QUESTIONS

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(1) Show that there exist arbitrarily large intervals that are free of primes, i.e., for every positive integer \( k \) there exist \( k \) consecutive positive integers none of which is a prime.

(2) Prove that \( \mu(n) \mu(n+1) \mu(n+2) \mu(n+3) = 0 \) if \( n \) is a positive integer.

(3) Find all \( n \) such that \( \phi(n) = 12 \) and \( n = 17 \phi(n) \).

(4) Find a positive integer \( n \) such that \( \mu(n) + \mu(n+1) + \mu(n+2) = 3 \).

(5) Show that \( H_n = 1 + 1/2 + ... + 1/n \) is not an integer for \( n \geq 2 \).

(6) Show that, for every positive integer \( n \geq 2 \),
\[
\sum_{1 \leq k \leq n-1 \atop (k,n)=1} k = \frac{n}{2} \phi(n)
\]

(7) Let \( f \) be a multiplicative function. We know that the Dirichlet inverse \( f^{-1} \) is then also multiplicative. Show that \( f^{-1} \) is completely multiplicative if and only if \( f(p^m) = 0 \) for all prime powers \( p^m \) with \( m \geq 2 \).

(8) If \( n \) is any even integer, prove that \( \sum_{d|n} \mu(d) \phi(d) = 0 \).

(9) Let \( f_k \) be defined as follows
\[
f_k(n) = \sum_{d|n \atop (k,d)=1} \mu(d),
\]
here \( k \) is a fixed positive integer, and the summation runs over those divisors of \( n \) that are relatively prime to \( k \). Show that \( f_k \) is the characteristic function of the set \( A_k = \{ n \in \mathbb{N} : p/n \rightarrow p/k \} \).

(10) Show that \( \exp(\log x/\log \log x) = o(x^\epsilon) \) for any \( \epsilon > 0 \).

(11) For any positive integer \( n \), prove that \( \phi(n) + \sigma(n) \geq 2n \) and the equality holds iff \( n = 1 \) or prime.

(12) Show that \( \psi(x) = \theta(x) + O(\sqrt{x}) \).

(13) Let \( \omega(n) \) be the number of distinct prime factors of \( n \). Show that \( \omega(n) \leq 2 \log n \).

(14) Let \( f \) be a multiplicative function and suppose that \( \lim_{p^m \to \infty} f(p^m) = 0 \). Show that \( \lim_{n \to \infty} f(n) = 0 \) also.

(15) Show that \( \frac{n}{\log n} << \phi(n) \) for \( n \geq 2 \).

(16) Let \( d(n) \) be the number of divisors of \( n \). Show that \( d(n) = O(n^\epsilon) \) for every \( \epsilon > 0 \).

(17) Show that \( d(n) = O(\log n) \) is not true.

(18) Show that \( \frac{n}{\phi(n)} = \sum_{d|n} \frac{\mu(n)^2}{\phi(n)} \).

(19) Show that \( \sum_{n \leq x} \frac{\mu(n)^2}{\phi(n)} \geq \log x \).

(20) Show that \( \sum_{n \leq x} \frac{n}{\phi(n)} << x \). Moreover show that for any fixed real number \( k \), \( \sum_{n \leq x} \left(\frac{n}{\phi(n)}\right)^k << x \). This means most of the time \( \phi(n) \) is very close to \( n \).

(21) Let \( P \) be a set of primes such that \( \sum_{p \in P} \frac{1}{p} \) is finite.
Define \( A_P = \{ n : p|n \rightarrow p \in P \} \), \( A_P(x) = |\{ n \leq x : n \in A_P \}| \),
\[ C = \{n : (n, p) = 1 \forall p \in P\}. \] Show that \( \sum_{n \in A_P} \frac{1}{n} \) is finite, \( A_P(x) = o(x) \), and \( C(x) \) is asymptotic to \( ax \) where \( a = \prod_{p \in P} (1 - 1/p) \).

(22) Show that \(|\{n \leq x : p|n \to p = 4k + 1\}| = o(x)\).

(23) Show that \( \pi_3(x) < \frac{x}{\log^2 x} \) where \( \pi_3(x) \) is the number of primes \( p \leq x \) such that \( p + 2 \) and \( p + 6 \) are also primes.

(24) Show that \( \sum_{p \leq x} d(p - 1) = O(x) \).

(25) Show that primes of the form \( n^2 + 1 \) with \( n \leq x \) is \(< \frac{x}{\log x} \).

(26) Prove that Selberg’s asymptotic formula
\[
\psi(x) \log x + \sum_{n \leq x} A(n) \psi(\frac{x}{n}) = 2x \log x + O(x)
\]
implies Chebyshev estimates. (In fact Selberg’s formula has a key role for the elementary proof of PNT.)

(27) Using Mertens’ estimates find the asymptotic of \( \sum_{pq \leq x} \frac{1}{pq} \). Using PNT prove that \( \sum_{pq \leq x} 1 \) is asymptotic to \( \frac{x \log \log x}{\log x} \).

(28) It is known that PNT is equivalent to \( \sum_{n \leq x} \mu(n) = o(x) \). Using this show that
\[
PNT \iff \lim_{x \to \infty} x \sum_{n > x} \frac{\mu(n)}{n} = 0
\]

(29) Let \( E = \lim \inf_{n} \frac{p_{n+1} - p_{n}}{\log p_{n}} \). Using PNT show that \( E \leq 1 \).

(30) Using Brun’s sieve what can you say about a lower bound for \( \pi(x) \) and \( \pi_2(x) \)?