A Mathematical Introduction to Modal Logic

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NESİN MATEMATİK KÖYÜ

Contents

Preface

Modal logic is a huge research area. Researchers from mathematics, philosophy, computer science, linguistics, political science and economics work on variety of modal logics focusing on numerous different topics with many amazingly different applications. Mathematicians approach it mostly from a model theoretical point of view. For philosophers, modal logic is a powerful tool for semantics. Many concepts in philosophy of language can be formalized in modal logic. Computer scientists, on the other hand, use modal logic to represent the programs. Model checking and temporal logic are very *hot* research areas in computer science which use modal logics extensively. In semantics theory that many linguists work on, modal logic helps a lot. Political scientists and economists try to come up with fair division algorithms and welfare theories that have game theoretical motivation with an underlying modal logical intuition. Moreover, game theory uses modal models to represent the knowledge of the players and their decisions under uncertainty. All these research areas show that modal logic is a promising field with many interdisciplinary research opportunities (see my webpage with many small notes discussing variety of modal logics).

These lecture notes were prepared for the *Turkish Mathematical Society 2009 Summer School*. The intended purpose is to cover the basics of modal logic from a rather mathematical perspective. Nevertheless, we will try our best not to lose our basic intuition by making occasional remarks to the philosophical and applied considerations. The intended course is a short one, and these notes will cover only the basics. For this reason, based on a subjective judgement, I left out several important subject such as algebraical and category theoretical treatment of modal logics.

There will be many exercise questions spread in the text and it is highly recommended that the reader spend enough time on them. One major difference between this lecture notes and any other mathematics text is that I did not include most proofs. There are several reasons for that. First is to invite the reader to attend the classes to learn the proofs with discussions. Second, once the reader gets stuck, is to encourage him to go check the basic articles and textbooks which would be considered as a small-scale research.

These notes were written in Summer 2009 and come with no guarantee. If you spot a typo, or an unconvincing argument along these lines, please let me know. You can contact me by e-mail at cbaskent@gc.cuny.edu or at www.canbaskent.net. You can also find the lecture notes at the latter address.

Enjoy!

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Lecture 1

Introduction

1.1 Introduction

Modal logic is the logic of *modalities*. There are variety of modalities one can think of. *Alethic* modalities are the modalities dealing with *possibility* (not to be confused with "probability") and *necessity*. *Epistemic* modalities intend to formalize *knowledge* while *doxastic* modalities do it for *belief*. Yet another well-studied modality, *temporal* modality discusses the *time* and *tense*. *Deontic* modalities, on the other hand, are studied for the logic of obligations and norms. *Dynamic* modalities formalize *actions* and *programs*.

Historical background of the modal logics can be traced back to Aristotle when he distinguished the statements with "necessary" and "possible" (Goldblatt, 2006). This is what we now call an alethic modality. Furthermore, according to the modality of the given statement, we can analyze it from a modal logical perspective. If the sentence is uttered in the form 'it is known to the agent that...' or "the agent knows that...", then it is easy to see that epistemic modalities should be used to express it. In a similar fashion, if we intend to formalize the time adverbs such as "henceforth", "eventually", "since" or "until", we will need temporal logic etc.

Modal logics became popular when Kripke gave his famous completeness proof which introduced his intuitive semantics which was named after him (Kripke, 1959). He also used the very same notions in his philosophy of language in a very bright way. His notion of *rigid designator* heavily depends on the modal notions.

1.2 Some Modal Logics

1.2.1 Temporal Logic

From a linguistics point of view, classical propositional logic is always problematic. Consider the following statement which we will call α: "I came home and had dinner". Now, consider another statement which we will call β:"I had dinner and came home". Apparently, they do have different meaning. The statement α says, first I came home, then I had dinner whereas β says, first I had dinner, then came home. But, from a logical point of view, α and β have the same truth value, thus $\alpha \equiv \beta$. What should we do?

In a similar fashion, how can we formalize the situation when φ *will* be the case, or φ *was* the case? The notion of *time* and *tense* enter the game.

The first step would be to have temporal modalities to express the situation (van Benthem, 1988):

$F\varphi : \varphi$ will be the case in the future (at least once)
$P\varphi : \varphi$ was the case in the past (at least once)

Notice that F stands for "future" while P stands for "past".

In a similar fashion, we can have some more inter-definable (we will see how) tense modalities:

$G\varphi : \varphi$ will always be the case in the future
$H\varphi : \varphi$ was always the case in the past

Some examples are in order.

Example 1.1. Let us see how we can formalize simple statements by using temporal logic. Let χ denote the statement "I come home" and ξ denote the statement "I have dinner" (notice that they are in present tense). Recalling the discussion at the beginning of this section, let us try to formalize α and β with their intended meaning. The statement $P(\chi \wedge F\xi)$ will describe α while the statement $P(\xi \wedge F\chi)$ will describe β. Obviously, when modalities are involved α and β are indeed different statements, i.e. $(\alpha \leftrightarrow \beta)$ is not a validity.

Exercise 1.2. *Convince yourself that $\varphi \to GP\varphi$ and $\varphi \to HF\varphi$ make sense.*

We mentioned earlier that the philosophical discussion on modalities go back to Aristotle. Let us now consider his famous sea battle argument.

Example 1.3. (van Benthem, 1988)

"If I give the order to attack (p), then, necessarily, there will be a sea battle tomorrow (q). If not, then, necessarily, there will not be one. Now, I give the order, or I do not. Hence, either it is necessary that there is a sea battle tomorrow, or it is necessary that none occurs."

This famous logical argument for determination of the future (the admiral does not "really" have a choice) has been revived again and again in the philosophical tradition by J. Łukasiewicz and R. Taylor. One first analysis employs a modal notation similar to that we have done along these lines. Let us denote the *necessity* by $\Box$. Then, we have the following two reading of the sea battle argument.

$$
\begin{array}{ll}
p \to \Box q & \Box(p \to q) \\
\neg p \to \Box \neg q & \Box(\neg p \to \neg q) \\
p \vee \neg p & p \vee \neg p \\
\therefore \quad \Box q \vee \Box \neg q & \therefore \quad \Box q \vee \Box \neg q
\end{array}
$$

The crucial point is the scope of the "necessarily" in the first premises. In the first reading (narrow scope), the argument is valid, but the (strong) premises beg the conclusion. In the second reading (wide scope), the premises are plausible, but the conclusion does not follow.

1.2.2 Provability Logic

If one interprets $\Box\varphi$ as "φ is provable in Peano Arithmetic" valid principles from the metamathematics of arithmetic turn into familiar modal axioms, notably $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and $\Box\varphi \to \Box\Box\varphi$, and, most striking, Lob's Axiom: $\Box(\Box\varphi \to \varphi) \to \Box\varphi$ (van Benthem, 1988).

Notice that $\Box\varphi \to \varphi$ is not generally provable in PA - even though this modal axiom is intuitively true.

1.2.3 Epistemic Logic

Let us have an epistemic modality K which stands for "knowledge". In this reading, the statement $K\varphi$ will read "the agent knows φ" (Hintikka, 1962).

Let us consider several axioms of epistemic logic and their respective readings.

$K\varphi \to \varphi$	what the agent knows is true
$K\varphi \to KK\varphi$	if the agent knows something, she knows that she knows it
$\neg K\varphi \to K\neg K\varphi$	if the agent does not know something, she knows that she does not know it

Exercise 1.4. *Give the epistemic reading of* $K(\varphi \to \psi) \to (K\varphi \to K\psi)$.

1.3 Possible Worlds

The idea of possible worlds to describe an ontology goes back to the famous German mathematician and philosopher Leibniz. Leibniz, while discussing the problem of evil, remarked that the *god created the world as the best out of all possible worlds*. In his picture, therefore, there are variety of possible worlds $w_0, w_1, w_2, \ldots$, but the god (according to some calculation) picked the best one, say w_0, and created the world according to it. In other words, we do consider the other worlds *possible*. Furthermore, there is a relation between our world w_0 and the other worlds w_1, w_2 etc. Following Leibniz's analogy, w_0 is the actual state we are in now as it is the best possible one.

1.4 A Note on First-Order Modal Logic

This course is intended as an absolute beginner course in modal logic. Thus, whenever we say "modal logic", we always mean "propositional unimodal logic", modal logics without quantification and with only one modality (recall that temporal logic has two modalities F and P).

The reader familiar with model theory might immediately ask: "What about the modal logic in a first-order language"? However, the answer is not very optimistic. The first-order modal logic is not so much well-studied and there is not much application related to it, yet. Furthermore, variety of philosophical and technical issues such as *de re / de dicto* distinction and Russellian definite descriptions still pose some problems. Nevertheless, we invite the reader to study the major textbook on that subject (Fitting & Mendelsohn, 1998).

Lecture 2

Basics

In this lecture we will introduce the basics of modal logic. We will define modal models, and discuss their syntactical and semantical structures. For a much more sophisticated account of the model theory of modal logic, we invite the reader to consult to (Goranko & Otto, 2007) and (Goldblatt, 2006). First, we will discuss the relational semantics of modal structures which were made popular by Kripke (Kripke, 1959).

2.1 Syntax

For simplicity, we will consider modal structures with one modality. First, let us construct our logical language $\mathcal{L}$. We will need a countable set of propositional variables $P = \{p, q, r, \dots\}$. We will use the *signature* of propositional logic together with a modal symbol $\Diamond$. We will call the diamond modality $\Diamond$ the *possibility* modality. The inter-definable *necessity* modality will be denoted as $\Box$ (reads *box*). However, for our current purposes, we only need to include one of them (in this case, $\Diamond$) to our signature.

The set of well-formed formulae of the language $\mathcal{L}$ of the propositional modal logic is recursively formed in the following fashion.

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi$$

Note that, the propositional variable p varies over P. The *truth* constant $\top$ is imported from the propositional logic. The unstated Boolean connectives $\vee$ and $\rightarrow$ are inter-definable obviously.

The binding strength of the symbols will be as same as in the propositional logic. The additional symbol $\Diamond$ will bind strongest. Thus, we will omit the parenthesis where there is no ambiguity.

Exercise 2.1. *Verify that the following are well-formed formulae in the language of modal logic: (i) $\Diamond\Diamond\Diamond p$, (ii) $p \wedge \Diamond\top$, (iii) $\Diamond q \wedge \neg\Diamond\neg p$*

Now, we can define modal models with one single modality. A modal model $M = \langle W, R, V \rangle$ is triple where W, the *domain*, is a non-empty set, $R \subseteq W \times W$ is a binary relation on W, and V is a valuation function that takes every propositional variable p from P and assigns a subset $V(p) \subseteq W$ to it. This is what we usually call a *relational model*. The relation will soon be able to express our notions of necessity and possibility. This definition can easily be extended to multi-modal cases

when there are more than one modalities involved. For self-containment of our discussion here, let us give the definition of multi-modal models. A multi-modal model $\langle W, \{R_0, R_1, \ldots R_n\}, V \rangle$ is a structure where W and V are as before, and each R_i is a binary relation on W for all $i \leq n$.

We call the structure $F = \langle W, R \rangle$ a *frame*. A model, therefore, is a frame equipped with a valuation.

Exercise 2.2. *Let W be a set with the property that $|W| = n$. How many different frames can be defined by taking W as the carrier set?*

2.2   Semantics

Now, we have a language and a set of well-formed formulae and a model. We can now give a meaning to these formulae. For a given model $M = \langle W, R, V \rangle$ and a point $w \in W$, we define the satisfaction relation $\models$ as follows.

$$
\begin{array}{lll}
M, w \models p & \text{iff} & w \in V(p) \\
M, w \models \top & \text{iff} & \text{always} \\
M, w \models \neg\varphi & \text{iff} & \text{not-}(M, w \models \varphi) \text{ (notation: } M, w \not\models \varphi) \\
M, w \models \varphi \wedge \psi & \text{iff} & M, w \models \varphi \text{ and } M, w \models \psi \\
M, w \models \Diamond\varphi & \text{iff} & \exists v \in W(wRv \wedge M, v \models \varphi)
\end{array}
$$

We define $\Box$ as the *dual* of $\Diamond$, i.e. $\Box\varphi \equiv \neg\Diamond\neg\varphi$. The semantics of $\Box\varphi$ then can be given similarly.

$$
M, w \models \Box\varphi \quad \text{iff} \quad \forall v \in W(wRv \rightarrow M, v \models \varphi)
$$

Since $\Box$ is inter-definable in terms of $\Diamond$, we did not need to include it in the signature of $\mathcal{L}$.

Exercise 2.3. *Verify that the semantics of $\Box\varphi$ is indeed correct.*

Exercise 2.4. *Give an informal semantic argument for why $\Box\varphi \equiv \neg\Diamond\neg\varphi$.*

Exercise 2.5. *Show that F and G modalities are duals of each other. Similarly, show that P and H modalities are duals of each other.*

Exercise 2.6. *Show that $\varphi \rightarrow GP\varphi$ and $\varphi \rightarrow HF\varphi$ are sound in temporal logic.*

We write $M \models \varphi$ if φ is true at all points in M. For instance, $M \models \top$ and $M \models \Box r \vee \neg\Box r$. A formula is called *satisfiable* in a model M if there is a point in M at which it is true.

Exercise 2.7. *Show that $M \models \varphi$ for all modal models M if φ is a propositional tautology.*

Let us see how the semantics work with an example.

Example 2.8. Consider the model given in the following figure (Blackburn & van Benthem, 2006). Let us call it M.
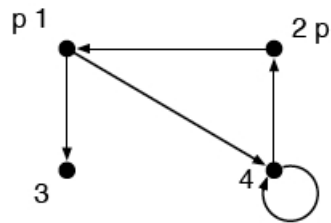
Observe that the propositional variable p is true only at the points 1 and 2. Let us first write down the relation R represented by the arrows. Thus, $R = \{(1,3),(1,4),(2,1),(4,2),(4,4)\}$. Observe that $M,1 \models \Diamond\Box\Diamond p$.

Exercise 2.9. *Consider the model M given in Example 2.8. Verify that $M,2 \not\models \Diamond\Box\Diamond p$ and $M,3 \not\models \Diamond\Box\Diamond p$.*

In this course, we will not touch the validity and satisfiability problems or any other complexity theory subjects. But for the reader who is familiar with the notions, note that the validity problem for the basic modal logic **K** (that we will define in the next section) is *PSPACE*. Recall that the validity problem for classic propositional logic is *NP-COMPLETE*.

2.3 Tableaus

In the previous section, we have defined the truth at the modal models. Now, we can give the proof theoretical structure of modal logics.

The set of all modal validities can be axiomatized by a Hilbert-style proof system called **K**. The axioms of **K** are the following.

1. All propositional tautologies,

2. $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

Exercise 2.10. *Prove that the axiom scheme $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ is sound.*

The second axiom is called *Kripke axiom* or *normality axiom*. The logics which possess the normality axiom are surprisingly called *normal modal logics*.

We have two proof rules. The first one, *modus ponens*, is a familiar one: If $\vdash \varphi$ and $\vdash \varphi \to \psi$, then $\vdash \psi$.

The second one is unique to modal models, and called the *necessitation rule*: If $\vdash \varphi$, then $\vdash \Box\varphi$. A brief explanation for the necessitation rule is in order here. The assumption $\vdash \varphi$ means that the formula φ is provable with no additional axiom required. A proof is a syntactic notion, and independent from the valuation and semantics (at least, until we prove the completeness). In other words, naively, one can consider proofs as a symbol manipulation. Thus, if φ is provable, then it is provable at each state regardless of the valuation. Thus, wherever you go, it will be necessary, namely it will stay provable at each accessible state. Hence, $\vdash \Box\varphi$.

Let us see an incorrect application of the necessitation rule. Consider the following "proof".

1. φ Assumption
2. $\Box\varphi$ Necessitation: 1
3. $\varphi \rightarrow \Box\varphi$ Discharge assumption

Neither of our axioms says that $\varphi \rightarrow \Box\varphi$ is valid. So, there is something wrong: we cannot use assumptions as input to necessitation rule! The reason for that is the fact that the necessitation rule does not preserve satisfiability. Necessitation rule enables us to generate new validities from the old ones (Blackburn *et al.* , 2001).

Exercise 2.11. *Show that the deduction theorem is not valid in modal models.*

Exercise 2.12. *Recall the definition of a proof.*

Now, we can start discussing a major proof method for modal logics. Tableau systems are "one of the common analytical proof procedures" (Fitting & Mendelsohn, 1998). The important aspect of tableau systems is that they are refutation systems. As it was remarked, in tableau based proofs, "to prove a formula, we begin by negating it, analyze the consequences of doing so, using a kind of tree structure, and, if the consequences turn out to be impossible, we conclude the original formula has been proved" (Fitting & Mendelsohn, 1998). Namely, we construct an argument based on *reductio ad absurdum*. The tableau methods we will present here are due to Fitting. Most of the arguments we will cite here are taken from the aforecited book, thus we will not mention it over and over again to maintain the readability of this section.

Recall that in the semantics of the modal logic, we had states which are accessible from some other states. We will simply mimic this idea in this proof procedure. We will define a *prefix* as a finite sequence of positive integers such as 1.2 or 1.1.2.3. If σ is a prefix and n is an integer, then $\sigma.n$ is the prefix composed of σ followed by a period and followed by the integer n. Prefixes will help us to keep track of the states we are considering. Also note that if σ is a prefix then $\sigma.n$ is accessible directly from σ. Namely, $\sigma.n$ is a possible world for σ. This semantic argument should not be taken as a proof method as we have not shown the completeness of modal logic yet. Nevertheless, the tableaus reflect the same intuition and many completeness proofs are given by using tableaus (Chagrov & Zakharyaschev, 1997).

An attempt to construct a tableau proof of a formula φ begins by creating a tree with $1.\neg\varphi$ as its root. This step says, the formula φ is false. We thus start by the negation of the formula to get a contradiction. We will construct the tree depending on the form of the formula φ by following the rules. Depending on the form, we can have multiple branches as well. If we can spot a contradiction along a branch, this will *close* the branch. As our aim was to get a contradiction for each possibility, thus our aim is to close each branch (notation $\otimes$). Once this is done, this will prove the statement. Let us first see the rules, then the examples will clarify the matter.

Definition 2.13 (Tableau Rules for Booleans)**.** Let σ be any prefix. The rule of double negation is as follows.

$$\frac{\sigma \ \neg\neg\varphi}{\sigma \ \varphi}$$

Conjunction rules are given as follows.

$$\frac{\sigma \ \varphi \wedge \psi}{\begin{array}{c} \sigma \ \varphi \\ \sigma \ \psi \end{array}} \qquad \frac{\sigma \ \neg(\varphi \vee \psi)}{\begin{array}{c} \sigma \ \neg\varphi \\ \sigma \ \neg\psi \end{array}} \qquad \frac{\sigma \ \neg(\varphi \rightarrow \psi)}{\begin{array}{c} \sigma \ \varphi \\ \sigma \ \neg\psi \end{array}} \qquad \frac{\sigma \ \varphi \leftrightarrow \psi}{\begin{array}{c} \sigma \ \varphi \rightarrow \psi \\ \sigma \ \psi \rightarrow \varphi \end{array}}$$

Disjunction rules are the following.

$$\frac{\sigma\ \varphi \vee \psi}{\sigma\ \varphi \quad \sigma\ \psi} \qquad \frac{\sigma\ \neg(\varphi \wedge \psi)}{\sigma\ \neg\varphi \quad \sigma\ \neg\psi} \qquad \frac{\sigma\ \varphi \rightarrow \psi}{\sigma\ \neg\varphi \quad \sigma\ \psi} \qquad \frac{\sigma\ \neg(\varphi \leftrightarrow \psi)}{\sigma\ \neg(\varphi \rightarrow \psi) \quad \sigma\ \neg(\psi \rightarrow \varphi)}$$
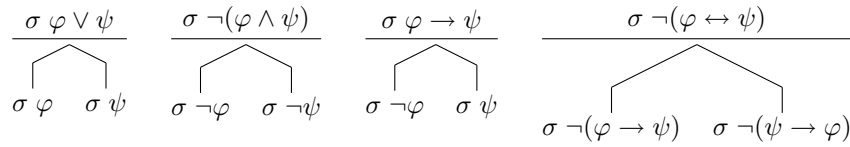
Definition 2.14 (Tableau Rules for Modalities). If the prefix $\sigma.n$ is new to the branch,
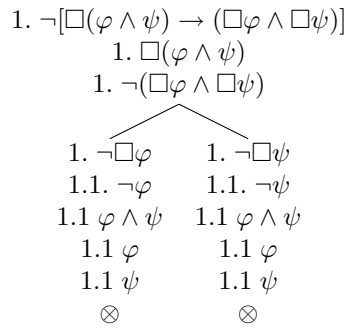
$$\frac{\sigma\ \Diamond\varphi}{\sigma.n\ \varphi} \qquad \frac{\sigma\ \neg\Box\varphi}{\sigma.n\ \neg\varphi}$$

If the prefix $\sigma.m$ already occurs on the branch,

$$\frac{\sigma\ \Box\varphi}{\sigma.m\ \varphi} \qquad \frac{\sigma\ \neg\Diamond\varphi}{\sigma.m\ \neg\varphi}$$

Let us now see in an example how tableau proofs work.

Example 2.15. Let us prove $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$ with a tableau.

$$
\begin{array}{c}
1.\ \neg[\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)] \\
1.\ \Box(\varphi \wedge \psi) \\
1.\ \neg(\Box\varphi \wedge \Box\psi)
\end{array}
$$

$1.\ \neg\Box\varphi$	$1.\ \neg\Box\psi$
$1.1.\ \neg\varphi$	$1.1.\ \neg\psi$
$1.1\ \varphi \wedge \psi$	$1.1\ \varphi \wedge \psi$
$1.1\ \varphi$	$1.1\ \varphi$
$1.1\ \psi$	$1.1\ \psi$
$\otimes$	$\otimes$

Exercise 2.16. *Give a tableau proof of $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$.*

Exercise 2.17. *Give a tableau proof of $(\Box\varphi \wedge \Diamond\psi) \rightarrow \Diamond(\varphi \wedge \psi)$.*

Exercise 2.18. *By constructing a tableau, observe that $\Box(\varphi \vee \psi) \rightarrow (\Box\varphi \vee \Box\psi)$ is not provable. Give a counter-model out of the non-closed tableau you constructed.*

Exercise 2.19. *Show that if $\Box\varphi$ has a tableau proof, so does φ.*

Notice that the tableau proofs are significantly easier than Hilbert-style proofs. For example consider the following example which was proved in Example 2.15 (Blackburn *et al.* , 2001).

Example 2.20. Consider the formula $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$. This formula is provable in the basic modal logic. Consider the following sequence of formulae.

1. $\vdash \varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi))$		Tautology
2. $\vdash \Box(\varphi \rightarrow (\psi \rightarrow (\varphi \wedge \psi)))$		Generalization: 1
3. $\vdash \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$		K-axiom

4. $\vdash \Box(\varphi \to (\psi \to (\varphi \land \psi))) \to (\Box\varphi \to \Box(\psi \to (\varphi \land \psi)))$ Uniform Substitution for 3

5. $\vdash \Box\varphi \to \Box(\psi \to (\varphi \land \psi))$ Modus Ponens: 2, 4

6. $\vdash \Box(\varphi \to (\varphi \land \psi)) \to (\Box\varphi \to \Box(\varphi \land \psi))$ Uniform Substitution for 3

7. $\vdash \Box\varphi \to (\Box\psi \to \Box(\varphi \land \psi))$ Propositional Logic: 5, 6

8. $\vdash (\Box\varphi \land \Box\psi) \to \Box(\varphi \land \psi)$ Propositional Logic: 7

Strictly-speaking, the given sequence of formulae is not a proof in the modal logic. It is a subsequence of the proof consisting of the most important items. To get the full proof, the reader should fill in the gaps to get the last three lines.

Lecture 3

Bisimulations

Recall that in model theory, we have *homomorphisms* (Hodges, 2002).

Definition 3.1. Let M and N be two structures in a common language. By a homomorphism f from M to N we will mean a function f from $dom(M)$ to $dom(N)$ with the following three properties.

1. For each constant c in the language, $f(c^M) = c^N$.

2. For each $n > 0$ and each n-ary relation symbol R in the language, and n-tuple $\vec{a}$ from M, if $\vec{a} \in R^M$, then $f\vec{a} \in R^N$.

3. For each $n > 0$ and each n-ary function symbol F in the language, and n-tuple $\vec{a}$ from M, $f(F^M(\vec{a})) = F^N(f\vec{a})$.

where for $\vec{a} = (a_0, \ldots, a_{n-1})$, by $f\vec{a}$, we mean $(fa_0, \ldots, fa_{n-1})$

Theorem 3.2. *Let M and N be two models and f be a mapping from M to N. f is a homomorphism if and only if, for every atomic formula $\varphi(\vec{x})$, and tuple $\vec{a}$ of M,*

$$M \models \varphi(\vec{a}) \Rightarrow N \models \varphi[f\vec{a}]$$

Proof. Exercise, see (Hodges, 2002) ∎

Thus, homomorphism can help us in dealing with atomic formulae in modal models. However, when it comes to modalities, we need some other constructions. Considering modal models as processes or transition systems, we need a construction that will enable us to represent modal model equivalence.

Before going into details, let us first consider several semantic mappings which preserve truth in modal models.

3.1   Truth Preserving Operations

3.1.1   Disjoint Unions

Let us assume we have two separate modal models M and N with different domains. What would happen if we consider these two separate models as one single structure put together. Disjoint

unions address this construction. Let us now give a general definition. Note that we will call two models disjoint if their domains are disjoint.

Definition 3.3 (Disjoint Unions)**.** Let $M_i = \langle W_i, R_i, V_i \rangle$ be a family of disjoint models. Their disjoint union is the structure $\biguplus_i M_i = \langle W, R, V \rangle$ where $W = \bigcup_i W_i$, $R = \bigcup_i R_i$ and $V(p) = \bigcup_i V_i(p)$.

Disjoint union operation preserve the truth.

Theorem 3.4. *Let $M_i = \langle W_i, R_i, V_i \rangle$ be a family of disjoint models. Then for each modal formula φ, and for each i, and each $w \in M_i$, we have the following.*

$$M_i, w \models \varphi \text{ if and only if } \biguplus_i M_i, w \models \varphi$$

Proof. In class. This theorem shows that modal satisfaction is invariant under disjoint unions. ■

3.1.2 Generated Submodels

The idea of *accessibility* is crucial for modal logic. There are points which are accessible from the current state, and similarly there are points which are not accessible from the current state. Therefore, we can throw away the points which are inaccessible from the current state. Since, the modality will not be able to access them, it is safe to get rid off them.

Let us first start with defining *submodels*. Let $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ be two models. We say M' is a submodel of M if $W' \subseteq W$, $R' = R \cap (W' \times W')$ and $V'(p) = V(p) \cap W'$.

Thus, to obtain a submodel, we take a subset of the carrier set, and restrict the relation and valuation to that set. Now, we can define generated submodels.

Definition 3.5 (Generated Submodels)**.** Let $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ be two models. We say M' is a generated submodel of M if the following closure condition holds:

$$\text{If } w \in M' \text{ and } wRv, \text{ then } v \in M'$$

We have an invariance result.

Theorem 3.6. *Let M' be a generated submodel of M. Then, for each formula φ in the language of basic modal logic, and for each point $w \in M'$, we have the following.*

$$M, w \models \varphi \text{ if and only if } M', w \models \varphi$$

Proof. In class. This theorem shows that modal satisfaction is invariant under generated submodels.
■

3.1.3 Bounded Morphism

We already remarked that homomorphisms are weak to establish modal satisfiability invariance as they do not cover modal cases. Now, our task is to generalize this concept to modalities.

Let us first learn what bounded morphisms are. Let $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ be two models, and $f : M \to M'$ be a mapping with the following conditions.

1. w and $f(w)$ satisfy the same propositional variables.

2. If wRv then $f(w)R'f(v)$.

3. If $f(w)R'v'$, then there exists v such that wRv and $f(v) = v'$.

The expected theorem follows.

Theorem 3.7. *Let M and M' be two models, and $f : M \to M'$ a bounded morphism. Then, for each point in M and for each formula φ in the language of basic modal logic, we have the following.*

$$M, w \models \varphi \text{ if and only if } M', f(w) \models \varphi$$

Proof. In class. This theorem shows that modal satisfaction is invariant under bounded morphisms. ∎

3.2 Bisimulations

As it was put in (Blackburn *et al.*, 2001), "a bisimulation is a relation between two models in which related states have identical atomic information and matching transition possibilities".

Definition 3.8 (Bisimulation)**.** Let $M = \langle W, R, V \rangle$ and $M' = \langle W', R', V' \rangle$ be two models. A non-empty binary relation $\Gamma \subseteq W \times W'$ is called a bisimulation if the following conditions are satisfied.

1. If $w\Gamma w'$, then w and w' satisfy the same propositional variables.

2. If $w\Gamma w'$ and wRv, then there exists $v' \in W'$ such that $v\Gamma v'$ and $w'R'v'$.

3. If $w\Gamma w'$ and $w'R'v'$, then there exists $v \in W$ such that $v\Gamma v'$ and wRv.

Let us now introduce a piece of notation. If there is a bisimulation between two states w and w', we denote it as $w \simeq w'$. If two states w and w' satisfy the same formulae, i.e. $w \models \varphi$ if and only if $w' \models \varphi$, then we denote it as $w \sim w'$. Notice that these two notions are not identical.

The expected theorem is as follows.

Theorem 3.9. *Let M and M' be two models. Then, for every $w \in dom(M)$ and $w' \in dom(M')$, we have $w \simeq w'$ implies $w \sim w'$.*

Proof. In class. This theorem shows that modal satisfaction is invariant under bisimulations. ∎

What about the converse then? When does $w \sim w'$ imply $w \simeq w'$?
We call a model *image-finite*, if for each state w, the set of accessible states from w is finite.

Theorem 3.10. *Let M and M' be two image-finite models. Then, for every $w \in dom(M)$ and $w' \in dom(M')$, we have $w \simeq w'$ if and only if $w \sim w'$.*

Proof. In class! ∎

Lecture 4

Alternative Semantics

The semantics we have presented so far is by far the most common semantics that is being used. However, there some other semantics that are worth discussing.

4.1 Topological Semantics

Topological semantics of modal logic is historically the first one and can be traced back to the 1930s (Goldblatt, 2006). The most influential series of works on the topological interpretation of modal logic was initiated by J. C. C. McKinsey whom was later joined by A. Tarski (McKinsey & Tarski, 1944). Let us now start by reviewing the basics.

Definition 4.1 (Topological Space). A topological space $\mathcal{S} = \langle S, \sigma \rangle$ is a structure with a set S and a collection σ of subsets of S satisfying the following axioms:

1. The empty set and S are in σ.

2. The union of any collection of sets in σ is also in σ.

3. The intersection of a finite collection of sets in σ is also in σ.

The collection σ is said to be a *topology* on S. The elements of S are called *points* and the elements of σ are called *opens*. The complements of open sets are called *closed* sets.

A topological model M is a triple $\langle S, \sigma, V \rangle$ where $\langle S, \sigma \rangle$ is a topological space, and V is a valuation function. When we are in topological models, we will use the notation $\mathbb{I}$ for $\Box$ after the term *interior*. Likewise, we will put $\mathbb{C}$ for $\Diamond$ after the term *closure*.

In topological models, the $\mathbb{I}$ operator is interpreted as the topological *interior* operator Int. Recall that the interior of a set X, $Int(X)$, is the largest open subset of X. We then define the valuation of $\mathbb{I}$ modality as follows
$$V(\mathbb{I}\varphi) = Int(V(\varphi))$$

Recall now that the topological interior operator Int satisfies the following properties for each $X, Y \subseteq S$: (i) $Int(X) \subseteq X$, (ii) $Int(X \cap Y) = Int(X) \cap Int(Y)$, (iii) $Int(Int(X)) = Int(X)$.

Exercise 4.2. *Recall the properties of the closure operator Clo.*

Consequently, we define the semantics as follows.

Definition 4.3 (Topological Semantics). Truth of modal formulae in topological semantics is defined inductively at a point s for a topological model $M = \langle S, \sigma, V \rangle$:

$$
\begin{array}{lll}
M, s \models p & \text{if and only if} & s \in V(p) \text{ for } p \in P \\
M, s \models \neg\varphi & \text{if and only if} & M, s \not\models \varphi \\
M, s \models \varphi \wedge \psi & \text{if and only if} & M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models \mathbb{I}\varphi & \text{if and only if} & \exists U \in \sigma \ (s \in U \wedge (\forall t \in U), M, t \models \varphi)
\end{array}
$$

There is a very well known connection between the topological spaces and the Kripke models. Namely, every reflexive and transitive Kripke frame $\langle S, R \rangle$ gives rise to a topological space $\langle S, \sigma_R \rangle$, where σ_R is the set of all upward closed subsets of the given frame. It is easy to see that the empty set and S are in σ_R, and furthermore arbitrary unions and finite intersections of upward closed sets are still upward closed. Hence, σ_R is a topology.

The topology σ_R we obtained out of the Kripke frame $\langle S, R \rangle$ is a special one and is called an Alexandroff topology. Alexandroff topologies are those in which each point has a *least* neighborhood. It is evident that, in Kripke frames the least neighborhood of a point is the set of points which are accessible in one step. In other words, the least neighborhood of a point s is the set $\{t \in W : sRt\}$.

However, Alexandroff topologies can be characterized in several ways. Another equivalent definition states that Alexandroff spaces are those topological spaces in which intersection of any family of opens is again an open. It is then a nice exercise to see the equivalence of these two definitions.

Exercise 4.4. *Verify that the two given definitions of Alexandroff topologies are indeed equivalent.*

We just briefly explained how to obtain a topology out of a Kripke frame. We can also get a Kripke frame from a topological space. The reason for that is the fact that each topological space $\langle S, \sigma \rangle$ induces a partial order R_σ defined as $sR_\sigma t$ for $s \in Clo(t)$. It is now straight forward to see that R_σ is reflexive and transitive. We then have $R = R_{\sigma_R}$ if and only if the topological space is Alexandroff (van Benthem & Bezhanishvili, 2007).

4.2   Game Theoretical Semantics

We will here consider two kinds of games. *Modal Evaluation Games* will evaluate whether a given formula φ holds at a given state w in a given model M. The *Modal Bisimulation Games*, on the other hand, will evaluate whether two states are bisimular or not.

The players of both games will be Eloise (denoted by $\exists$) and Abelard (denoted by $\forall$) by following the common terminology.

We will calculate whether a given formula φ holds at a given state w in the model M by playing modal evaluation games. The positions in the game will be of the form (φ, w) where φ is a well formed formula in the language of basic modal logic and w is a state. We will work with formulae in the positive normal form. Recall that a formula φ is in positive normal form if φ has no negation symbol, or $\varphi \equiv \neg\psi$ where ψ has no negation symbol. Observe that, $\neg$ symbol changes the roles of $\forall$ and $\exists$.

Exercise 4.5. *What is a subformula? What are the subformulae of $\Box(p \wedge (\Diamond q \vee \neg r))$?*

Hence the modal evaluation game $\mathcal{E}(\varphi, w)$ for modal model M is a board game with players $\exists$ and $\forall$ moving a token around the positions of the form (ψ, v) where ψ is a subformula of φ and v is a state in M. The game starts at the state w which is the designated state at which we evaluate the formulae φ. As the game evolves, we move to the different accessible states and unravel φ. Once a player gets stuck, the game ends. Let us now consider the rules of the game.

The rules of the game is given below.

Position	Player	Admissible Moves
$(\bot, w)$	$\exists$	$\emptyset$
$(\top, w)$	$\forall$	$\emptyset$
(p, w) with $w \in V(p)$	$\forall$	$\emptyset$
(p, w) with $w \notin V(p)$	$\exists$	$\emptyset$
$(\psi_1 \wedge \psi_2, w)$	$\forall$	$\{(\psi_1, w), (\psi_2, w)\}$
$(\psi_1 \vee \psi_2, w)$	$\exists$	$\{(\psi_1, w), (\psi_2, w)\}$
$(\Diamond \psi, w)$	$\exists$	$\{(\psi, v) : wRv\}$
$(\Box \psi, w)$	$\forall$	$\{(\psi, v) : wRv\}$

Negations change the roles of $\forall$ and $\exists$ in the game. Observe that, since the language of modal logic is an extension of propositional logic, the game will be finite and terminate.

Winning conditions can be formulated as follows: $\exists$ wins if $\forall$ gets stuck, and dually, $\forall$ wins if $\exists$ gets stuck. As a matter of notation, we will denote the winning positions for $\exists$ by $Win_\exists(\mathcal{E}(\varphi, w))$.

What is the intuition behind introducing the game semantics? One can feel the basic intuition behind the evaluation games by considering the winning positions. Note that if a position x is a *winning position* for the player P in a game X, then P has a winning strategy starting from x in X. We will observe in the next theorem that, if a position (φ, w) is a winning position for $\exists$, then it is equivalent to say $w \models \varphi$.

Theorem 4.6. $(\varphi, w) \in Win_\exists(\mathcal{E}(\varphi, w))$ *if and only if* $w \models \varphi$.

Proof. In class. ∎

Modal bisimulation games, on the other hand, will provide an alternative semantics to approach to bisimulations. In this game, $\forall$ and $\exists$ will compare the states across the respective models. $\exists$ will win if the given two neighborhood situations are bisimular, $\forall$ will win otherwise.

Assume we are given w and w' in the models M and M' respectively. $\forall$ starts. He picks an accessible point to either v or v'. For simplicity, let us assume that $\forall$ picked v such that wRv. Now, $\exists$ has to pick a point v' with $w'Rv'$ such that v and v' are bisimular as well. She loses immediately, if she cannot find a corresponding point v in M.

A modal bisimulation game of length n, then can be defined as a game which can distinguish formulas of depth at most n. It is then easy to observe that, $\exists$ has a winning strategy in the bisimulation game of length n for w and v if and only if these two states are actually bisimular for formulas of depth at most n.

Theorem 4.7. $w \leftrightarrows_n v$ *if and only if* $\exists$ *has a winning strategy in the modal bisimulation game of length* n.

Proof. Exercise! ∎

Exercise 4.8. *Hintikka defined Ehrenfeucht - Fraïssé games first (Hintikka, 1962). Learn them.*

Lecture 5

Defining Formulae

In this section, we have two aims. First, we will establish the connection between first-order predicate logic and modal logic. Second, we will discuss the characteristic formula for some special frames.

5.1 Standard Translation

We will immediately start with the standard translation from basic modal language to the first-order language.

Let x be a first-order variable. Standard translation Θ_x is defined as follows.

$$
\begin{aligned}
\Theta_x(p) &= Px \\
\Theta_x(\top) &= x = x \\
\Theta_x(\neg\varphi) &= \neg\Theta_x(\varphi) \\
\Theta_x(\varphi \wedge \psi) &= \Theta_x(\varphi) \wedge \Theta_x(\psi) \\
\Theta_x(\Diamond\varphi) &= \exists y(xRy \wedge \Theta_y(\varphi)) \\
\Theta_x(\Box\varphi) &= \forall y(xRy \rightarrow \Theta_y(\varphi))
\end{aligned}
$$

where y is a fresh variable.

Exercise 5.1. *Compute $\Theta_x(\Box p \rightarrow p)$, $\Theta_x(\Diamond p \rightarrow p)$ and $\Theta_x(\Box\Box p \rightarrow p)$.*

Exercise 5.2. *Verify the following. $\Theta_x(\Diamond(\Box p \rightarrow q)) = \exists y_1(xRy_1 \wedge (\forall y_2(y_1 Ry_2 \rightarrow Py_2) \rightarrow Qy_1))$.*

Note that for any modal formula φ, $\Theta_x(\varphi)$ will have exactly one free variable, x. The role of this free variable is to mark the current state (Blackburn *et al.* , 2001). Let us now introduce a piece of notation. We will write $M \models \Theta_x(\varphi)[w]$ to mean that the first-order formula $\Theta_x(\varphi)$ is satisfied in the model M when w is assigned to the free variable x. The following theorem establishes the invariance result.

Theorem 5.3. *For all models M, and all states w of M and any modal formulae φ, we have:*

$$M, w \models \varphi \text{ if and only if } M \models \forall x\Theta_x(\varphi)[w]$$

Proof. In class! ∎

Exercise 5.4. *Show that for all models M, and all modal formulae φ,*

$$M \models \varphi \text{ if and only if } M \models \forall x \Theta_x(\varphi)$$

The famous van Benthem characterization theorem establishes the connection between modal and first-order languages.

Theorem 5.5 (van Benthem). *Let $\varphi(x)$ be a first-order formula in the language of first-order logic with one variable. Then, $\varphi(x)$ is invariant for bisimulations if and only if it is equivalent to the standard translation of a modal formula.*

Proof. Skipped. ∎

5.2   Defining Formulae

Recall that the frames are structure with no valuation attached. Yet, we can utilize modal formulae to characterize them.

Exercise 5.6. *Recall the definitions of reflexive, symmetric, transitive and serial relations.*

In modal frames, we have only two components: a set and a relation defined on this set. Therefore, most of our job will be to characterize the properties of the relation by using modal formulae. For instance, we can ask which the modal formula characterize the frames with reflexive relations or symmetric relations. In our section, we will discuss this.

Proposition 5.7. *The formula $\Box p \to p$ characterizes the reflexive frames.*

Proof. In class. ∎

Exercise 5.8. *Similarly, the formula $p \to \Diamond p$ characterizes the reflexive frames.*

Proposition 5.9. *The formula $\Box p \to \Box \Box p$ characterizes the transitive frames.*

Proof. In class. ∎

Exercise 5.10. *Similarly, the formula $\Diamond \Diamond p \to \Diamond p$ characterizes the transitive frames.*

Exercise 5.11. *The formula $p \to \Box \Diamond p$ characterizes the symmetric frames.*

Exercise 5.12. *The formula $\Diamond \top$ characterizes the serial frames.*

Let us now introduce the terminology. The basic modal logic is **K** describes the class of all frames. The logic **K4** characterizes the transitive frames and is axiomatized by the axioms of basic modal logic and $\Box p \to \Box \Box p$. The logic **T** characterizes the reflexive frames and is axiomatized by the axioms of basic modal logic and $\Box p \to p$. The logic **S4** characterizes the transitive and transitive frames and is axiomatized by the axioms of basic modal logic and $\Box p \to p$ with $\Box p \to \Box \Box p$. Finally, the logic **S5** characterizes the frames whose relation is an equivalence relation and is axiomatized by the axioms of basic modal logic and $\Box p \to p$ together with $\Box p \to \Box \Box p$ and $p \to \Box \Diamond p$.

There are many other intermediate logics that define some other class of frames and are axiomatized with some other additional axioms, too. However, for our current concerns here, these logics are sufficient.

Lecture 6

Completeness

The completeness proof of modal logic uses a very common method. Given a logic, we show that every consistent set of formulae has a model. Then the question is "how can we construct such a model"? This question invites a constructive method. Therefore, given a set of formulae, we will build the model step by step.

6.1   Maximal Consistency

The first notion we need is maximal consistency. A set S of formulae is maximal consistent if (i) S is consistent, and (ii) any set of formulae properly containing S is inconsistent. In other words, maximal consistency breaks down if you add one formula to a maximal consistent set. It is maximal in the sense that it cannot admit further formulae.

Exercise 6.1. *The theory at the state w in a model M is a maximal consistent set. Note that the theory at the state w in a model M is defined as $\{\varphi : M, w \models \varphi\}$.*

Let us now see some important properties of maximal consistent sets.

Theorem 6.2. *Let S be a maximal consistent set. Then,*

1. *S is closed under modus ponens. In other words, if $\varphi, \varphi \rightarrow \psi \in S$, then $\psi \in S$.*

2. *For all formulae φ, either $\varphi \in S$ xor $\neg\varphi \in S$.*

Proof. Exercise! ∎

Then, one can ask: "How can we construct maximal consistent sets?" The following famous lemma gives a precise algorithm for this task.

Lemma 6.3 (Lindenbaum's Lemma). *Let S be a consistent set of formulae. Then there is a maximal consistent set of formulae S^+ such that $S \subseteq S^+$. Here, S^+ is called the maximal consistent extension of S.*

Proof. In class! ∎

6.2 Canonical Models

Once we obtained maximal consistent sets, we can construct canonical models by lifting the relation and the valuation to the level of maximal consistent sets. Here is the definition.

Definition 6.4 (Canonical Models)**.** Canonical model M^+ for a normal modal logic is the triple $\langle W^+, R^+, V^+ \rangle$ where W^+ is the set of all maximal consistent sets in the given logic, R^+ is defined by xR^+y if for all formulae φ, $\varphi \in y$ implies $\Diamond\varphi \in x$, and finally $V^+(p) = \{x \in W^+ : p \in x\}$

There is a very important motivation behind the construction of canonical models. The valuation in the canonical model identifies the truth of a propositional variable at x with its membership in x. Thus, for all $x \in W^+$, and for all propositional variables p, we have $M^+, x \models p$ if and only if $p \in x$. Our aim now is to extend this idea to any modal formula.

Before satisfying our expectations, let us first consider the following lemma which we will need soon.

Lemma 6.5. *For any $x \in W^+$, if $\Diamond\varphi \in x$, then there is a state $y \in W^+$ such that xR^+y and $\varphi \in y$.*

Proof. In class. ■

Now we can state and prove the famous "truth lemma" which is the foundation of the completeness proof.

Lemma 6.6 (Truth Lemma)**.** *For any formula φ, we have $M^+, x \models \varphi$ if and only if $\varphi \in x$.*

Proof. In class. ■

Here is the final touch.

Theorem 6.7 (Completeness)**.** *Any normal modal logic is strongly complete with respect to its canonical model.*

Proof. In class. ■

Theorem 6.8. **K** *is strongly complete with respect to the class of all frames.*

Proof. Exercise! ■

This is the end.

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