \sum_{i=2}^n (1/2i + 1/6i^2 + g(i)/i^3) = \frac{1}{2}\sum_{i=2}^n 1/i + \frac{1}{6}\sum_{i=2}^n 1/i^2 + \sum_{i=2}^n g(i)/i^3$. Since $g(i)$ is bounded and since $\sum_{i=2}^{\infty} 1/i^2$ and $\sum_{i=2}^{\infty} 1/i^3$ converge, for n large enough $\frac{1}{6}\sum_{i=2}^n 1/i^2 + \sum_{i=2}^n g(i)/i^3$ can be made as close to a constant as we wish to. On the other hand $\sum_{i=2}^n 1/i = \sum_{i=1}^n 1/i - 1$, can be made as close to $\ln n - 1$ as we wish to (Theorem 15.5.2). Thus s_n can be made close to $A + \frac{1}{2}\ln n$ for some constant A.)

15.5.3 Euler's Γ Function

Consider the integral

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt$$

where $z \in \mathbb{C}$ (but the reader may opt to take $z \in \mathbb{R}$) and, for $0 < a < b$, the function

$$F(z) = F_{a,b}(z) = \int_a^b e^{-t} t^{z-1} dt,$$

which is a continuous function of z by Theorem 15.4.1.

Fixing $a > 0$ and regarding this as a family of functions parametrized by $b > a$, we claim that for z varying over a set whose real part is bounded above, $F(z)$ converges uniformly when $b \to \infty$. We have to show that for all $\epsilon > 0$ there is b_0 such that for all b, $b' > b_0$ and all z in the right domain, $\left|\int_b^{b'} e^{-t}t^{z-1}dt\right| < \epsilon$. Indeed, let $\epsilon > 0$ and A be the upper bound for the real part of z. Let b_0 be such that $t^{A-1} < e^{t/2}$ for $t > b_0$ and $e^{-b_0/2} < \epsilon/4$. Then we have

$$|e^{-t}t^{z-1}| = e^{-t}t^{x-1}|t^{iy}| = e^{-t}t^{x-1}|e^{iy\ln t}| = e^{-t}t^{x-1} \le e^{-t}t^{A-1} < e^{-t/2},$$

so that for b, $b' > b_0$,

$$\begin{array}{rcl} \left|\int_b^{b'} e^{-t}t^{z-1}dt\right| & \le & \int_b^{b'} \left|e^{-t}t^{z-1}\right| dt < \int_b^{b'} e^{-t/2}dt \\ & = & -2e^{-b'/2} + 2e^{-b/2} < 4e^{-b_0/2} < \epsilon. \end{array}$$

It follows that

$$\int_a^\infty e^{-t}t^{z-1}dt$$

is a continuous function of z for $a > 0$.

Now we fix b and view $F(z)$ as a family of functions parametrized by $0 < a < b$. We claim that the family $F(z)$ converges uniformly when $a \to 0$ on any set in which the real part of z is bounded below. For this we have to show that for any $\epsilon > 0$ there is a $\delta > 0$ such that for all $0 < a < a' < \delta$ and for all $z \in \mathbb{C}$ in the right domain (the real part above some fixed $A > 0$), $\left|\int_a^{a'} e^{-t}t^{z-1}dt\right| < \epsilon$. Indeed let $\epsilon > 0$. Let A be the lower bound for the real part of z. For $t \in [0, 1)$, we have

$$|e^{-t}t^{z-1}| = e^{-t}t^{x-1} < t^{x-1} \le t^{A-1}.$$

Let $\delta < 1$ be such that $\delta^A < \epsilon A/2$. Then for all $0 < a < a' < \delta < 1$,

$$\begin{array}{rcl} \left|\int_a^{a'} e^{-t}t^{z-1}dt\right| & \le & \int_a^{a'} \left|e^{-t}t^{z-1}\right| dt \le \int_a^{a'} t^{A-1}dt \\ & = & a'^A/A - a^A/A < 2a'^A/A < 2\delta^A/A < \epsilon. \end{array}$$

Thus

$$\int_0^b e^{-t}t^{z-1}dt$$

is a continuous function of z for $a > 0$.

It follows from above that $\Gamma(z)$ is well-defined and is a continuous function of z on the domain $\{z \in \mathbb{C} : \Re(z) > 0\}$.

By integrating by parts, we have ($du = e^{-t}dt$, $v = t^z$),

$$\begin{array}{rcl} \Gamma(z+1) & = & \int_0^\infty e^{-t}t^z dt = [-e^{-t}]_0^\infty - \int_0^\infty (-e^{-t})zt^{z-1}dt \\ & = & [-1+1] + z\int_0^\infty e^t t^{z-1}dt = z\Gamma(z). \end{array}$$

We deduce that for $n \in \mathbb{N} \setminus \{0\}$,

$$\Gamma(z+n) = (z+n-1)(z+n-2)\cdots z\Gamma(z).$$

On the other hand $\Gamma(1) = \int_0^\infty e^{-t}dt = [-e^{-t}]_0^\infty = 1$. Thus, setting $z = 1$ above we get,

$$\Gamma(n+1) = n!$$

We proved the following:

Theorem 15.5.9 *The function*

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt$$

where $z \in \mathbb{C}$ is well-defined and continuous for $\Re(z) > 0$. Furthermore

$$\Gamma(z+n) = (z+n-1)(z+n-2)\cdots z\Gamma(z)$$

and $\Gamma(n+1) = n!$

The function Γ is called **Euler's Gamma function**. We will soon extend its domain of definition from $\Re(z) \in \mathbb{R}^{>0}$ to $\Re(z) \in \mathbb{R} \setminus (-\mathbb{N})$. For this purpose we write

$$\Gamma(z) = \int_0^1 e^{-t}t^{z-1}dt + \int_1^\infty e^{-t}t^{z-1}dt.$$

As we have seen above the second part is a continuous function of z for all $z \in \mathbb{C}$. We deal with the first one. Let us replace e^{-t} by its Taylor series $\sum_{n=0}^\infty (-1)^n t^n/n!$. This series converges uniformly for $t \in [0,1]$, thus also when we multiply it with t^{z-1}, hence we can integrate it term per term to get:

$$\begin{array}{rcl} \int_0^1 e^{-t}t^{z-1}dt & = & \int_0^1 (\sum_{n=0}^\infty (-1)^n t^{n+z-1}/n!)dt = \sum_{n=0}^\infty [\int_0^1 (-1)^n t^{n+z-1}/n!]dt \\ & = & \sum_{n=0}^\infty [(-1)^n t^{n+z}/n!(n+z)]_0^1 \\ & = & \sum_{n=0}^\infty (-1)^n/n!(n+z) \end{array}$$

Such a series converges uniformly on every compact subset ($\equiv$ closed bounded subset) not containing the nonpositive integers according to Weierstrass M-test (Theorem 9.3.3), because then $|n+z| > \delta > 0$ for some δ and so $|(-1)^n/n!(n+z)| = 1/n!|n+z| < 1/\delta n!$ and $\sum_{n=0}^\infty 1/\delta n!$ converges. Thus

$$\Gamma(z) = \sum_{n=0}^\infty (-1)^n/n!(n+z) + \int_1^\infty e^{-t}t^{z-1}dt.$$

But this new expression is well-defined and continuous for all $z \in \mathbb{C} \setminus (-\mathbb{N})$. We still call it $\Gamma(z)$.

Now we check that the functional equality

$$\Gamma(z+n) = (z+n-1)(z+n-2)\cdots z\Gamma(z)$$

holds for the new definition for all $z \in \mathbb{C} \setminus (-\mathbb{N})$. It is enough to show it for $n = 1$. Indeed,

$$\begin{aligned}
\Gamma(z+1) &= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+z+1)} + \int_1^{\infty} e^{-t} t^z dt \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n((n+z+1)-z)}{(n+1)!(n+z+1)} + \int_1^{\infty} e^{-t} t^z dt \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n((n+z+1)-z)}{(n+1)!(n+z+1)} + [-e^{-t} t^z]_1^{\infty} + z \int_1^{\infty} e^{-t} t^{z-1} dt \\
&= \textstyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} + z \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+z)} + e^{-1} + z \int_1^{\infty} e^{-t} t^{z-1} dt \\
&= z \textstyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n+z)} + z \int_1^{\infty} e^{-t} t^{z-1} dt = z\Gamma(z).
\end{aligned}$$

Chapter 16

Supplements

16.1 Stone-Weierstrass Theorem

Theorem 16.1.1 *Let K be a compact metric space. Let $A \leq C(K)$ be a subalgebra. Suppose*

1) A separates points.

2) A contains the constant functions.

Then A is dense in $C(K)$ with respect to the uniform norm.

Ex: Find maximal ideals of $C([a, b])$. Answer: Only the expected ones.

Chapter 17

Fourier Series

17.1 Hilbert Spaces

Let H be a real or complex vector space with a scalar product (,). Then $|x| = \sqrt{(x,x)}$ defines a normed vector space. Suppose H is separable (has a countable dense subset) and complete. H is called a **separable Hilbert space**. Choose a countable dense subset, then a countable maximal linearly independent subset, then orthogonalize this subset by Gram-Schmidt.

Theorem 17.1.1 *Let H be a Hilbert space and $M \leq H$ be a closed subspace. Then there is a unique $y \in M$ such that $|x-y| = d(x,M)$.*

Theorem 17.1.2 *Any Hilbert space is isomorphic to ℓ^2.*

17.2 Fourier Series

Theorem 17.2.1 *The function*

$$d_2(f,g) = |f-g|_2 = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)-g(x)|^2 dx}$$

defines a pseudometric on the set of integrable functions on the interval $[-\pi,\pi]$.

Theorem 17.2.2 *Let f be an integrable function on $[-\pi,\pi]$. Let $\phi_n(x) = \exp(inx)$ and*

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\exp(-inx)dx = \langle f, \phi_n \rangle.$$

Let $s_n(f;x) = \sum_{-N}^{N} c_n\phi_n(x)$. Then

$$\lim_{N\to\infty}\left(\sum_{-\pi}^{\pi} |f(x) - s_N(f;x)|^2 dx\right) = 0.$$

Furthermore

$$\langle \phi_m, \phi_n \rangle = \delta_{n,m}$$

for all n, $m \in \mathbb{Z}$

The L^2-norm: $|f|_2 = \int_{-\pi}^{\pi} |f(x)|^2 dx$.

Lemma 17.2.3 *f bounded integrable. $\epsilon > 0$. Then there is a polygonal function p on $[-\pi, \pi]$ such $|f - p|_2 < \epsilon$.*

Proof: Trivial. □

Theorem 17.2.4 *Let f and g be two integrable functions on $[-\pi, pi]$. Let $f(x) \sim \sum_{-\infty}^{\infty} c_n \phi_n$ and $g \sim \sum_{-\infty}^{\infty} d_n \phi_n$ be its Fourier series. Then*

$$\langle f, g \rangle = \sum_{n=\bar{i}nfty}^{\infty} c_n \overline{d_n}$$

and

$$|f|_2 = \sum_{n=\bar{i}nfty}^{\infty} |c_n|^2.$$

Lemma 17.2.5 *If f is a bounded function on $[-\pi, \pi]$ which is integrable then for all $\epsilon > 0$ there is an $h \in C([-\pi, \pi])$ such that $|f - h|_2 < \epsilon$.*

$R =$ Integrable functions.
$C([-\pi, \pi]) =$ Continuous functions.
$F =$ Finite linear combinations of $\phi_n = \langle \phi_n : n \in \mathbb{Z} \rangle$.
$s_N(f; x) \in F_N = \langle \phi_n : -N \leq n \leq N \rangle$.
$L^2 = \{f : \int_{-\pi}^{\pi} |f|^2 dx < \infty\}$.
It $t_N \in F_N$ then $|f - s_N| \leq |f - t_N|$.

Chapter 18

Topological Spaces (continued)

18.1 Product Topology

For each i in an index set I, let X_i be a topological space. Let $X = \prod_{i\in I} X_i$. Consider the smallest topology on X that makes all the projection maps $\pi_i : X \longrightarrow X_i$ continuous. This is called the **product topology**.

Exercises.

i. The open sets

$$\{x \in \prod_{i\in I} X_i : x_{i_1} \in U_{i_1}, \dots, x_{i_n} \in U_{i_n}\}$$

($n \in \mathbb{N}$ and $U_{i_j} \subseteq X_{i_j}$ open) form a base of the topology.

ii. If the topological spaces X_i are discrete, then the open sets

$$\{x \in \prod_{i\in I} X_i : x_{i_1} = a_{i_1}, \dots, x_{i_n} = a_{i_n}\}$$

($n \in \mathbb{N}$ and $a_{i_j} \in X_{i_j}$) form a base of the product topology.

iii. If $I = \mathbb{N}$ and the topological spaces X_i are discrete, then the open sets

$$\{x \in \prod_{i\in I} X_i : x_1 = a_1, \dots, x_n = a_n\}$$

($n \in \mathbb{N}$ and $a_i \in X_i$) form a base of the product topology.

iv. If $X_i = \mathbb{R}$ (with the usual topology), then the open sets

$$\{x \in \prod_{i\in I} X_i : x_{i_1} \in (a_{i_1}, b_{i_1}), \dots, x_{i_n} \in (a_{i_n}, b_{i_1})\}$$

($n \in \mathbb{N}$ and $a_{i_j}, b_{i_j} \in X_{i_j}$) form a base of the product topology.

v. Assume I is finite and each X_i is a metric space. Show that the product topology on $\prod_I X_i$ is given by any of the product metrics.

vi. Suppose $I = \omega = \mathbb{N}$ and $X_i = \mathbb{N}$ has the discrete topology. Then, viewing $\prod_\omega X_i$ as the set of function from $\mathbb{N}$ into $\mathbb{N}$, we have:

a. $\prod_\omega X_i$ is metrisable by the following metric $d(x, y) = 1/(n+1)$ if n is the smallest integer for which $x_n \neq y_n$.

b. The set of injective functions from ω into ω is a closed subset of $\prod_\omega X_i$.

c. Neither the set of surjective functions nor $\mathrm{Sym}(\omega)$ is a closed subset of $\prod_\omega X_i$.

Problem 18.1.1 *What is the closure of* $\mathrm{Sym}(\omega)$ *in* $\prod_\omega \omega$*?*

Problem 18.1.2 *Let* $\mathrm{Sym}^f(\omega)$ *be the set of permutations of* $\mathrm{Sym}(\omega)$ *that move only finitely many elements of* ω*. Clearly* $\mathrm{Sym}^f(\omega)$ *is a subgroup of* $\mathrm{Sym}(\omega)$*. What is its closure in* $\mathrm{Sym}(\omega)$*?*

Hausdorff Spaces. Let X be a topological space. If for any two distinct points x and y of X there are disjoint open sets U_x and U_y containing x and y, then we say that X is **Hausdorff**.

The coarsest topology on a set with at least two points is not Hausdorff.

Proposition 18.1.3 *A metric space is Hausdorff.*

Exercises.

i. Assume that each X_i is a Hausdorff space. Show that the product topology $\prod_{i \in I} X_i$ is Hausdorff.

18.2 Homeomorphisms

The notion of isomorphism between topological spaces is defined as follows: Two topological spaces X and Y are called **homeomorphic** (i.e. isomorphic) if there is continuous bijection $f : X \longrightarrow Y$ whose inverse is also continuous. Such a map is called a homeomorphism.

18.3 Sequences in Topological Spaces

Let X be a set, $(x_n)_n$ a sequence in X and $x \in X$. We say that x is a **limit** of the sequence $(x_n)_n$ if for any open subset U containing x, there is a natural number N such that $x_n \in U$ whenever $n > N$.

Lemma 18.3.1 *Let* $A \subseteq X$ *be closed,* $(a_n)_n$ *a sequence from* A *and* x *a limit of* $(a_n)_n$*. Then* $x \in A$*.*

Examples. 1. If the sequence is eventually constant, i.e. if there is an x and a natural number N such that $x_n = x$ for $n > N$, then x is a limit of the sequence $(x_n)_n$.

2. Let X have the coarsest topology. Then any point of X is a limit of any sequence of X.

3. Let X have the discrete topology. Then a sequence has a limit if and only if the sequence is eventually constant, i.e. if there is an x and a natural number N such that $x_n = x$ for $n > N$.

Proposition 18.3.2 *In a Hausdorff topological space a sequence has at most one limit.*

In this case we say that $(x_n)_n$ is a converging sequence and we write $\lim_{n\to\infty} x_n = x$.

Proposition 18.3.3 *Let X and Y be two topological spaces. Let $f : X \longrightarrow Y$. Show that f is continuous if and only if for any convergent sequence $(x_n)_n$ of X, $(f(x_n))_n$ is convergent and $f(\lim_{n\to\infty} x_n) = f(\lim_{n\to\infty} f(x_n))$.*

18.4 Sequential Compactness

Theorem 18.4.1 *A metric space is compact if and only if it is sequentially compact.*

The converse fails:

Example.

18.5 Supplements

18.5.1 T_0-Identification

A topological space X is called T_0 if for any distinct x, $y \in X$ there is an open subset that contains only one of the two points.

Lemma 18.5.1 *A topological space X id T_0 if and only if for any two distinct x and y in X, $\overline{\{x\}} \neq \overline{\{x\}}$.*

Proof: Left as an exercise. $\square$

In a topological space define $x \sim y$ if and only if $\overline{\{x\}} = \overline{\{x\}}$. This is an equivalence relation. On $X/\sim$ put the largest topology that makes the projection map $X \longrightarrow X/\sim$ continuous.

Proposition 18.5.2 *$X/\sim$ is a T_0-topological space.*

Proof: HW. $\square$

Problem 1. Find the universal property of $X/\sim$ that characterizes it.
Problem 2. Try to do the same with T_1-spaces.

Chapter 19

Exams

19.1 Midterm Math 121 (November 2002)

i. Which of the following are not vector spaces over $\mathbb{R}$ (with the componentwise addition and scalar multiplication) and why?

$$\begin{array}{l} V_1 = \{(x,y,z) \in \mathbb{R}^3 : xy \geq 0\} \\ V_2 = \{(x,y,z) \in \mathbb{R}^3 : 3x - 2y + z = 0\} \\ V_3 = \{(x,y,z) \in \mathbb{R}^3 : xyz \in \mathbb{Q}\} \\ V_4 = \{(x,y) \in \mathbb{R}^3 : x + y \geq 0\} \\ V_5 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 0\} \\ V_6 = \{(x,y) \in \mathbb{C}^2 : x^2 + y^2 = 0\} \end{array}$$

(2+2+2+2+2+5 pts.)

Answers. V_1 is not a vector space because e.g. $(-1,0,0) \in V_1$, $(0,-1,0) \in V_2$ but their sum $(-1,-1,0) \notin V_1$.

V_2 is a vector space.

V_3 is not a vector space because e.g. $(1,1,1) \in V_3$, but $\sqrt{2}(1,1,1) \notin V_3$.

V_4 is not a vector space because e.g. $(1,2,1) \in V_4$, but $-(1,2,1) \notin V_4$.

V_5 is a vector space because $V_5 = \{(0,0,0)\}$.

V_6 is not a vector space because e.g. $(1,i) \in V_6$, $(1,-i) \in V_6$, but their sum $(1,0) \notin V_6$.

ii. On the set $X = \{2, 3 \ldots, 100\}$ define the relation $x \prec y$ by "$x \neq y$ and x divides y".

a) Show that this defines a partial order on X. (3 pts.)

b) Is this a linear order? (2 pts.)

c) Find all the maximal and minimal elements of this poset. (5 pts.)

Answers. a) Yes, this is a partial order: Clearly $x \not\prec x$ for any x. Since division is transitive, $\prec$ is transitive as well. (The details are left).

b) No, because 2 and 3 are not comparable.

c) The prime numbers are minimal elements. The maximal elements are the numbers which are greater than 50. For example 53 is both minimal and maximal.

iii. On $\mathbb{R} \times \mathbb{R}$ define the relation $\prec$ as follows $(x, y) \prec (x_1, y_1)$ by "either $y < y_1$, or $y = y_1$ and $x < x_1$".

a) Show that this is a linear order. (5 pts.)

b) Does every subset of this linear order which has an upper bound has a least upper bound? (5 pts.)

Answers. a) Yes!

b) No. For example the set $\mathbb{R} \times \{0\}$ is bounded above by $(0, 1)$ but it does not have a least upper bound.

iv. For each $n \in \mathbb{N}$, let a_n and b_n be two real numbers. Assume that for each n, $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$. Show that $\cap_{n\in\mathbb{N}}[a_n, b_n] = [a, b]$ for some real numbers a and b. (10 pts.)

Proof: Since the set $\{a_n : n \in \mathbb{N}\}$ is bounded above by b_0, it has a least upper bound, say a. Similarly the set $\{b_n : n \in \mathbb{N}\}$ has a greatest lower bound, say b. I claim that $\cap_{n\in\mathbb{N}}[a_n, b_n] = [a, b]$.

If $x \geq a$, then $x \geq a_n$ for all n. Likewise, if $x \leq b$, then $x \leq b_n$ for all n. Hence, if $x \in [a, b]$, the $x \in [a_n, b_n]$ for all n.

Conversely, let $x \in \cap_{n\in\mathbb{N}}[a_n, b_n]$. Then $a_n \leq x \leq b_n$ for all n. Thus x is an upper bound for $\{a_n : n \in \mathbb{N}\}$ and a lower bound for $\{b_n : n \in \mathbb{N}\}$. Hence $a \leq x \leq b$.

v. Show that for any natural number n and for any real number $x \in [0, 1)$,

$$(1 - x)^n \leq 1 - nx + \frac{n(n-1)}{2}x^2.$$

(10 pts.)

Proof: We proceed by induction on n.

If $n = 0$, then both sides are equal to 1.

Suppose we know the result for n, i.e. suppose we know that for any real number $x \in [0, 1)$,

$$(1 - x)^n \leq 1 - nx + \frac{n(n-1)}{2}x^2.$$

We will prove that for any real number $x \in [0, 1)$,

$$(1 - x)^{n+1} \leq 1 - (n+1)x + \frac{(n+1)n}{2}x^2.$$

Let $x \in [0,1)$. Since $(1-x)^n \leq 1 - nx + \frac{n(n-1)}{2}x^2$ and since $1 - x > 0$, multiplying by $1-x$ both sides we get $(1-x)^{n+1} = (1-x)^n(1-x) \leq (1 - nx + \frac{n(n-1)}{2}x^2)(1-x) = 1 - (n+1)x + \frac{(n+1)n}{2}x^2 - \frac{n(n-1)}{2}x^3$. Thus $(1-x)^{n+1} \leq 1-(n+1)x + \frac{(n+1)n}{2}x^2 - \frac{n(n-1)}{2}x^3$. Since $x \geq 0$, $(1-x)^{n+1} \leq 1 - (n+1)x + \frac{(n+1)n}{2}x^2 - \frac{n(n-1)}{2}x^3 \leq 1 - (n+1)x + \frac{(n+1)n}{2}x^2$.

vi. a) Show that for any complex number α there is a polynomial of the form $p(X) = X^2 + aX + b \in \mathbb{R}[X]$ such that $p(\alpha) = 0$. (Note: a and b should be real numbers). (10 pts.)

b) What can you say about a and b if $\alpha = u + iv$ for some u, $v \in \mathbb{Z}$? (5 pts.)

Proof: a) Let α be a complex number. Then $p(x) := (x-\alpha)(x-\overline{\alpha}) = x^2 - (\alpha + \overline{\alpha})x + \alpha\overline{\alpha} \in \mathbb{R}[x]$ and it is easy to check that $p(\alpha) = 0$.

b) It is clear that if $\alpha = u + iv$ for some u, $v \in \mathbb{Z}$, then $p(x) \in \mathbb{Z}[x]$.

Second Proof of part a. Write $\alpha = u + iv$ where u and v are real numbers. Then $\alpha^2 = u^2 - v^2 + 2uvi$. Thus $\alpha^2 - 2u\alpha = (u^2 - v^2 + 2uvi) - 2u(u + iv) = -u^2 - v^2$, so that $\alpha^2 - 2u\alpha + (u^2 + b^2) = 0$. Hence α is a root of the polynomial $p(x) = x^2 - 2ux + (u^2 + b^2) \in \mathbb{R}[x]$.

Part b follows from this immediately.

vii. a) Show that for any $\alpha \in \mathbb{C}$ there is a $\beta \in \mathbb{C}$ such that $\beta^2 = \alpha$. (15 pts.)

b) Show that for any α, $\beta \in \mathbb{C}$ there is an $x \in \mathbb{C}$ such that $x^2 + \alpha x + \beta = 0$. (10 pts.)

Proof: **a.** Let $\alpha = a + bi$. We try to find $\beta \in \mathbb{C}$ such that $\beta^2 = \alpha$, i.e. we try to find two real numbers x and y such that $(x + iy)^2 = a + bi$. We may assume that $\alpha \neq 0$ (otherwise take $\beta = 0$). Thus a and b cannot be both 0. After multiplying out, we see that this equation is equivalent to the system

$$\begin{array}{rcl} x^2 - y^2 & = & a \\ 2xy & = & b \end{array}$$

Since $x = 0$ implies $a = 0 = y = b$, x must be nonzero. Thus we have $y = b/2x$ and so the above system is equivalent to the following:

$$\begin{array}{rcl} x^2 - (b/2x)^2 & = & a \\ y & = & b/2x \end{array}$$

Equalizing the denominators in the first one, we get the following equivalent system:

$$\begin{array}{rcl} 4x^4 - 4ax^2 - b^2 & = & 0 \\ y & = & b/2x \end{array}$$

So now the problem is about the solvability of the first equation $4x^4 - 4ax^2 - b^2 = 0$. (Once we find x, which is necessarily nonzero, we set $y = b/2x$). Setting $z = x^2$, we see that the solvability of $4x^4 - 4ax^2 - b^2 = 0$

is equivalent to the question of whether $4z^2-4az-b^2=0$ has a nonnegative solution. Since the last one is a quadratic equation over $\mathbb{R}$, it is easy to answer this question. There are two possible solutions: $z = a \pm \sqrt{a^2+b^2}$ and one of them $z = a + \sqrt{a^2+b^2}$ is nonnegative (even if a is negative). Thus we can take

$$x = \sqrt{a + \sqrt{a^2+b^2}}$$

and

$$y = b/2x.$$

b. We first compute as follows: $0 = x^2+\alpha x+\beta = x^2+\alpha x+\alpha^2/4+(\beta-\alpha^2/4) = (x+\alpha/2)^2+(\beta-\alpha^2/4)$. Thus if $z \in \mathbb{C}$ is such that $z^2 = \alpha^2/4-\beta$ (by part a there is such a z), then $x := z-\alpha/2$ is a root of $x^2+\alpha x+\beta$.

viii. Suppose X and Y are two subsets of $\mathbb{R}$ that have least upper bounds. Show that the set $X+Y := \{x+y : x \in X,\ y \in Y\}$ has a least upper bound and that $\sup(X+Y) = \sup(X)+\sup(Y)$. (15 pts.)

Proof: Let a and b be the least upper bounds of X and Y respectively. Thus $x \le a$ for all $x \in X$ and $y \le b$ for all $y \in Y$. It follows that $x+y \le a+b$ for all $x \in X$ and $y \in Y$, meaning exactly that $a+b$ is an upper bound of $X+Y$. Now we show that $a+b$ is the least upper bound of $X+Y$. Let $\epsilon > 0$ be any. We need to show that $a+b-\epsilon < x+y$ for some $x \in X$ and $y \in Y$. Since a is the least upper bound of X, there is an $x \in X$ such that $a-\epsilon/2 < x$. Similarly there is a $y \in Y$ such that $b-\epsilon/2 < y$. Summing these two, we get $a+b-\epsilon < x+y$.

ix. We consider the subset $X = \{1/2^n : n \in \mathbb{N}\} \cup \{0\}$ of $\mathbb{R}$ together with the usual metric, i.e. for $x, y \in X$, $d(x,y)$ is defined to be $|x-y|$. Show that the open subsets of X are the cofinite subsets[1] of X and the ones that do not contain 0. (20 pts.)

Proof: We first show that the singleton set $\{1/2^n\}$ is an open subset of X. This is clear because $B(1/2^n, 1/2^{n+1}) = \{1/2^n\}$. It follows that any subset of X that does not contain 0 is open. Now let U be any cofinite subset of X. We proceed to show that U is open. If $0 \notin U$, then we are done by the preceding. Assume $0 \in U$. Since U is cofinite, there is a natural number $n_\circ$ such that for all $n \ge n_\circ$, $1/2^n \in U$, i.e. $B(0, 1/2^{n_\circ}) \subseteq U$. Now U is the union of $B(0, 1/2^{n_\circ})$ and of a finite subset not containing 0. Thus U is open.

For the converse, we first show that a nonempty open ball is of the form described in the statement of the question. If the center of the ball is 0, then the ball is cofinite by the Archimedean property. If the center of the ball is not 0, then either the ball does not contain 0 or else it does contain 0, in which case the ball must be cofinite.

To finish the proof, we must show that an arbitrary union of open balls each of which does not contain 0 cannot contain 0. But this is clear!

[1] A subset Y of X is called **cofinite** if $X \setminus Y$ is finite.

19.2 Final Math 121 (January 2003)

Justify all your answers. A nonjustified answer will not receive any grade whatsoever, even if the answer is correct. DO NOT use symbols such as $\forall$, $\exists$, $\Rightarrow$. Make full sentences with correct punctuation. You may write in Turkish or in English.

i. *Let $(a_n)_n$ be a convergent sequence of real numbers.*

a. *Does the sequence $(a_{2n})_n$ converge necessarily?* (2 pts.)

Since $(a_{2n})_n$ is a subsequence of the converging sequence $(a_n)_n$, both sequences converge to the same limit.

Remarks. We have seen in class that a subsequence of a converging sequence converges.

Contrary to what some of you think, $a_{2n} \neq 2a_n$!

b. *Assume $a_n \neq 0$ for all n. Does the sequence $(a_n/a_{n+1})_n$ converge necessarily?* (2 pts.)

No, the sequence $(a_n/a_{n+1})_n$ may not converge if $\lim_{n\to\infty} a_n = 0$. For example, choose

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1/n^2 & \text{if } n \text{ is odd} \end{cases}$$

Clearly $\lim_{n\to\infty} a_n = 0$, but

$$\frac{a_n}{a_{n+1}} = \begin{cases} \frac{(n+1)^2}{n} & \text{if } n \text{ is even} \\ \\ \frac{n+1}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

And the subsequence $\frac{(n+1)^2}{n}$ diverges to ∞, although the subsequence $\frac{n+1}{n^2}$ converges to 0.

On the other hand, if the limit of the sequence $(a_n)_n$ is nonzero, say ℓ, then the sequence $(a_n/a_{n+1})_n$ converges to 1 because $\lim_{n\to\infty} a_n/a_{n+1} = \lim_{n\to\infty} a_n/\lim_{n\to\infty} a_{n+1} = \ell/\ell = 1$. Note that the last part uses the fact that ℓ is nonzero.

ii. *Find the following limits and prove your result using only the definition.* (30 pts.)

a. $\lim_{n\to\infty} \frac{2n-5}{5n+2}$.

We claim that the limit is $2/5$. Let $\epsilon > 0$. Since $\mathbb{R}$ is Archimedean, there is an N such that $2 < \epsilon N$. Now for $n > N$, $|\frac{2n-5}{5n+2} - \frac{2}{5}| = \frac{29}{5(5n+2)} < \frac{29}{25n} < 2/n < 2/N < \epsilon$.

b. $\lim_{n\to\infty} \frac{2n^2-5}{-5n+2}$.

We claim that the limit is $-\infty$. For this it is enough to prove that $\lim_{n\to\infty} \frac{2n^2-5}{5n-2} = \infty$. Let A be any real number. Let $N = \max(5A, 2)$.

For $n > N$ we have $\frac{2n^2-5}{5n-2} > \frac{2n^2-5}{5n} > \frac{2n^2-n^2}{5n} = \frac{n^2}{5n} = \frac{n}{5} > \frac{N}{5} \geq A$. This proves that $\lim_{n\to\infty} \frac{2n^2-5}{5n-2} = \infty$.

c. $\lim_{n\to\infty} \frac{2n^2-5}{n^3+2}$.

We claim that the limit is 0. Let $\epsilon > 0$. Let N_1 be such that $2 > \epsilon N_1$ (Archimedean property of $\mathbb{R}$). Let $N = \max(1, N)$. Then for all $n > N$,

$$\left|\frac{2n^2-5}{n^3+2}\right| = \frac{2n^2-5}{n^3+2} < \frac{2n^2}{n^3+2} \leq \frac{2n^2}{n^3} = \frac{2}{n} < \frac{2}{N} < \epsilon.$$

Note that the first equality is valid because $n^3 - 5 \geq 0$, the second inequality is valid because $n^3 + 2 > 0$.

iii. *Find* (16 pts. Justify your answers).

a. $\lim_{n\to\infty}(1/2 + 1/n)^n$.

Note that for $n \geq 3$, $0 < 1/2+1/n \leq 1/2+1/3 = 5/6$. Thus the sequence $((1/2+1/n)^n)_n$ is eventually squeezed between the zero constant sequence and the sequence $((5/6)^n)_n$. Since $\lim_{n\to\infty}(5/6)^n = 0$ (because $5/6 < 1$), $\lim_{n\to\infty}(1/2 + 1/n)^n = 0$.

b. $\lim_{n\to\infty}(3/2 - 7/n)^n$.

Since $3/2 > 1$ and $\lim_{n\to\infty} 7/n = 0$, there is an N such that $7/N < 1/2 = 3/2 - 1$. In fact, it is enough to take $N = 15$. Then for all $n \geq N$, $3/2-7/n \geq 3/2-7/N > 1$ and so $(3/2-7/n)^n \geq (3/2-7/N)^n$. Therefore the sequence $((3/2-7/n)^n)_n$ is greater than the sequence $((3/2-7/N)^n)_n$. Since $3/2 - 7/N > 1$, the sequence $((3/2 - 7/N)^n)_n$ diverge to ∞. Hence $\lim_{n\to\infty}(3/2 - 7/n)^n = \infty$.

iv. *Find* $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}}$. (10 pts. Justify your answer).

Assume $n > 5$. Then we have, $n^3-n-5 > n^3-2n > n^3-n^3/2 = n^3/2 > 0$. Therefore,

$$\left|\frac{n^2-1}{n^3-n-5}\right| = \frac{n^2-1}{n^3-n-5} < \frac{n^2}{n^3-n-5} < \frac{n^2}{n^3/2} = 2/n.$$

Also $\frac{n^2-1}{2n-3} > \frac{n^2-1}{2n} > \frac{n^2-n}{2n} = \frac{n-1}{2} > 2$. Hence

$$\left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} < (2/n)^2.$$

It follows that $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$.

v. *Show that the series* $\sum_{n=1}^{\infty} (1/n)^n$ *converges. Find an upper bound for the sum.* (10 pts.)

Since for $n \geq 2$, $1/n \leq 1/2$, we have $\sum_{n=1}^{\infty}(1/n)^n \leq 1 + \sum_{n=2}^{\infty}(1/n)^n \leq 1 + \sum_{n=1}^{\infty}(1/2)^n = 1 + \frac{1}{2}\sum_{n=0}^{\infty}(1/2)^n = 1 + 1/2 = 3/2$.

vi. *Let $(a_n)_n$ be a sequence of nonnegative real numbers. Suppose that the sequence $(a_n^2)_n$ converges to a. Show that the sequence $(a_n)_n$ converges to $\sqrt{a}$.* (15 pts.)

Note first that, since $a_n \geq 0$, $a \geq 0$ as well.

Let $\epsilon > 0$.

Case 1: $a > 0$**.** Since $\lim_{n\to\infty} a_n^2 = a$, there is an N_2 such that for all $n > N$, $|a_n^2 - a| < \epsilon a$. Now for all $n > N$, $|a_n - \sqrt{a}| = \frac{|a_n^2 - a|}{a_n + a} \leq \frac{|a_n^2 - a|}{a} < \epsilon$.

Case 2. $a = 0$**.**

Since the sequence $(a_n^2)_n$ converges to 0, there is an N such that for all $n > N$, $a_n^2 < \epsilon^2$. So $(\epsilon + a_n)(\epsilon - a_n) = \epsilon^2 - a_n^2 > 0$. Since $\epsilon > 0$ and $a_n \geq 0$, we can divide both sides by $\epsilon + a_n$ to get $\epsilon - a_n > 0$, i.e. $a_n < \epsilon$. Since $a_n \geq 0$, this implies $|a_n| < \epsilon$.

vii. *We have seen in class that the sequence given by $a_n = ((1 + 1/n)^n)_n$ converges to a real number > 1. Let e be this limit. Do the following sequences converge? If so find their limit.* (15 pts. Justify your answers).

a) $\lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^n$.

$\lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^n = \lim_{n\to\infty} \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{1+\frac{1}{n+1}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to\infty} 1+\frac{1}{n+1}} = \frac{e}{1} = e$. The first equality is algebraic. The second equality holds because the limits of the numerator and the denominator exist and they are nonzero. The third equality holds because $\left(\left(1 + \frac{1}{n+1}\right)^{n+1}\right)_n$ is a subsequence of $\left(\left(1 + \frac{1}{n}\right)^n\right)_n$.

b) $\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{3n}$.

$\lim_{n\to\infty} \left(1 + \frac{1}{2n}\right)^{3n} = \lim_{n\to\infty} \left(\left(1 + \frac{1}{2n}\right)^{2n}\right)^{3/2} = e^{3/2}$ by Question vi.

c) $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{n^2}$.

Since $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e > 1$, there is an N_1 such that for all $n > N_1$, $|\left(1 + \frac{1}{n}\right)^n - e| < \frac{e-1}{2}$, so $1 < r := \frac{e}{2} + \frac{1}{2} = e - \frac{e-1}{2} < \left(1 + \frac{1}{n}\right)^n$. Thus, $r^n < \left(\left(1 + \frac{1}{n}\right)^n\right)^n = \left(1 + \frac{1}{n}\right)^{n^2}$. Thus $\infty = \lim_{r\longrightarrow\infty} r^n \leq \left(1 + \frac{1}{n}\right)^{n^2}$ (because $r > 1$). It follows that $\left(1 + \frac{1}{n}\right)^{n^2} = \infty$.

19.3 Resit of Math 121, February 2003

i. Find a sequence neither decreasing nor increasing that converges to 1. (2 pts.)

ii. Let $(a_n)_n$ be a convergent sequence of real numbers. Suppose that $a_n \in \mathbb{Z}$ for all n. Is it true that $\lim_{n\to\infty} a_n \in \mathbb{Z}$? (4 pts.)

iii. Let $(a_n)_n$ be a convergent sequence of real numbers. Suppose that $a_n \in \mathbb{Q}$ for all n. Is it true that $\lim_{n\to\infty} a_n \in \mathbb{Q}$? (3 pts.)

iv. Let $(a_n)_n$ be a convergent sequence of real numbers. Suppose that $5a_n/2 \in \mathbb{N}$ for all n. What can you say about $\lim_{n\to\infty} a_n$? (2 pts.)

v. Let $(a_n)_n$ be a sequence of real numbers such that the subsequence $(a_{2n})_n$ converges. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

vi. Let $(a_n)_n$ be a sequence of real numbers such that the sequence $(a_n^2)_n$ converges to 1. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

vii. Let $(a_n)_n$ be a sequence of real numbers such that the subsequences $(a_{2n})_n$ and $(a_{2n+1})_n$ both converge. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

viii. Let $(a_n)_n$ be a sequence of real numbers such that the sequence $(a_n^2)_n$ converges to 0. Does the sequence $(a_n)_n$ converge necessarily? (8 pts.)

ix. Let $(a_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. Is it true that $\lim_{n\to} a_{2n} = \infty$? (3 pts.)

x. Assume $\lim_{n\to\infty} a_n$ exists and $a_n \neq 0$ for all n. Does the sequence $(a_{2n}/a_{2n+1})_n$ converge necessarily? (5 pts.)

xi. Find the following limits and prove your result using **only** the definition. (30 pts.)

a. $\lim_{n\to\infty} \frac{3n+105}{5n-79}$

b. $\lim_{n\to\infty} \frac{n^2-5n+3}{-100n+2}$

c. $\lim_{n\to\infty} \frac{n-8}{2n^3-89}$

xii. Find (16 pts. Justify your answers).

a. $\lim_{n\to\infty} \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n}$

b. $\lim_{n\to\infty} \left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n}$

xiii. Find $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}}$. (10 pts. Justify your answer).

xiv. Show that the series $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n/3}$ converges. Find an upper bound for the sum. (10 pts.)

xv. Let $(a_n)_n$ be a sequence of real numbers. Assume that there is an $r > 1$ such that $|a_{n+1}| \geq r|a_n|$ for all n. What can you say about the convergence or the divergence of $(a_n)_n$? (6 pts.)

19.4 Correction of the Resit of Math 121, February 2003

i. Find a sequence neither decreasing nor increasing that converges to 1. (2 pts.)

   Answer: Let $a_n = 1 + \frac{(-1)^n}{n}$. It is clear that $\lim_{n\to\infty} a_n = 1$. Since the subsequence $(a_{2n})_n$ is decreasing and converges to 1 and the subsequence $(a_{2n})_n$ is increasing and converges to 1, the sequence (a_n) is neither increasing nor decreasing.

ii. Let $(a_n)_n$ be a convergent sequence of real numbers. Suppose that $a_n \in \mathbb{Z}$ for all n. Is it true that $\lim_{n\to\infty} a_n \in \mathbb{Z}$? (4 pts.)

   Answer: Yes, it is true. In fact this is true even for Cauchy sequences: A Cauchy sequence $(a_n)_n$ whose terms are in $\mathbb{Z}$ is eventually constant, i.e. there is an N such that $a_n = a_N$ for all $n \geq N$, and this implies of course that $\lim_{n\to\infty} a_n = a_N \in \mathbb{Z}$. So, let us show that the Cauchy sequence $(a_n)_n$ is eventually constant.

   In the definition of Cauchy sequences, take $\epsilon = 1/2$. Thus, there is an M such that for all n, $m > M$, $|a_n - a_m| < 1/2$. But since a_n and a_m are in $\mathbb{Z}$, this means that for all n, $m > M$, $|a_n - a_m| = 0$, i.e. that $a_n = a_m$. Now take $N = M + 1$.

iii. Let $(q_n)_n$ be a convergent sequence of real numbers. Suppose that $q_n \in \mathbb{Q}$ for all n. Is it true that $\lim_{n\to\infty} q_n \in \mathbb{Q}$? (3 pts.)

   Answer: Of course not! In fact every real number is the limit of a rational sequence. Indeed, let $r \in \mathbb{R}$. Let $n \in \mathbb{N} \setminus \{0\}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational number $q_n \in (r - 1/n, r)$. Since $r - 1/n < q_n < r$, by the Sandwich Lemma, $\lim_{n\to\infty} q_n = r$.

iv. Let $(a_n)_n$ be a convergent sequence of real numbers. Suppose that $5a_n/2 \in \mathbb{N}$ for all n. What can you say about $\lim_{n\to\infty} a_n$? (4 pts.)

   Answer: Let $\lim_{n\to\infty} a_n = r$. Then $\lim_{n\to\infty} 5a_n/2 = 5r/2$. By hypothesis and by part ii, $5r/2 \in \mathbb{Z}$. Thus $r = 2n/5$ for some $n \in \mathbb{N}$.

v. Let $(a_n)_n$ be a sequence of real numbers such that the subsequence $(a_{2n})_n$ converges. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

   Answer: Of course not! We can have $a_{2n} = 1/n$ and $a_{2n+1} = n$.

vi. Let $(a_n)_n$ be a sequence of real numbers such that the sequence $(a_n^2)_n$ converges to 1. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

   Answer: Of course not! We can have $a_n = (-1)^n$. Then $(a_n)_n$ is a sequence of alternating ones and minus ones, so that it diverges. And since $a_n^2 = 1$, the sequence $(a_n^2)_n$ converges to 1.

vii. Let $(a_n)_n$ be a sequence of real numbers such that the subsequences $(a_{2n})_n$ and $(a_{2n+1})_n$ both converge. Does the sequence $(a_n)_n$ converge necessarily? (2 pts.)

Answer: Of course not! We can have $a_n = (-1)^n$. Then $(a_n)_n$ is a sequence of alternating ones and minus ones, so that it diverges. And since $a_{2n} = 1$ and $a_{2n+1} = -1$, the sequence $(a_{2n})_n$ converges to 1 and the sequence $(a_{2n+1})_n$ converges to -1.

viii. Let $(a_n)_n$ be a sequence of real numbers such that the sequence $(a_n^2)_n$ converges to 0. Does the sequence $(a_n)_n$ converge necessarily? (8 pts.)

Answer: Yes! Let $\epsilon > 0$. Let $\nu = \sqrt{\epsilon}$. Since the sequence $(a_n^2)_n$ converges to 0, there is an N such that for all $n > N$, $|a_n^2| < \nu$, i.e. $|a_n|^2 < \epsilon^2$. Since $|a_n|$ and ν are positive, this implies that $|a_n| < \epsilon$. Thus there is an N such that for all $n > N$, $|a_n| < \epsilon$; i.e. the sequence $(a_n)_n$ converges to 0.

ix. Let $(a_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. Is it true that $\lim_{n\to} a_{2n} = \infty$? (3 pts.)

Answer: Yes! Let A be any real number. $\lim_{n\to\infty} a_n = \infty$, there is an N such that for all $n > N$, $a_n > A$. Then for $2n > N$, $a_{2n} > A$.

x. Assume $\lim_{n\to\infty} a_n$ exists and $a_n \neq 0$ for all n. Does the sequence $(a_{2n}/a_{2n+1})_n$ converge necessarily? (5 pts.)

Answer: No, the sequence $(a_{2n}/a_{2n+1})_n$ may not converge if $\lim_{n\to\infty} a_n = 0$. For example, choose

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 1/n^2 & \text{if } n \text{ is odd} \end{cases}$$

Clearly $\lim_{n\to\infty} a_n = 0$, but

$$\frac{a_n}{a_{n+1}} = \begin{cases} \frac{(n+1)^2}{n} & \text{if } n \text{ is even} \\ \\ \frac{n+1}{n^2} & \text{if } n \text{ is odd} \end{cases}$$

And the subsequence $\frac{(n+1)^2}{n}$ diverges to ∞, although the subsequence $\frac{n+1}{n^2}$ converges to 0.

On the other hand, if the limit of the sequence $(a_n)_n$ is nonzero, say ℓ, then the sequence $(a_{2n}/a_{2n+1})_n$ converges to 1 because $\lim_{n\to\infty} a_{2n}/a_{2n+1} = \lim_{n\to\infty} a_{2n}/\lim_{n\to\infty} a_{2n+1} = \ell/\ell = 1$. Note that the last part uses the fact that ℓ is nonzero.

xi. Find the following limits and prove your result using **only** the definition. (30 pts.)

a. $\lim_{n\to\infty} \frac{3n+105}{5n-79}$

Answer: $\lim_{n\to\infty} \frac{3n+105}{5n-79} = \frac{3}{5}$.

Proof: Let $\epsilon > 0$. Let N_1 be such that $32 < \epsilon N_1$. Let $N = \max(N_1, 395)$. Now for $n > N$, we have,

$$\left|\frac{3n+105}{5n-79} - \frac{3}{5}\right| = \left|\frac{762}{25n-395}\right| = \frac{762}{25n-395} \leq \frac{762}{24n} < \frac{32}{n} < \frac{32}{N_1} < \epsilon.$$

The first equality is simple computation. The second equality follows from the fact $n > N \geq 395 > 16$ (so that $25n - 395 > 0$). The third inequality follows from the fact that $n > N \geq 395$, so that $25n - 395 \geq 25n - n = 24n$.The fourth inequality is also a simple computation.

b. $\lim_{n\to\infty} \frac{n^2-5n+3}{-100n+2}$

Answer: $\lim_{n\to\infty} \frac{n^2-5n+3}{-100n+2} = -\infty$.

Proof: It is enough to show that $\lim_{n\to\infty} \frac{n^2-5n+3}{100n-2} = \infty$.

We first note that the two roots of $n^2 - 5n + 3$ are $\frac{5\pm\sqrt{25-12}}{2} = \frac{5\pm\sqrt{13}}{2}$, so that if $n \geq 5 > \frac{9}{2} = \frac{5+\sqrt{16}}{2} > \frac{5+\sqrt{13}}{2}$, then $n^2 - 5n + 3 > 0$.

Now let $A \in \mathbb{R}$ be any real number. Let $N = \max(100A + 5, 5)$. Now, for all $n > N$,

$$\frac{n^2-5n+3}{100n-2} > \frac{n^2-5n+3}{100n} > \frac{n^2-5n}{100n} = \frac{n-5}{100} > \frac{N-5}{100} = A.$$

Here, the first inequality follows from the fact that $n > N \geq 5$, so that $n^2 - 5n + 3 > 0$.

c. $\lim_{n\to\infty} \frac{n-8}{2n^3-89}$.

Answer: $\lim_{n\to\infty} \frac{n-8}{2n^3-89} = 0$.

Proof: Let $\epsilon > 0$. Let $N = \max(1/\epsilon, 89)$. Now for $n > N$, $\left|\frac{n-8}{2n^3-89}\right| = \frac{n-8}{2n^3-89} < \frac{n}{2n^3-n} = \frac{1}{2n^2-1} < \frac{1}{n^2} < \frac{1}{n} < \epsilon$.

xii. Find (16 pts. Justify your answers).

a. $\lim_{n\to\infty} \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n}$.

Answer: $\lim_{n\to\infty} \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n} = 0$.

Proof: We use the fact that $2/3 < 1$. Since $\lim_{n\to\infty} \frac{6n}{n^2+1} = 0$, there is an N such that for all $n > N$, $\frac{6n}{n^2+1} < 1/6$. Then $0 \leq \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n} = (2/3+1/6)^{3n} = (5/6)^{3n}$. By Sandwich Lemma $\lim_{n\to\infty} \left(\frac{2}{3} + \frac{6n}{n^2+1}\right)^{3n} = 0$.

b. $\lim_{n\to\infty} \left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n}$.

Answer: $\lim_{n\to\infty} \left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n} = \infty$.

Proof: We use the fact that $5/4 > 1$. Since $\lim_{n\to\infty} \frac{7}{n^5} = 0$, there is an N such that for all $n > N$, $\frac{7}{n^5} < 1/8$. Then $\left(\frac{5}{4} - \frac{7}{n^5}\right)^{n^n} > (5/4 - 1/8)^{n^n} = (9/8)^{n^n} \geq (9/8)^n$. Since $(9/8) > 1$, $\lim_{n\to\infty}(9/8)^n = \infty$. The result follows.

xiii. Find $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}}$. (10 pts.).

Answer: $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$.

Proof: Since $\lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right) = 0$, there is an N_1 such that for all $n > N_1$, $\frac{n^2-1}{n^3-n-5} < 1/2$. On the other hand, for $n > 3$, $\frac{n^2-1}{2n-3} < \frac{n^2-1}{n} < n$. Let $N = \max(3, N_1)$. Now for $n > N$, $\left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} < (1/2)^{\frac{n^2-1}{2n-3}} < (1/2)^n$. Since the right hand side converges to 0, by Sandwich Lemma, $0 \leq \lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}} = 0$. (For the first inequality, one needs the fact that $n^3 - n - 5 > 0$ for $n \geq 2$. This follows from the facts that $2^3 - 2 - 5 = 1 > 0$ and $n^3 - n - 5 < (n+1)^3 - (n+1) - 5$. And this last inequality is easy to show).

xiv. Show that the series $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n/3}$ converges. Find an upper bound for the sum. (10 pts.)

Answer: $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n/3} = \sum_{n=1}^{\infty}(1/n)^{n/3} < \sum_{n=1}^{\infty} 1/2^{n/3} =$
$\sum_{n=1}^{\infty} 1/2^{3n/3} + \sum_{n=0}^{\infty} 1/2^{\frac{3n+1}{3}} + \sum_{n=0}^{\infty} 1/2^{\frac{3n+2}{3}}$
$= \sum_{n=1}^{\infty} 1/2^n + \frac{1}{2^{1/3}} \sum_{n=0}^{\infty} 1/2^n + \frac{1}{2^{1/3}} \sum_{n=0}^{\infty} 1/2^n = 1/2 + 2^{-1/3} + 2^{-2/3} < 5$.

xv. Let $(a_n)_n$ be a sequence of real numbers. Assume that there is an $r > 1$ such that $|a_{n+1}| \geq r|a_n|$ for all n. What can you say about the convergence or the divergence of $(a_n)_n$? (6 pts.)

Answer: The sequence diverges. Furthermore the sequence diverges to ∞ if it is eventually positive and to $-\infty$ if it is eventually negative.

Proof: One can show by induction on n that $|a_n| > r^n|a_0|$. Thus $\lim_{n\to} |a_n| = \infty$ (because $r > 1$). It should now be clear that the answer is valid.

19.5 Second Resit of Math 121, March 2003

i. Let $(a_n)_n$ be a converging sequence of nonzero real numbers. Do the sequences $\left(\frac{a_n}{a_{n+1}}\right)_n$, $\left(\frac{1-a_n}{1-a_{n+1}}\right)_n$ and $\left(\frac{a_n}{1+a_{n+1}^2}\right)_n$ converge? (6 pts.)

ii. Let $(a_n)_n$ be a convergent sequence of natural numbers. Is it true that $\lim_{n\to\infty} a_n \in \mathbb{N}$? (4 pts.)

iii. Let $x > 1$. Discuss the convergence of $(x^n/n!)_n$ (4 pts.)

iv. Let $x \in \mathbb{R}$. Discuss the convergence of $(x^{n!}/n!)_n$ (8 pts.)

v. Let $(a_n)_n$ be a convergent sequence of real numbers. Suppose that $\lim_{n\to\infty} a_n \in \mathbb{Q}$. Is it true that $a_n \in \mathbb{Q}$ for infinitely many n? (3 pts.)

vi. Let $(a_n)_n$ be a sequence of real numbers such that the subsequences $(a_{2n})_n$, $(a_{2n+1})_n$ and $(a_{7n+1})_n$ all converge. Does the sequence $(a_n)_n$ converge necessarily? (5 pts.)

vii. Let $(a_n)_n$ be a sequence of real numbers such that the sequence $(a_n^2)_n$ converges. Discuss the convergence of the sequence $(a_n)_n$. (5 pts.)

viii. Let $(a_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. Is it true that $\lim_{n\to\infty} a_{2n} = \infty$? (2 pts.)

ix. Let $(a_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} a_n = \infty$. Discuss the convergence of $\lim_{n\to\infty} 1/a_{2n} = 0$. (10 pts.)

x. Find the following limits and prove your result using **only** the definition. (18 pts.)

   a. $lim_{n\to\infty} \frac{3n^2-5}{5n^2+3n}$.

   b. $lim_{n\to\infty} \frac{n^2-5n+3}{-4n+2}$

   c. $lim_{n\to\infty} \frac{5n^2-4}{2n^3+2}$

xi. Find $lim_{n\to\infty} \left(\frac{n^2-1}{n^3-n-5}\right)^{\frac{n^2-1}{2n-3}}$. (10 pts. Justify your answer).

xii. Show that the series $\sum_{n=0}^{\infty}(-1)^n z^{2n+1}/(2n+1)!$ converges for all $z \in \mathbb{R}$. (4 pts.)

xiii. Show that the series $\sum_{n=1}^{\infty} \left(\frac{n}{n^2+1}\right)^{n/3}$ converges. Find an upper bound for the sum. (10 pts.)

xiv. Let $(a_n)_n$ be a sequence of real numbers. Assume that there is an $r > 1$ such that $|a_{n+1}| \geq r|a_n|$ for all n. What can you say about the convergence or the divergence of $(a_n)_n$? (6 pts.)

19.6 Midterm of Math 152, April 2004

i. Decide the convergence of the series

$$\sum_n \frac{1}{\sqrt{|n^2-2|}}.$$

ii. Decide the convergence of the series

$$\sum_n \frac{1}{\sqrt{n^2+1}}.$$

iii. Decide the convergence of the series

$$\sum_n \frac{1}{\sqrt{|n^4-6|}}.$$

iv. Suppose that the series $\sum_n a_n$ is convergent. Show that $\lim_{n\to\infty} a_n = 0$.

v. Suppose that $(a_n)_n$ is a positive and decreasing sequence and that the series $\sum_n a_n$ is convergent. Show that $\lim_{n\to\infty} na_n = 0$.

vi. Find a positive sequence $(a_n)_n$ such that the series $\sum_n a_n$ is convergent but that $\lim_{n\to\infty} na_n \neq 0$.

vii. Suppose that series $\sum_n a_n$ is absolutely convergent and that the sequence $(b_n)_n$ is Cauchy. Show that the series $\sum_n a_n b_n$ is absolutely convergent.

viii. Let $(a_n)_n$ be a sequence. Suppose that $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ converges. Such a sequence is called of **bounded variation**. Show that a sequence of bounded variation converges.

19.7 Final of Math 152, June 2004

I. Convergent Sequences. For each of the topological spaces (X, τ), describe the convergent sequences and discuss the uniqueness of their limits.

i. $\tau = \wp(X)$. ($\wp(X)$ is the set of all subsets of X, 2 pts.).

Answer: Only the eventually constant sequences converging to that constant.

ii. $\tau = \{\emptyset, X\}$ (2 pts.).

Answer: All sequences converge to all elements.

iii. $a \in X$ is a fixed element and τ is the set of all subsets of X that do not contain a, together with X of course. (5 pts.)

Answer: First of all, all sequences converge to a. Second: If a sequence converges to $b \neq a$, then the sequence must be eventually the constant b.

iv. $a \in X$ is a fixed element and τ is the set of all subsets of X that contain a, together with $\emptyset$ of course. (5 pts.)

Answer: Only the eventually constant sequences converge to a. A sequence converge to $b \neq a$ if and only if the sequence eventually takes only the two values a and b.

v. τ is the set of all cofinite subsets of X, together with the $\emptyset$ of course. (6 pts.)

Answer: All the sequences without infinitely repeating terms converge to all elements. Eventually constant sequences converge to the constant. There are no others.

II. Subgroup Topology on $\mathbb{Z}$. Let $\tau = \{n\mathbb{Z}+m : n, m \in \mathbb{Z}, n \neq 0\} \cup \{\emptyset\}$. We know that $(\mathbb{Z}, \tau)$ is a topological space.

i. Let $a \in \mathbb{Z}$. Is $\mathbb{Z} \setminus \{a\}$ open in τ? (5 pts.)

Answer: Yes. $\cup_{n\neq 0,\pm 1} n\mathbb{Z} \cup (3\mathbb{Z}+2) = \mathbb{Z} \setminus \{1\}$.Translating this set by $a-1$, we see that $\mathbb{Z} \setminus \{a\}$ is open.

ii. Find an infinite non open subset of $\mathbb{Z}$. (5 pts.)

Answer: The set of primes is not an open subset. Because otherwise, for some $a \neq 0$ and $b \in \mathbb{Z}$, the elements of $a\mathbb{Z}+b$ would all be primes. So b, $ab+b$ and $2ab+b$ would be primes, a contradiction.

iii. Let $a, b \in \mathbb{Z}$. Is the map $f_{a,b} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f_{a,b}(z) = az + b$ continuous? (Prove or disprove). (10 pts.)

Answer: Translation by b is easily shown to be continuous. Let us consider the map $f(z) = az$. If $a = 0, 1, -1$ then clearly f is continuous. Assume $a \neq 0, \pm 1$ and that f is continuous. We may assume that $a > 1$ (why?) Choose a b which is not divisible by a. Then $f^{-1}(b\mathbb{Z})$ is open, hence contains a subset of the form $c\mathbb{Z}+d$. Therefore $a(c\mathbb{Z}+d) \subseteq b\mathbb{Z}$. Therefore $ac = \pm b$ and so a divides b, a contradiction. Hence f is not continuous unless $a = 0, \pm 1$.

iv. Is the map $f_{a,b} : \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f(z) = z^2$ continuous? (Prove or disprove). (5 pts.)

Answer: No! Left as an exercise.

v. Is the topological space $(\mathbb{Z}, \tau)$ compact? (Prove or disprove).(15 pts.)

Answer: First Proof: Note first the complement of open subsets of the form $a\mathbb{Z}+b$ are also open as they are unions of the form $a\mathbb{Z}+c$ for $c = 0, 1, \ldots, a-1$ and $c \not\equiv b \bmod a$. Now consider sets of the form $U_p = p\mathbb{Z} + (p-1)/2$ for p an odd prime. Then $\cap_p U_p = \emptyset$ because if $a \in \cap_p U_p$ then for some $x \in \mathbb{Z} \setminus \{-1\}$, $2a+1 = px+p$, so that a is divisible by all primes p and $a = 0$. But if $a = 0$ then $(p-1)/2$ is divisible by p, a contradiction. On the other hand no finite intersection of the U_p's can be emptyset as $(a\mathbb{Z}+b)\cap(c\mathbb{Z}+b)\neq\emptyset$ if a and b are prime to each other (why?) Hence $(U_p^c)_p$ is an open cover of $\mathbb{Z}$ that does not have a finite cover. Therefore $\mathbb{Z}$ is not compact.

First Proof: Let p be a prime and $a = a_0 + a_1p + a_2p^2 + \ldots$ be a p-adic integer which is not in $\mathbb{Z}$. Let

$$b_n = a_0 + a_1p + a_2p^2 + \ldots + a_{n-1}p^{n-1}.$$

Then $\cap_n p^n\mathbb{Z} + b_n = \emptyset$ but no finite intersection is empty. We conclude as above.

III. Miscellaneous.

i. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the squaring map. Suppose that the arrival set is endowed with the usual Euclidean topology. Find the smallest topology on the domain that makes f continuous. (5 pts.)

   Answer: The smallest such topology is the set

   $$\{U \cap -U : U \text{ open in the usual topology of } \mathbb{R}\}.$$

ii. Let τ be the topology on $\mathbb{R}$ generated by $\{[a,b) : a,\, b \in \mathbb{R}\}$. Compare this topology with the Euclidean topology. (3 pts.) Is this topology generated by a metric? (20 pts.)

   Answer: Any open subset of the Euclidean topology is open in this topology because $(a,b) = \cup_{n=1}^{\infty}[a+1/n, b)$. But of course $[0,1)$ is not open in the usual topology.

   Assume a metric generates the topology. Note that $[0,\infty)$ is open as it is the union of open sets of the form $[0,n)$ for $n \in \mathbb{N}$. Thus the sequence $(-1/n)_n$ cannot converge to 0. In fact for any $b \in \mathbb{R}$, no sequence can converge to b from the left. Thus for any $b \in \mathbb{R}$ there is an $\epsilon_b > 0$ such that $B(b,\epsilon_b) \subseteq [b,\infty)$. Let b_0 be any point of $\mathbb{R}$. Let $\epsilon_0 > 0$ be such that $B(b_0,\epsilon_0) \subseteq [b_0,\infty)$. Since $\{b_0\}$ is not open, there is $b_1 \in B(b_0,\epsilon_0) \setminus \{b_0\}$. Let $0 < \epsilon_1 < \epsilon_0/2$ be such that $B(b_1,\epsilon_1) \subseteq [b_1,\infty) \cap B(b_0,\epsilon_0)$. Inductively we can find $(b_n)_n$ and $(\epsilon_n)_n$ such that $B(b_n,\epsilon_n) \subseteq [b_n,\infty) \cap B(b_{n-1},\epsilon_{n-1}) \setminus \{b_{n-1}\}$ and $\epsilon_n < \epsilon_0/2^n$. Then $(b_n)_n$ is a strictly increasing convergent sequence, a contradiction.

iii. Show that the series $\sum_{i=0}^{n} x^n/n!$ converges for any $x \in \mathbb{R}$ (10 pts.) Show that the map $\exp : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\exp(x) = \sum_{i=0}^{n} x^n/n!$ is continuous. (10 pts.)

   Answer: For the first part show that the sequence of partial sums is Cauchy. The second part is easy as well, just write down.

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